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A GENERALIZATION OF m-CONVEXITY AND A SANDWICH THEOREM

Teodoro Lara, Janusz Matkowski, Nelson Merentes, Roy Quintero, Małgorzata Wróbel

Abstract. Functional inequalities generalizing m-convexity are considered. A result of a sandwich type is proved. Some applications are indicated.

1. Introduction

We consider some notions of convexity. To be more detailed assume that $\alpha\colon [0,1] \to \mathbb{R}$ is a given function and $I \subset \mathbb{R}$ is an interval such that $tI+\alpha(t)I \subset I$ for all $t \in [0,1]$, where $tI + \alpha(t)I$ denotes the set $\{tx + \alpha(t)y : x, y \in I\}$. In Section 2 we deal with functions satisfying the inequality

$$f\left(tx + \alpha(t)y\right) \leq tf(x) + \alpha(t)f(y)$$

for all $x, y \in I$, $t \in [0,1]$, and referred to as a convexity with respect to α (convex wrt α). It turns out that, under some general conditions on α, if f is convex wrt α, then f has to be convex; and under a little stronger conditions, f is convex wrt α if and only if it is convex (Proposition 2.1). We note that

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this notion is "closer" to the classical convexity if α is a decreasing involution ($\alpha \circ \alpha = \mathrm{id}|_{[0,1]}$). It occurs, in particular if, for some $p > 0$,

$$\alpha(t) = (1 - t^p)^{1/p}, \quad t \in [0, 1].$$

Moreover, given a number $m > 0$, we say that f is m-convex with respect to an involution α, if

$$f(tx + m\alpha(t)y) \le tf(x) + m\alpha(t)f(y), \quad x, y \in I,\ t \in [0, 1].$$

For $\alpha(t) = 1 - t,\ \ t \in [0, 1]$, this notion coincides with the concept of m-convexity introduced by Toader [15] 1984 (see also [5, 7, 8, 13]). We compare the m-convexity with the convexity with respect to a mean (Aumann [2], 1933).

In Section 3 we deal with m-convex functions when $0 < m < 1$. We note that, in general, the m-convex functions do not share the properties of convex ones (Corollary 3.3). However, we show that a function is affine, if it is m-affine (Remark 3.4). For every $m \in (0, 1)$ we construct a polynomial h of degree 4 such that $f := h|_{[0,+\infty)}$ has the following properties: f is a diffeomorphic m-convex self-mapping of $[0, +\infty)$, but not convex in $[0, +\infty)$. It shows that the m-convex functions do not have the property that their graphs are placed above the supporting straight-lines. On the other hand, for any sequence $(t_n \in (0, 1) : n \in \mathbb{N})$ such that $\lim_{n\to+\infty} t_n = 1$ there is a sequence $(s_n \in (0, 1) : n \in \mathbb{N})$, with $\lim_{n\to+\infty} s_n = 0,\ \ t_n + s_n < 1$ for every $n \in \mathbb{N}$, and

$$f(t_n x + s_n y) \le t_n f(x) + s_n f(y), \quad x, y \in [0, +\infty), n \in \mathbb{N};$$

so m-convex functions are, to some extent, quite close to convex ones.

In Section 4, assuming that $0 < m < 1$, we prove the following result of a sandwich type: *if* $f\colon (0, +\infty) \to \mathbb{R}$ *is* m-convex*, then there exists a convex function* $h\colon I \to \mathbb{R}$ *such that*

$$f(x) \le h(x) \le mf\left(\frac{x}{m}\right), \quad x > 0.$$

The main result of the last section says that every m-convex function $f\colon (0, +\infty) \to \mathbb{R}$ such that $\liminf_{x\to 0+} f(x) \le 0$, where $m > 1$, is a linear function.

