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AN EXTENSION OF A GER'S RESULT

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Abstract. The aim of this paper is to extend a result presented by Roman Ger during the 15th International Conference on Functional Equations and Inequalities. First, we present some necessary and sufficient conditions for a continuous function to be convex. We will use these to extend Ger's result. Finally, we make some connections with other mathematical notions, as g-convex dominated function or Bregman distance.

1. Introduction

During the 15th International Conference on Functional Equations and Inequalities, Ger proposed the following functional inequality (see [8]):

Proposition 1.1. *Let $g\colon (a,b) \to \mathbb{R}$ be a differentiable function. If $f : (a,b) \to \mathbb{R}$ is a convex solution of functional inequality*

$$\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) \le \frac{g(x)+g(y)}{2} - g\left(\frac{x+y}{2}\right), \quad x, y \in (a,b), \tag{1.1}$$

then f is differentiable and $|f'(x) - f'(y)| \le |g'(x) - g'(y)|$ holds for any $x, y \in (a,b)$.

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In fact, Proposition 1.1 represents an extension of the problem 11641 (see [2]). The solution of this problem can be found in [7].

The aim of this paper is to present new results of the same type, but more general. For the sake of clearness, we divide our work into two parts. The second section contains the results only. The first original contributions are included in Theorem 2.3, where we present some necessary and sufficient conditions for a continuous function to be convex. It is the main tool which we will use to prove the main results, Theorems 2.5–2.7, and also some consequences. Finally, we obtain some connection with other mathematical notions, namely g-convex dominated function or Bregman distance. The original proofs are included in the third section of this paper.

2. The results

In this paper, I stands for an open real interval. Recall that a function $f\colon I \to \mathbb{R}$ is convex if

$$f\left(tx+(1-1)\,y\right) \leq tf\left(x\right)+(1-t)\,f\left(y\right), \tag{2.1}$$

for any $x, y \in I$ and $t \in [0,1]$. Some of the properties of these functions are included in the next proposition (for example, see Theorem 1.3.3 from [10]).

Proposition 2.1. *Let $f\colon I \longrightarrow \mathbb{R}$ be a convex function. Then, f is continuous on I and has finite left and right derivatives at each point of I. Particularly, both f'_- and f'_+ are nondecreasing on I and, for any $x \in I$,*

$$f'_-\left(x\right) \leq f'_+\left(x\right).$$

For the functions which do not satisfy the inequality (2.1), Páles (*see* [11]) proved the following proposition:

Proposition 2.2. *Let $f\colon I \to \mathbb{R}$ be a continuos and non-convex function. Then there exist $a, b \in I, a < b$, such that*

$$f\left(ta+(1-t)\,b\right) > tf\left(a\right)+(1-t)\,f\left(b\right),$$

for any $t \in (0,1)$.

Hence, the condition of the convexity can be weakened as it can be seen in the next theorem:

THEOREM 2.3. *Let $f\colon I \to \mathbb{R}$ be a continuous function. The following three statements are equivalent:*
(a) *The function f is convex;*
(b) *For any $x, y \in I, x < y$, there exists a $z \in (x, y)$ such that*

$$(y-x)\,f(z) \le (y-z)\,f(x) + (z-x)\,f(y)\,; \tag{2.2}$$

(c) *For any $x, y \in I$, there exists a $t \in (0,1)$ such that*

$$f(tx + (1-1)\,y) \le tf(x) + (1-t)\,f(y)\,. \tag{2.3}$$

As a consequence of the equivalence $a) \Leftrightarrow c)$ from the previous theorem, we obtain a classic result from the convex functions theory:

COROLLARY 2.4. *Let $f\colon I \to \mathbb{R}$ be a continuous function. Then f is a convex function if and only if it is a midconvex function.*

Hence, the main results of this paper are:

THEOREM 2.5. *Let $g\colon I \to \mathbb{R}$ be a differentiable function and $f\colon I \to \mathbb{R}$ be a convex function such that for any $x, y \in I, x < y$, there exists a $z \in (x, y)$ with the property*

$$\begin{aligned}(y-z)\,f(x) + (z-x)\,f(y) - (y-x)\,f(z)& \\ \le (y-z)\,g(x) + (z-x)\,g(y) - (y-x)\,g(z)&\,.\end{aligned} \tag{2.4}$$

Then, the function f is differentiable and $|f'(x) - f'(y)| \le |g'(x) - g'(y)|$ holds for any $x, y \in I$.

