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MULTI PING-PONG AND AN ENTROPY ESTIMATE IN GROUPS

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Abstract. We provide an entropy estimate from below for a finitely generated group of transformation of a compact metric space which contains a ping-pong game with several players located anywhere in the group.

1. Introduction

In [9], we provided entropy estimates for a finitely generated group G of transformations of a compact metric space X which contains two maps (*ping-pong players*) which transform a subset A of X into two disjoint subsets A_1 and A_2 of A. The players are located anywhere in G. Here, we improve that estimate in the more general case: G contains an arbitrary finite number of ping-pong players located anywhere in G.

The notion of entropy for finitely generated groups of transformations of compact metric spaces has been introduced (in the wider context of pseudogroups and foliations) by Ghys, Langevin and the second author [4] (see either [2, Chapter 13] or [10] for more detailed expositions). It corresponds to the topological entropy of single transformations, depends on the choice of a generating set but its vanishing (or, non-vanishing) is independent of such a choice.

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Ping-pong in transformation groups is attributed (see [5, Chapter II.B]) to Feliks Klein who used it to study Kleinian groups. It implies some complexity of the dynamics, in particular, positive entropy and – in 1-dimensional dynamics – arises always when the dynamics of the system is complicated enough (see, for example, [8] and the bibliography therein). In some sense, in one dimensional dynamics, ping-pong is related to horseshoes which can be used to estimate (or even, to calculate) entropies of the systems (see [7] and, again, the bibliography therein).

It is known (see, for example, Prop. 2.4.10 in [10]) that ping-pong in a group (with two players) implies the entropy estimate from below: entropy is greater or equal to the product of $\log 2$ by the inverse of the maximum of distances (in the metric determined by a given generating set) of ping-pong players from the identity. In the same way, ping-pong with N players provides entropy greater or equal to $\log N$ divided again by the maximum of their distances from the identity. Here, we produce a better estimate: we replace the denominator in the above by a quantity which arises from a well known lower bound for binomial distribution (see, for example, [1]) and is strictly larger than the quantity (maximal distance) of the estimate mentioned above.

Note that our estimates can be adapted to pseudogroups and foliations to relate the value of entropy with the "strength" of a resilient orbit (or, of a resilient leaf) which can be defined and related to the entropy and expressed in terms of the "length" of a piece of the orbit (or, of a leaf curve) providing ping-pong in the corresponding space (for foliations, *via* holonomy, on a transversal), see [6]. We expect (and try to get) a similar estimate from above in the case of 1-dynamical dynamics, that is when our space X coincides with a segment, a circle or, more generally, a graph (see [7] again), also when a foliation has codimension 1. The work in this direction is in progress. A reader interested in such topics is referred also to Chapters 2 and 3 of [10].

2. Preliminaries

Throughout the paper, X is a compact metric space with distance d, G a finitely generated group of continuous transformations of X and G_1 a fixed finite symmetric (i.e., such that $e \in G_1$ and $G^{-1} = \{g^{-1}; g \in G_1\} \subset G_1$) set of generators for G. For any $n \in \mathbb{N}$, we put $G_n = \{g_1 \circ \ldots \circ g_n; g_1, \ldots, g_n \in G_1\}$. Note that since $e \in G_1$, $G_1 \subset G_2 \subset G_3 \subset \ldots$.

Definition 2.1 (Ping-pong). Let G be a group acting on a compact metric space X. We say that $f_1, f_2 \in G$ are *playing ping-pong* if there exist sets

$A, A_1, A_2 \subset X$ such that $A_1, A_2 \subset A$, $\mathrm{dist}(A_1, A_2) > 0$, $f_1(A) \subset A_1$ and $f_2(A) \subset A_2$.

DEFINITION 2.2 (Multi Ping-pong). Let G be a group acting on a compact metric space X. We say that $f_1, \ldots, f_N \in G$ are *playing multi ping-pong* if there exist sets $A, A_1, \ldots, A_N \subset X$ such that $A_1, \ldots, A_N \subset A$, and for all $i \neq j, 1 \leq i, j \leq N$ $\mathrm{dist}(A_i, A_j) > 0$ and $f_i(A) \subset A_i$.