2. Convexity with respect to a function and m-convexity

Let us begin with the following

PROPOSITION 2.1. *Let $\alpha\colon [0,1] \to \mathbb{R}$ be a continuous function and $I \subset \mathbb{R}$ be an open nonempty interval such that $tI + \alpha(t)I \subset I$ for all $t \in [0,1]$. Suppose that a function $f\colon I \to \mathbb{R}$ is convex with respect to α, i.e., f satisfies the inequality*

$$f(tx + \alpha(t)y) \le tf(x) + \alpha(t)f(y), \quad x, y \in I,\ t \in [0,1]. \tag{2.1}$$

(i) *If there exists $t_0 \in (0,1)$ such that $t_0 + \alpha(t_0) = 1$, then f is convex in the classical sense; moreover, if $0 \in I$, $f(0) \le 0$ and $0 \le t + \alpha(t) \le 1$ for all $t \in [0,1]$, then f satisfies (2.1) if and only if it is convex.*

(ii) *If there are $t_1, t_2 \in [0,1]$ such that $t_1 + \alpha(t_1) < 1$ and $t_2 + \alpha(t_2) > 1$, then $(0, +\infty) \subset I$ and $f(x) = f(1)x$ for all $x \in I$.*

PROOF. (i) By the assumption we have

$$f(t_0x + (1 - t_0)y) \le t_0f(x) + (1 - t_0)f(y), \quad x,\ y \in I,$$

so f is Jensen convex [4].

Note that there are $x, y \in I$, $x \ne y$, such that the function $[0,1] \ni t \mapsto tx + \alpha(t)y$ is not constant.

Indeed, in the opposite case, for every pair $(x, y) \in I^2$, $x \ne y$, there would exist a constant $c(x, y)$ such that $tx + \alpha(t)y = c(x, y)$ for all $t \in [0,1]$, whence $y \ne 0$ and

$$\alpha(t) = \frac{c(x,y)}{y} - \frac{x}{y}t, \quad t \in [0,1].$$

Since α does not depend on x and y, it follows that $x = y$. This contradiction proves the claim.

Take $x, y \in I$, $x \ne y$, such that the function $[0,1] \ni t \mapsto tx + \alpha(t)y$ is not constant. Since it is continuous, its range is a nontrivial interval $I(x, y)$. Moreover, applying (2.1) and the Weierstrass Theorem for the continuous function $[0,1] \ni t \mapsto tx + \alpha(t)f(y)$, we get the boundedness from above of f on the interval $I(x, y)$. Now, the Bernstein-Doetsch Theorem (cf. [6, Theorem 6.4.2]) implies that f is convex.

To prove the "moreover" part note first that if f is convex and $f(0) \le 0$ then f is starshaped, i.e., $f(\lambda x) \le \lambda f(x)$ for all $\lambda \in [0,1]$ and $x \in I$. Indeed,

$$f(\lambda x + (1 - \lambda)0) \le \lambda f(x) + (1 - \lambda)f(0) \le \lambda f(x).$$

Hence, for all x, $y \in I$, $t \in [0,1]$, we get

$$\begin{aligned} f(tx+\alpha(t)y) &= f\left(\frac{t}{t+\alpha(t)}(t+\alpha(t))x + \frac{\alpha(t)}{t+\alpha(t)}(t+\alpha(t))y\right) \\ &\le \frac{t}{t+\alpha(t)} f((t+\alpha(t))x) + \frac{\alpha(t)}{t+\alpha(t)} f((t+\alpha(t))y) \\ &\le tf(x) + \alpha(t)f(y). \end{aligned}$$

(ii) By the Darboux property of α, between t_1, t_2 there is $t_0 \in (0,1)$ that $t_0+\alpha(t_0) = 1$. In view of (i), the function f is convex, so the function $I \ni x \mapsto \frac{f(x)}{x}$ is either monotonic or, for some $x_0 \in I$, decreasing in $I \cap (-\infty, x_0)$ and increasing in $I \cap (x_0, +\infty)$ (see [1] where this "modality" property of convex functions, conjectured by M. Kuczma, has been proved). Since, by (2.1),

$$\frac{f((t_1+\alpha(t_1))x)}{(t_1+\alpha(t_1))x} \le \frac{f(x)}{x}, \quad x \in I,$$

and

$$\frac{f((t_2+\alpha(t_2))x)}{(t_2+\alpha(t_2))x} \le \frac{f(x)}{x}, \; x \in I,$$

the function $x \mapsto \frac{f(x)}{x}$ is non-decreasing and non-increasing, so it must be constant. □