THEOREM 2.6. *Let $g\colon I \to \mathbb{R}$ be a differentiable function and $f\colon I \to \mathbb{R}$ be a convex function such that for any $x, y \in I, x < y$, there exists a $t \in (0,1)$ with the property*

$$\begin{aligned}tf(x) + (1-t)\,f(y) - f(tx + (1-t)\,y)& \\ \le tg(x) + (1-t)\,g(y) - g(tx + (1-1)\,y)&\,.\end{aligned} \tag{2.5}$$

Then, the function f is differentiable and $|f'(x) - f'(y)| \le |g'(x) - g'(y)|$ holds for any $x, y \in I$.

Now, we observe that Proposition 1.1 is a particular case of the previous theorem. Indeed, if we choose $t = \frac{1}{2}$, we obtain the mentioned result.

THEOREM 2.7. *Let $g\colon I \to \mathbb{R}$ be a differentiable function and $f\colon I \to \mathbb{R}$ be a convex function such that there exist $s, t \in (0,1)$ with the property*

$$\begin{aligned} (2.6)\quad & sf(x) + (1-s) f(y) - f(tx + (1-t) y) \\ & \qquad \leq sg(x) + (1-s) g(y) - g(tx + (1-t) y), \end{aligned}$$

for any $x, y \in I$. Then, the function f is differentiable and $|f'(x) - f'(y)| \leq |g'(x) - g'(y)|$ holds for any $x, y \in I$.

The case $s = t$ gives us to the next corollary:

COROLLARY 2.8. *Let $g\colon I \to \mathbb{R}$ be a differentiable function and $f\colon I \to \mathbb{R}$ be a convex function. If there exists a $t \in (0,1)$ such that*

$$\begin{aligned} (2.7)\quad & tf(x) + (1-t) f(y) - f(tx + (1-t) y) \\ & \qquad \leq tg(x) + (1-t) g(y) - g(tx + (1-t) y), \end{aligned}$$

for any $x, y \in I$, then, the function f is differentiable and $|f'(x) - f'(y)| \leq |g'(x) - g'(y)|$ holds for any $x, y \in I$.

If the function f from Theorems 2.5–2.7 and Corollary 2.8 is continuous instead of convex, the results of these facts remain valid if the left-hand sides of inequalities (2.4)-(2.7) are positive, due to Theorem 2.3. Another consequence is represented by the functional equation included in the next corollary.

COROLLARY 2.9. *Let $f\colon I \to \mathbb{R}$ be a continuous function such that*

$$tf(x) + (1-t) f(y) = f(tx + (1-t) y),$$

for any $x, y \in I$ and $t \in [0,1]$. Then, f is an affine function.

An interesting fact concerning the inequalities (2.4)–(2.7) is represented by the connections with an important notion from the inequalities theory. It is connected with the notion of g-convex dominated functions. For a given convex function $g\colon I \to \mathbb{R}$, the function $f\colon I \to \mathbb{R}$ is called *g-convex dominated* on I if

$$\begin{aligned} & |tf(x) + (1-t) f(y) - f(tx + (1-t) y)| \\ & \qquad \leq tg(x) + (1-t) g(y) - g(tx + (1-t) y), \end{aligned}$$

for any $x, y \in I$ and $t \in [0,1]$. Lemma 1 from [6] gives us some necessary and sufficient conditions for a function $f\colon I \to \mathbb{R}$ to be g-convex dominated on I. For example, the functions $g - f$ and $g + f$ must be convex on I. In fact, this lemma shows us that the class of g-convex dominated functions is equal to the class of *delta convex functions*. The references [9] or [13] are relevant.