DEFINITION 2.3 (Entropy). Let $\varepsilon > 0$ and $n > 0$, $n \in \mathbb{N}$. We say that points $x, y \in X$ are (n, ε)-*separated* if there exists a continuous map $f \in G_n$ such that $d(f(x), f(y)) \geq \varepsilon$.

A set $A \subset X$ is (n, ε)-*separated* if all the pairs of points $x, y \in A$, $x \neq y$, have this property.

Since X is compact, every (n, ε)-separated set is finite and we may put

$$s(n, \varepsilon, G_1) := \max\{\#A; A \subset X \text{ is } (n, \varepsilon)\text{-separated}\}$$

and

$$s(\varepsilon, G_1) := \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, G_1).$$

The number $h(G, G_1) := \lim_{\varepsilon \to 0} s(\varepsilon, G_1)$ is called the (topological) *entropy* of G with respect to G_1.

For simplicity, in the sequel we avoid writing G_1 in all these formulae because we are interested in only one, fixed, set of generators.

In our calculations, we shall use the following (see, [3, Chapter 11]) *lower bound* for the binomial distribution.

LEMMA 2.4. *If* $k_1 + \ldots + k_N = n$, $k_i = np_i$ *and* $P = (p_1, p_2, \ldots, p_N)$, *then*

$$\binom{n}{k_1, k_2, \ldots, k_N} \geq \frac{1}{(n+1)^N} e^{nH(P)},$$

where

$$\binom{n}{k_1, k_2, \ldots, k_N} := \frac{n!}{k_1! k_2! \ldots k_N!} = \binom{n}{k_1} \binom{n - k_1}{k_2} \cdots \binom{n - \sum_{\xi=1}^{N-1} k_\xi}{k_N}$$

and $H(P) := -\sum_{i=1}^{N} p_i \log p_i$.

3. Multi ping-pong

THEOREM 3.1. *Let G be a group of transformations of X containing N continuous maps $f_1, \ldots, f_N$ playing multi ping-pong. If $f_i \in G_{m_i}$, $m_1 \le m_2 \le \ldots \le m_N$, then the entropy $h(G)$ satisfies*

$$h(G) \ge -\log p,$$

where $p \in (0,1)$ and $\sum_{i=1}^N p^{m_i} = 1$.

PROOF. Let X be a compact metric space. Take $f_1, \ldots, f_N, A_1, \ldots, A_N$ as in Definition 2.2 and choose ε such that for all $i \neq j$ $\operatorname{dist}(A_i, A_j) > \varepsilon$. Choose any $c \in X$. Define the set

$$E_{n,\Bbbk} := \{f_{i_1} \circ \ldots \circ f_{i_n}(c); i_j \in \{1, \ldots, N\}, \forall_{\xi < N}\ \#\{j : i_j = \xi\} \ge k_\xi\},$$

$$\text{where } \Bbbk = (k_1, \ldots, k_N),\ 0 \le k_\xi \le n,\ \sum_{\xi=1}^{N-1} k_\xi \le n,\ k_N := n - \sum_{\xi=1}^{N-1} k_\xi.$$

From $A_i \cap A_j = \emptyset$ for all $i \neq j$ we gain that points $f_{i_1} \circ \ldots \circ f_{i_n}(c)$ and $f_{j_1} \circ \ldots \circ f_{j_n}(c)$ are different when $\{i_1, \ldots, i_n\} \neq \{j_1, \ldots, j_n\}$. Therefore, we obtain the inequality

$$\#E_{n,\Bbbk} \ge$$
$$\sum_{i_1=0}^{k_N} \binom{n}{k_1 + i_1} \sum_{i_2=0}^{k_N - i_1} \binom{n - i_1}{k_2 + i_2} \cdots \sum_{i_{N-1}=0}^{k_N - (i_1 + \ldots + i_{N-2})} \binom{n - (i_1 + \ldots + i_{N-2})}{k_{N-1} + i_{N-1}}.$$