It follows that in some generalizations of the convexity notion in the form (2.1) it can be reasonable to assume that (see below, Corollary 5.3)

$$t+\alpha(t) \le 1, \quad t \in [0,1].$$

Moreover, taking in this proposition $\alpha\colon [0,1] \to [0,1]$,

$$\alpha(t) := 1-t, \quad t \in [0,1],$$

the function f satisfies (2.1) if and only if it is convex. Since in this case we have $(\alpha \circ \alpha)(t) = t$ for all $t \in [0,1]$, it may be sometimes convenient to assume that α is an involution.

We propose the following generalizations of the notion of m-convex function introduced by Toader [15].

Definition 2.2. Let $\alpha\colon [0,1]\to[0,1]$ be a function and $m>0$ be fixed. A subset X of a linear space is said to be *convex with respect to* α (convex wrt α), if

$$x,y\in X \implies tx+\alpha(t)y\in X;$$

m-convex wrt α, if

$$x,y\in X \implies tx+m\alpha(t)y\in X.$$

We say that a function $f\colon X\to\mathbb{R}$ is *convex* (*concave, affine*) *wrt* α, if X is convex wrt α and

$$f(tx+\alpha(t)y)\le tf(x)+\alpha(t)f(y),\quad x,y\in X,\ t\in[0,1],$$

(respectively, if converse inequality or equality holds).

We say that a function $f\colon X\to\mathbb{R}$ is *m-convex* (*m-concave, m-affine*) *wrt* α, if X is m-convex wrt α and

$$f(tx+m\alpha(t)y)\le tf(x)+m\alpha(t)f(y),\quad x,y\in X,\ t\in[0,1], \tag{2.2}$$

(respectively, if converse inequality or equality holds).

Remark 2.3. A function $f\colon X\to\mathbb{R}$ is m-convex wrt α if and only if its epigraph

$$E(f):=\{(x,y)\in X\times\mathbb{R}: f(x)\le y\}$$

is m-convex wrt α.

Indeed, assume that f is m-convex wrt α and take arbitrary $(x_1,\ y_1)$, $(x_2,y_2)\in E(f)$. Then $f(x_1)\le y_1$, $f(x_2)\le y_2$ and, for arbitrary $t\in[0,1]$,

$$f(tx_1+m\alpha(t)x_2)\le tf(x_1)+m\alpha(t)f(x_2)\le ty_1+m\alpha(t)y_2.$$

Hence

$$t(x_1,y_1)+m\alpha(t)(x_2,y_2)=(tx_1+m\alpha(t)x_2,ty_1+m\alpha(t)y_2)\in E(f),$$

which shows that the set $E(f)$ is m-convex wrt α. The converse implication is also easy to verify.

In the sequel we assume that $X=I\subset\mathbb{R}$ is a nonempty interval such that $tI+\alpha(t)I\subset I$ for every $t\in I$, i.e., I is convex wrt α (respectively, $tI+m\alpha(t)I\subset I$ for every $t\in I$).

Remark 2.4. If $\alpha\colon [0,1] \to [0,1]$ is a decreasing involution, that is

$$(\alpha \circ \alpha)(t) = t, \quad t \in [0,1],$$

then it is a continuous bijection of $[0,1]$. Moreover, replacing t by $\alpha(t)$ in (2.2), we get

$$f(\alpha(t)x + mty) \le \alpha(t)f(x) + mtf(y), \quad x, y \in I, \ t \in [0,1],$$

and repeating this procedure here, we return to (2.2), similarly as in the classical case.

If α is an involution and $m \in (0,1)$ then the interval I must be of the form $[0,b)$ or $(0,b)$ for some b such that $0 < b \le +\infty$.