Further, we use these results to prove the following theorem.

THEOREM 2.10. *Let $g\colon I \to \mathbb{R}$ be a differentiable function. A sufficient condition for a convex function $f\colon I \to \mathbb{R}$ to be g-convex dominated is to satisfy at least one of the next three conditions:*

(a) *For any $x, y \in I, x < y$, there exists a $z \in (x, y)$ such that*

$$\begin{aligned}(y-z)\, f(x) + (z-x)\, f(y) - (y-x)\, f(z)\\ \le (y-z)\, g(x) + (z-x)\, g(y) - (y-x)\, g(z);\end{aligned}$$

(b) *For any $x, y \in I$, there exists a $t \in (0,1)$ such that*

$$\begin{aligned}tf(x) + (1-t)\, f(y) - f(tx + (1-t)\, y)\\ \le tg(x) + (1-t)\, g(y) - g(tx + (1-1)\, y);\end{aligned}$$

(c) *There exist $s, t \in (0,1)$ such that*

$$\begin{aligned}sf(x) + (1-s)\, f(y) - f(tx + (1-t)\, y)\\ \le sg(x) + (1-s)\, g(y) - g(tx + (1-t)\, y),\end{aligned}$$

for any $x, y \in I$.

Finally, we obtain a characterization of Baillon-Haddad type for the inequality (1.1), under the differentiability restriction. First, for any differentiable function $h\colon I \to \mathbb{R}$, we denote

$$D_h(x,y) = h(x) - h(y) - h'(y)(x-y),$$

for any $x, y \in I$, also called *Bregman distance* ([4]). For any convex function h, we have

$$D_h(x,y) \ge 0,$$

for any $x, y \in I$.

THEOREM 2.11. *Let $f\colon I\to\mathbb{R}$ be a differentiable and convex function. Let $g\colon I\to\mathbb{R}$ be a differentiable function. The following statements are equivalent:*
(a) *For any $x,y\in I$ we have*

$$\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\leq\frac{g(x)+g(y)}{2}-g\left(\frac{x+y}{2}\right);$$

(b) *For any $x,y\in I$ we have*

$$D_f(x,y)\leq D_g(x,y);$$

(c) *The function g' is nondecreasing and*

$$|f'(x)-f'(y)|\leq|g'(x)-g'(y)|,$$

for any $x,y\in I$.

3. The proofs

In this section, we present the proofs of the most important results presented in this paper.

PROOF OF THEOREM 2.3. We prove the following chain of implications: $(a)\Rightarrow(b)\Rightarrow(c)\Rightarrow(a)$.

The implication $(a)\Rightarrow(b)$ is true due to the definition of the convexity. For $(b)\Rightarrow(c)$, let $x,y\in I$, $x<y$. There exists a $z\in(x,y)$ such that inequality (2.2) holds. Note that $z=\dfrac{y-z}{y-x}x+\dfrac{z-x}{y-x}y$. Denote $t=\dfrac{y-z}{y-x}$. We have $t\in(0,1)$ and $\dfrac{z-x}{y-x}=1-t$. Then (2.2) becomes

$$f(z)\leq\frac{y-z}{y-x}f(x)+\frac{z-x}{y-x}f(y),$$

i.e.

$$f(tx+(1-t)x)\leq tf(x)+(1-t)f(y),$$

and the inequality (2.3) holds.