Moreover, if $x = f_{i_1} \circ \ldots \circ f_{i_n}(c)$ and $y = f_{j_1} \circ \ldots \circ f_{j_n}(c)$ are different points of $E_{n,\Bbbk}$, then $d((f_{i_1} \circ \ldots \circ f_{i_m})^{-1}(x), (f_{j_1} \circ \ldots \circ f_{j_m})^{-1}(y)) \ge \varepsilon$, where m is the largest number satisfying the condition $i_1 = j_1, \ \ldots, \ i_m = j_m$. Furthermore, $(f_{i_1} \circ \ldots \circ f_{i_m})^{-1} \in G_{(m_1 k_1 + \ldots + m_N k_N)}$. Thus the set $E_{n,\Bbbk}$ is $(m_1 k_1 + \ldots + m_N k_N, \varepsilon)$-separated and

$$s(\sum_{i=1}^N k_i m_i, \varepsilon) \ge \#E_{n,\Bbbk} \ge \prod_{j=1}^{N-1} \sum_{i_j=0}^{k_N - (\sum_{\xi=1}^{j-1} i_\xi)} \binom{n - (\sum_{\xi=1}^{j-1} i_\xi)}{k_j + i_j}.$$

Remember that for any $n \in \mathbb{N}$ we have $\sum_{\xi=1}^N k_\xi = n$, so our sequences $\Bbbk$ consist of numbers depending on n.

We obtain the estimate

$$h(G) \geq \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{N} k_i m_i} \log \prod_{j=1}^{N-1} \sum_{i_j=0}^{k_N-(\sum_{\xi=1}^{j-1} i_\xi)} \binom{n-(\sum_{\xi=1}^{j-1} i_\xi)}{k_j+i_j}.$$

Of course, the whole sum in the above is larger than its first term. So by Lemma 2.4 and putting $k_i := np_i$ we obtain

$$\begin{aligned} h(G) &\geq \lim_{n\to\infty} \frac{1}{\sum_{i=1}^{N} k_i m_i} \log \binom{n}{k_1, k_2, \ldots, k_N} \\ &\geq \frac{-\sum_{i=1}^{N} p_i \log p_i}{\sum_{i=1}^{N} p_i m_i} =: \phi(p_1, \ldots, p_N), \quad \text{where} \sum_{i=1}^{N} p_i = 1,\ p_i \geq 0. \end{aligned}$$

The best estimate is obtained for the maximal value of our function ϕ. One can check by the method of Lagrange multipliers that the best value $\phi(p_1, \ldots, p_N) = -\log p$ is attained for $p_i = p^{m_i}$, where $p \in (0,1)$ and $\sum_{i=1}^{N} p^{m_i} = 1$. □

Corollary 3.2. *For N continuous maps $f_1, \ldots, f_N$ playing multi ping-pong we have the following.*

(1) *If $f_i \in G_1$ for all $1 \leq i \leq N$, the entropy satisfies $h(G) \geq \log N$.*
(2) *If $f_i \in G_m$ for all $1 \leq i \leq N$, the entropy satisfies $h(G) \geq \frac{\log N}{m}$.*

Finally, we present some computer aided numerical estimates:

Example 3.3. For $(m_1, \ldots, m_N)$ we assume that there exist N continuous maps f_i such that $f_i \in G_{m_i}$ for $1 \leq i \leq N$ whose are playing multi ping-pong. Then, we receive the following:

(1) for $(m_1; m_2; m_3) = (1; 5; 5)$, $p \approx 0,689139$ and $h(G) \geq 0,372312$,
(2) for $(m_1; m_2; m_3) = (1; 5; 7)$, $p \approx 0,715802$ and $h(G) \geq 0,334352$,
(3) for $(m_1; m_2; m_3) = (1; 7; 7)$, $p \approx 0,745072$ and $h(G) \geq 0,294274$,
(4) for $(m_1; m_2; m_3) = (2; 5; 7)$, $p \approx 0,76488$ and $h(G) \geq 0,268036$,
(5) for $(m_1; m_2; m_3) = (7; 7; 7)$, $p \approx 0,85475$ and $h(G) \geq 0,1569446$,
(6) for $(m_1; m_2; m_3; m_4) = (1; 5; 5; 5)$, $p \approx 0,65052$ and $h(G) \geq 0,429983$,
(7) for $(m_1; m_2; m_3; m_4) = (1; 7; 7; 7)$, $p \approx 0,714581$ and $h(G) \geq 0,336059$,
(8) for $(m_1; \ldots; m_5) = (1; 2; 3; 4; 5)$, $p \approx 0,50866$ and $h(G) \geq 0,675975$,

and so on.

As we can see, if one of the ping-pong player is of "shorter length" (smaller value of m_i), we gain more entropy. Bigger entropy is also received for larger amount of players in a ping-pong game.

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