Example 2.5. For arbitrarily fixed $p > 0$, the function $\alpha\colon [0,1] \to [0,1]$,

$$\alpha(t) := (1 - t^p)^{1/p}, \quad t \in [0,1],$$

is an involution. Moreover,

$$t + m(1 - t^p)^{1/p} \le 1, \quad t \in [0,1], \quad p \in (0,1], \ m \le 1.$$

For $p = 1$ we get $\alpha(t) := 1 - t$ $(t \in [0,1])$, and the inequality in Definition 2.2 reduces to

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y), \quad x, y \in I, \ t \in [0,1], \tag{2.3}$$

which means that the function f is m-convex in the sense considered by Toader [15] (see also [5, 7, 16]).

Some generalizations of the classical notion of the convex function are strictly related to the notion of mean.

Let $I \subset \mathbb{R}$ be an interval, and a function $M\colon I \times I \to I$ be a mean in I, that is

$$\min(x,y) \le M(x,y) \le \max(x,y), \quad x, y \in I.$$

Clearly, if $J \subset I$ is a subinterval, then $M(J \times J) \subset J$ and M is reflexive, that is

$$M(x,x) = x, \quad x \in I.$$

A lot of (already classical) generalizations of the convex function read as follows: A function $f\colon J \to I$ is convex (concave, affine) with respect to a mean M in the interval J (Aumann, [2]), if

$$f(M(x,y)) \leq M(f(x), f(y)), \quad x, y \in J,$$

(respectively, the opposite inequality, equality) holds.

Note that this definition is correct due to the inclusion $M(J \times J) \subset J$ for every subinterval $J \subset I$, that is equivalent to the mean property of M. For $I = \mathbb{R}$ and $M = A$, where A is the arithmetic mean $A(x,y) := \frac{x+y}{2}$, we get the notion of Jensen convex (concave, affine) function in an interval $J \subset \mathbb{R}$; for $I = (0, +\infty)$ and $M = G$, where $G(x,y) = \sqrt{xy}$, we obtain the definition of Jensen geometrically convex function in an interval $J \subset (0, +\infty)$ (cf. for instance [10]).

Remark 2.6. Let $\alpha\colon [0,1] \to [0,1]$ be an involution, $m > 0$, and an interval I be m-convex wrt α. For arbitrarily fixed $t \in (0,1)$ such that $\alpha(t) \neq t$, let $N\colon I^2 \to \mathbb{R}$ be given by

$$N(x,y) := tx + m\alpha(t)y, \quad x, y \in I.$$

Then

(i) N is a mean in I if and only if $m = 1$ and $\alpha(t) = 1 - t$ (in this case m-convexity with respect to α coincides with m-convexity);

(ii) if $m < 1$ and $N(I \times I) \subset I$, then 0 must belong to the closure of I; in particular, if $I \subset [0, +\infty)$, then I must be of the form $[0, b)$ or $(0, b)$ for some b such that $0 < b \leq +\infty$.

To see (i) note that, if N is a mean then its reflexivity implies $t + m\alpha(t) = 1$. Replacing here t by $\alpha(t)$ and taking into account $\alpha(\alpha(t)) = t$ we get $\alpha(t) + mt = 1$. These equalities imply that $(1-m)(\alpha(t) - t) = 0$, so $m = 1$ and, consequently, $\alpha(t) = 1 - t$. Part (ii) is obvious.

3. Some properties of m-convex functions and an example

In this section we consider the m-convex functions in the sense of Toader [15], that is, we assume in Definition 2.2 that $m < 1$ and $\alpha(t) = 1 - t$ for all $t \in [0,1]$.

Remark 3.1 ([15, 16]). Let an interval I be as in Definition 2.2 (m-convex wrt α).

(i) If $m > 0$ and $f\colon I \to \mathbb{R}$ is m-convex, then, for all $x, y, z \in I$,

$$x < z < my \implies \frac{f(x) - f(z)}{x - z} \le \frac{f(z) - mf(y)}{z - my};$$

$$my < z < x \implies \frac{f(x) - f(z)}{x - z} \ge \frac{f(z) - mf(y)}{z - my}.$$

It follows that f is continuous and locally Lipschitzian in $\operatorname{int} I$.