For the proof of the implication $(c) \Rightarrow (a)$, we assume by contradiction that the function f is non convex. Using Proposition 2.2, we find $x, y \in I$, $x < y$, such that

$$f(tx + (1-t)y) > tf(x) + (1-t)f(y),$$

for any $t \in (0,1)$. This contradicts the inequality (2.3) and therefore concludes the proof. $\square$

Proof of Theorem 2.5. The inequality (2.4) is equivalent to

$$(y-x)(g(z) - f(z)) \le (y-z)(g(x) - f(x)) + (z-x)(g(y) - f(y)).$$

Theorem 2.3 shows that the function $g - f$ is convex. We obtain that g is convex using the hypothesis. Let us denote $h = g - f$. Then, there exists $h'_-(x)$ and $h'_+(x)$, for any $x \in I$ and $h'_-(x) \le h'_+(x)$. For any $x \in I$, we have $g'_+(x) - f'_+(x) \ge g'_-(x) - f'_-(x)$. The function g is differentiable, thus $-f'_+(x) \ge -f'_-(x)$ and $f'_+(x) \le f'_-(x)$. The convexity of f yields $f'_+(x) = f'_-(x)$, so f is differentiable. In addition, the function h is differentiable.

Moreover, f', g' and h' are nondecreasing. Let $x, y \in I, x < y$. Then

$$\begin{aligned}
|f'(x) - f'(y)| &= f'(y) - f'(x) \\
&= g'(y) - h'(y) - g'(x) + h'(x) \\
&= g'(y) - g'(x) - (h'(y) - h'(x)) \\
&\le g'(y) - g'(x) \\
&= |g'(x) - g'(y)|.
\end{aligned}$$

The case $x > y$ is similar and the proof is complete. $\square$

Proof of Theorem 2.6. The inequality (2.5) is equivalent to

$$\begin{aligned}
&g(tx + (1-t)y) - f(tx + (1-t)y) \\
&\qquad \le t(g(x) - f(x)) + (1-t)(g(y) - f(y)).
\end{aligned}$$

Theorem 2.3 shows that the function $g - f$ is convex. Then, the proof is similar to the previous one. $\square$

Proof of Theorem 2.7. In this proof, we explore an idea of A. Olbryś from [12]. First, we rewrite the inequality (2.6) in the following form:

$$g(tx+(1-t)y)-f(tx+(1-t)y) \\ \le s(g(x)-f(x))+(1-s)(g(y)-f(y)).$$

Let $x, y \in I$. Using Daróczy–Páles identity (see [5]), we have

$$\frac{x+y}{2}=t\left(t\frac{x+y}{2}+(1-t)y\right)+(1-t)\left(tx+(1-t)\frac{x+y}{2}\right),$$

for any $t \in [0,1]$. Further

$$\begin{aligned}
&g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right)\\
&\quad=g\left(t\left(t\frac{x+y}{2}+(1-t)y\right)+(1-t)\left(tx+(1-t)\frac{x+y}{2}\right)\right)\\
&\qquad-f\left(t\left(t\frac{x+y}{2}+(1-t)y\right)+(1-t)\left(tx+(1-t)\frac{x+y}{2}\right)\right)\\
&\quad\le s\left(g\left(t\frac{x+y}{2}+(1-t)y\right)-f\left(t\frac{x+y}{2}+(1-t)y\right)\right)\\
&\qquad+(1-s)\left(g\left(tx+(1-t)\frac{x+y}{2}\right)-f\left(tx+(1-t)\frac{x+y}{2}\right)\right)\\
&\quad\le s\left(s\left(g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right)\right)+(1-s)(g(y)-f(y))\right)\\
&\qquad+(1-s)\left(s(g(x)-f(x))+(1-s)\left(g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right)\right)\right)\\
&\quad=\left(s^2+(1-s)^2\right)\left(g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right)\right)\\
&\qquad+s(1-s)(g(y)-f(y))+s(1-s)(g(x)-f(x)).
\end{aligned}$$

Now, we obtain

$$\left(1-s^2-(1-s)^2\right)\left(g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right)\right)$$
$$\leq s(1-s)(g(y)-f(y))+s(1-s)(g(x)-f(x)),$$

which is equivalent to

$$2s(1-s)\left(g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right)\right)$$
$$\leq s(1-s)(g(y)-f(y))+s(1-s)(g(x)-f(x)).$$

We cancel $s(1-s)$ and we get

$$g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right)\leq\frac{(g(x)-f(x))+(g(y)-f(y))}{2}$$

$$\Leftrightarrow\quad\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\leq\frac{g(x)+g(y)}{2}-g\left(\frac{x+y}{2}\right).$$