(ii) If $0 \le m_1 < m_2 \le 1$, then every m_2-convex function is m_1-convex.

If $f\colon [a, b] \to \mathbb{R}$ is convex in the classical sense in the compact real interval $[a, b]$, then the values of f at a and b can be increased without any harm for the convexity of f, so f need not be continuous at the endpoints a, b. (Therefore, in the classical theory of convexity one assumes that the functions are defined on open convex sets.)

In general, the m-convex functions do not have this property, and it follows from the following

Remark 3.2. Suppose that $0 < m < 1$ and f is m-convex in the sense of the above definition. Then

(i) if $0 \in I$, then $f(0) \le 0$;

(ii) if $a \in \operatorname{int} I$ and $f(a) \le 0$, then

$$f(x) \le 0, \quad x \in I \cap [0, a].$$

Indeed, from (2.3) with $x = y = a$ we get $f((t + m(1 - t))a) \le 0$ for all $t \in [0, 1]$, so $f(x) \le 0$ in the interval $[ma, a]$. Now, by induction, we obtain $f(x) \le 0$ in the interval $[m^n a, a]$ for all $n \in \mathbb{N}$.

Hence we get the following

Corollary 3.3. *Let $0 < m < 1$ and $0 < b < +\infty$. If $f\colon (0, b) \to \mathbb{R}$ is m-convex and there is a sequence $x_n \in (0, b)$ such that*

$$\lim_{n\to+\infty} x_n = b; \quad f(x_n) \le 0 \quad \text{for all } n \in \mathbb{N},$$

then

$$f(x) \le 0, \quad x \in (0, b).$$

This feature is not shared by the classical convex functions, as they have, important in different applications, the "modality" property.

In the sequel, we assume that $I = (0, +\infty)$.

To show that there are common properties of convex functions and m-convex functions, we prove the following

REMARK 3.4. Let $0 < m < 1$. If a function $f\colon (0, +\infty) \to \mathbb{R}$ is m-affine, then there are $a, b \in \mathbb{R}$ such that

$$f(x) = ax + b, \quad x > 0.$$

PROOF. Assume that f is m-affine, so

$$f(tx + m(1-t)y) = tf(x) + m(1-t)f(y), \quad x,\, y \in (0, +\infty),\ t \in [0, 1].$$

Taking arbitrarily fixed $x, y \in (0, +\infty)$, $y < x$, and setting here

$$z = tx + m(1-t)y, \quad t \in [0, 1],$$

we get

$$f(z) = az + b, \quad z \in [my, x],$$

where

$$a := \frac{f(x) - mf(y)}{x - my}, \quad b := m\frac{xf(y) - yf(x)}{x - my}.$$

Since x and y can be chosen arbitrarily, it follows that

$$f(z) = az + b, \quad z > 0. \qquad \square$$

This property is shared by the classical convex functions.

It is well known that a real function f defined in an open interval I is convex iff at every point $x_0 \in I$, the graph of f is located above a supporting straight-line passing by the point $(x_0, f(x_0))$.

The following example shows that this property is not shared by m-convex functions.

Example 3.5. Let $a \in (0, +\infty)$ and $b \in (0, \frac{a}{2})$ be two fixed real numbers. Then, the polynomial function $h \colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) := x^4 - 4(a+b)x^3 + 6a(a+2b)x^2 + a^2(11a+16b)x$$

has the following properties: its real roots are $-a$ and 0; its complex roots are

$$\frac{1}{2}(5a + 4b \pm i\sqrt{19a^2 + 24ab - 16b^2});$$

$h([0,+\infty)) = [0, +\infty)$; $f := h|_{[0,+\infty)}$ is strictly increasing and not convex on $[0,+\infty)$; f is m-convex for