Now, Theorem 2.6 completes the proof. □

Proof of Corollary 2.9. The hypothesis goes to conclusion that f is convex. Let $g\colon I\longrightarrow\mathbb{R}$ be an affine function. Then,

$$tg(x)+(1-t)g(y)=g(tx+(1-t)y),$$

for any $x,y\in I$ and $t\in[0,1]$. Consequently

$$tf(x)+(1-t)f(y)-f(tx+(1-t)y)$$
$$\leq tg(x)+(1-t)g(y)-g(tx+(1-1)y).$$

Theorem 2.6 shows that the function f is differentiable and

$$|f'(x)-f'(y)|\leq|g'(x)-g'(y)|,$$

for any $x,y\in I$. Since $g'(x)-g'(y)=0$, we find that f' is a constant function and the conclusion follows. □

PROOF OF THEOREM 2.10. First, we remark that the convexity of f, leads us to the convexity of g. Then $g+f$ is convex. On the other hand, any of the conditions (a), (b) or (c) implies that f is differentiable and

$$|f'(x)-f'(y)| \leq |g'(x)-g'(y)|,$$

for any $x,y \in I$, due to Theorems 2.5–2.7. Since f' and g' are nondecreasing, for $x>y$ we obtain $f'(x)-f'(y) \leq g'(x)-g'(y)$, which is equivalent to

$$g'(x)-f'(x) \geq g'(y)-f'(y).$$

This means that the function $g'-f'$ is nondecreasing, so $g-f$ is convex and the proof is complete. □

PROOF OF THEOREM 2.11. We prove the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. For the first implication, we rewrite the hypothesis in the following form:

$$g\left(\frac{x+y}{2}\right)-f\left(\frac{x+y}{2}\right) \leq \frac{(g(x)-f(x))+(g(y)-f(y))}{2},$$

for any $x,y \in I$. The function $g-f$ is convex, so $D_{g-f}(x,y) \geq 0$. Now it is enough to notice that

$$\begin{aligned} & D_{g-f}(x,y) \geq 0 \\ & \quad \Leftrightarrow \quad (g(x)-f(x))-(g(y)-f(y))-(g'(y)-f'(y))(x-y) \geq 0 \\ & \quad \Leftrightarrow \quad g(x)-g(y)-g'(y)(x-y) \geq f(x)-f(y)-f'(y)(x-y) \\ & \quad \Leftrightarrow \quad D_g(x,y) \geq D_f(x,y). \end{aligned}$$

For the implication $(b) \Rightarrow (c)$ we use the inequalities $D_f(x,y) \leq D_g(x,y)$ and $D_f(y,x) \leq D_g(y,x)$ for any $x,y \in I$. We obtain

$$f(x)-f(y)-f'(y)(x-y) \leq g(x)-g(y)-g'(y)(x-y)$$

and

$$f(y)-f(x)-f'(x)(y-x) \leq g(y)-g(x)-g'(x)(y-x).$$

Adding both sides of these inequalities, we get

$$(f'(x)-f'(y))(x-y) \leq (g'(x)-g'(y))(x-y).$$

The function f is convex, so f' is nondecreasing. Consequently

$$(f'(x) - f'(y))(x-y) \geq 0 \quad \text{and} \quad (g'(x) - g'(y))(x-y) \geq 0$$

for any $x, y \in I$. Then g' is nondecreasing and

$$|(f'(x) - f'(y))(x-y)| \leq |(g'(x) - g'(y))(x-y)|,$$

which leads to the conclusion.

For the implication $(c) \Rightarrow (a)$, we remark that f' is nondecreasing. If we choose $x > y$, we obtain $f'(x) - f'(y) \leq g'(x) - g'(y)$, i.e.

$$g'(x) - f'(x) \geq g'(y) - f'(y).$$

This means that the function $g' - f'$ is nondecreasing, so $g - f$ is convex. This concludes the proof. □

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