$$m \leq m(a,b) = \frac{(a-2b)(a+2b)^3}{(a-b)^2(a^2+6ab+11b^2)}.$$

Proof. Since

$$h(x) = x(x+a)(x^2 - (5a+4b)x + 11a^2 + 16ab),$$

$-a$ and 0 are roots of h. In turn, the quadratic polynomial given above has discriminant

$$\Delta = [-(5a+4b)]^2 - 4(11a^2 + 16ab) = -19a^2 - 24ab + 16b^2$$

and $\Delta < 0$ if and only if $b \in (\frac{3-2\sqrt{7}}{4}a, \frac{3+2\sqrt{7}}{4}a)$. But, by hypothesis, b belongs to $(0, \frac{a}{2})$ which is a proper subset of $(\frac{3-2\sqrt{7}}{4}a, \frac{3+2\sqrt{7}}{4}a)$. So, the other roots of h are complex and they are of the indicated form.

The next property follows from the facts that h is continuous,

$$\lim_{x \to +\infty} h(x) = +\infty,$$

and its only root in $[0,+\infty)$ is 0. Since

$$f''(x) = 12(x^2 - 2(a+b)x + a(a+2b)) = 12(x-a)(x-(a+2b)),$$

the function f is convex in $[0,a)$ and $(a+2b, +\infty)$ and concave in $(a, a+2b)$. Consequently, h is not convex.

Since f is the product of the identity and the polynomial of degree three which is strictly increasing in $[0,+\infty)$, it is strictly increasing.

To show the last property we apply formula (3) given in [16] with m instead of p denoted by $m(f)$; that is,

$$m(f) = \inf\left\{\frac{xf'(x)-f(x)}{yf'(x)-f(y)} : f''(x)=0, f'(x)=f'(y),\ x,y>0\right\}.$$

First we have to check that $xf'(x)-f(x)>0$ for all $x\in(0,+\infty)$ (i.e., f is strictly starshaped on $(0,+\infty)$). In fact,

$$\begin{aligned} xf'(x)-f(x) &= 3x^4-8(a+b)x^3+6a(a+2b)x^2 \\ &= 3x^2\left(x^2-\frac{8}{3}(a+b)x+2a(a+2b)\right) \\ &= 3x^2\left[\left(x-\frac{4}{3}(a+b)\right)^2+\frac{16}{9}\left(\frac{a}{2}-b\right)\left(\frac{a}{4}+b\right)\right]>0 \end{aligned}$$

for all $x\in(0,+\infty)$. We already know that $f''(x)=0$ if and only if $x=a$ or $x=a+2b$. Set $x_1=a$ and $x_2=a+2b$. Performing a simple calculation we get

$$f'(x_1)=15a^3+28a^2b,\quad f'(x_2)=15a^3+28a^2b-16b^3.$$

Solving for y on each of the equations

$$f'(y)=15a^3+28a^2b,\quad f'(y)=15a^3+28a^2b-16b^3,$$

we get the solutions $y_{11}=a+3b$ or $y_{12}=a$ and $y_{21}=a-b$ or $y_{22}=a+2b$, respectively. The next step consists in evaluating the function of two variables

$$\Phi(x,y):=\frac{xf'(x)-f(x)}{yf'(x)-f(y)}$$

at four points (x_1,y_{11}), (x_1,y_{12}), (x_2,y_{21}) and (x_2,y_{22}). In fact,

$$\Phi(x_1,y_{11})=\frac{a^3(a+4b)}{a^4+4a^3b+27b^4},\quad \Phi(x_2,y_{21})=\frac{(a-2b)(a+2b)^3}{(a-b)^2(a^2+6ab+11b^2)}$$

and

$$\Phi(x_1,y_{12})=\Phi(x_2,y_{22})=1.$$

To conclude, we have to compare all these values. Observe that all are positive. Set

$$A = a^3(a+4b), \quad B = a^4 + 4a^3b + 27b^4,$$

$$C = (a-2b)(a+2b)^3, \quad D = (a-b)^2(a^2+6ab+11b^2).$$

Then,

$$\Phi(x_{11}, y_{11}) > \Phi(x_2, y_{21}) \Leftrightarrow AD - BC > 0.$$

Since $AD - BC = 432(a+b)b^7$ and $\Phi(x_2, y_{21}) < 1$, we get

$$m(f) = \min\{\Phi(x_1, y_{11}), \Phi(x_2, y_{21}), 1\} = \Phi(x_2, y_{21}),$$

which completes the proof. □

Proposition 3.6. *For every* $m \in (0,1)$ *there is a polynomial* h *of degree* 4 *such that* $f := h|_{[0,+\infty)}$ *has the following properties:*

(i) $f(0) = 0$;
(ii) f *is a diffeomorphic mapping of* $[0,+\infty)$*;*
(iii) f *is* m*-convex in* $[0,+\infty)$*, and its epigraph* $E(f)$ *is an* m*-convex subset of* $\mathbb{R}^2$*;*
(iv) f *is not convex, and its epigraph* $E(f)$ *is not a convex subset of* $\mathbb{R}^2$*;*
(v) *for any sequence* $t_n \in (0,1)$*,* $n \in \mathbb{N}$*, such that*

$$\lim_{n\to+\infty} t_n = 1$$

there is a sequence $s_n \in (0,1)$*,* $n \in \mathbb{N}$*, such that*

$$\lim_{n\to+\infty} s_n = 0; \quad t_n + s_n < 1 \quad \textit{for every} \quad n \in \mathbb{N},$$

and

$$f(t_n x + s_n y) \le t_n f(x) + s_n f(y), \quad x, y \in [0,+\infty),\ n \in \mathbb{N}.$$

PROOF. Take arbitrarily fixed $m \in (0,1)$, $a > 0$ and put $b = \frac{a}{2}r$ where $r \in [0,1]$. Then, clearly, $b \in [0, \frac{a}{2}]$ and, in view of Example 3.5, we have

$$m(r) := m(a, \frac{a}{2}r) = \frac{16(1-r)(2+r)^3}{(2-r)^2(4+12r+11r^2)}, \quad r \in [0,1],$$

(so $m(r)$ does not depend on a). Since $m(0) = 1$, $m(1) = 0$, and the function $m(r)$ is continuous and one-to-one in $[0,1]$, there exists a unique $r_0 \in (0,1)$ such that $m(r_0) = m$. Applying the above example with $b = \frac{a}{2}r_0$ and Remark 2.3 we get the function f having properties (i)–(iv). Property (v) follows from (iii). $\square$

4. A result of a sandwich type

Now we shall prove a result of a sandwich type. But first notice that

REMARK 4.1. If I is $(0,+\infty)$ or $[0,+\infty)$ and $f\colon I \to \mathbb{R}$ is m-convex, then

$$f(mx) \leq mf(x), \quad x \in I.$$

THEOREM 4.2. *Let I be $(0,+\infty)$ or $[0,+\infty)$, and $0 < m < 1$. Assume that $f\colon I \to \mathbb{R}$ is m-convex. Then*

(i) *there exists a convex function $h\colon I \to \mathbb{R}$ such that*

$$f(x) \leq h(x) \leq mf\left(\frac{x}{m}\right), \quad x \in I,$$

or, equivalently,

$$\frac{1}{m}h(mx) \leq f(x) \leq h(x), \quad x \in I.$$

(ii) *If*

$$mf(x) \leq f(mx), \quad x \in I,$$

then

$$f(x) = f(1)x, \quad x \in I.$$

PROOF. (i) Replacing y in (2.3) by $\frac{y}{m}$ we obtain

$$f(tx+(1-t)y) \le tf(x) + m(1-t)f\left(\frac{y}{m}\right), \quad x, y \in I, \ t \in [0,1]. \tag{4.1}$$

Hence,

$$f(tx+(1-t)y) \le tmf\left(\frac{x}{m}\right) + (1-t)mf\left(\frac{y}{m}\right), \quad x, y \in I, \ t \in [0,1],$$

whence, setting

$$g(x) := mf\left(\frac{x}{m}\right), \quad x \in I,$$

we get

$$f(tx+(1-t)y) \le tg(x) + (1-t)g(x), \quad x, y \in I, \ t \in [0,1].$$

Applying the sandwich theorem [3] we conclude that there exists a (classical) convex function $h\colon I \to \mathbb{R}$ such that

$$f(x) \le h(x) \le g(x), \quad x \in I,$$

i.e., that

$$f(x) \le h(x) \le mf\left(\frac{x}{m}\right), \quad x \in I.$$

Since it is obvious that these inequalities are equivalent to

$$\frac{1}{m}h(mx) \le f(x) \le h(x), \quad x \in I,$$

the proof of (i) is complete.

(ii) In this case, by Remark 4.1, we have

$$f(mx) = mf(x), \ x \in I,$$

and,

$$f(tx+(1-t)y) \le tf(x) + m(1-t)f\left(\frac{y}{m}\right) = tf(x) + (1-t)f(y),$$

which means that f is convex. Moreover,

$$f(0+) = 0.$$

Now the convexity of f implies that the function

$$(0,+\infty) \ni x \mapsto \frac{f(x)}{x} \quad \text{is increasing.}$$

But then for any $x, y \in I$ arbitrary with $0 < x < y$,

$$\frac{f(x)}{x} \leq \frac{f(y)}{y}.$$

We assure f is a constant function. Indeed, if this is not the case we can find x_1, y_1 with $0 < x_1 < y_1$ and positive integer n such that $m^n y_1 < x_1$, consequently

$$\frac{f(m^n y_1)}{m^n y_1} = \frac{f(y_1)}{y_1} \leq \frac{f(x_1)}{x_1} < \frac{f(y_1)}{y_1}$$

which is impossible. □

In [12] it has been shown that an analogue of the sandwich theorem for convex functions (see [3]) is not true in the class of m-convex functions with $m \in (0,1)$.

Example 4.3 ([12]). Let us fix $m \in (0,1)$. For arbitrary fixed $a \in \mathbb{R}$ define the functions $f\colon [0,+\infty) \to \mathbb{R}$ and $g\colon [0,+\infty) \to \mathbb{R}$ by

$$f(x) := ax + 1, \quad g(x) := ax + \frac{1}{m}.$$

Then, for all $x, y \in [0,+\infty)$ and $t \in [0,1]$, we have

$$f(tx + m(1-t)y) \leq tg(x) + m(1-t)g(y),$$

and, of course, $f(x) \leq g(x)$ for all $x \in [0,+\infty)$. However, there is no m-convex function $h\colon [0,+\infty) \to \mathbb{R}$ such that

$$f(x) \leq h(x) \leq g(x), \quad x \geq 0.$$

5. Remarks on m-convex functions in the case $m > 1$

In this case the class of m-convex functions $f\colon (0,+\infty) \to \mathbb{R}$ such that $f(0+) \le 0$ is rather poor. Namely the following holds true.

PROPOSITION 5.1. *Let $m > 1$ be fixed. If $f\colon (0,+\infty) \to \mathbb{R}$ is m-convex and*

$$\liminf_{x\to 0+} f(x) \le 0,$$

then f is a linear function, i.e., $f(x) = f(1)x$ for all $x > 0$.

PROOF. By the assumption there is a positive decreasing sequence $(z_n : n \in \mathbb{N})$ such that $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} f(z_n) \le 0$. Let $(x_n : n \in \mathbb{N})$ be an arbitrary positive sequence such that $\lim_{n\to\infty} x_n = 0$. Without loss of generality, we can assume that $x_1 \le z_1$. Since $\lim_{n\to\infty} z_n = 0$, for every $n \in \mathbb{N}$, there exist $k_n, l_n \in \mathbb{N}$, $k_n < l_n$, such that

$$mz_{l_n} \le x_n \le z_{k_n}, \quad \lim_{n\to\infty} k_n = \infty.$$

Note that

$$t_n := \frac{x_n - mz_{l_n}}{z_{k_n} - mz_{l_n}} \in [0,1], \quad n \in \mathbb{N},$$

and

$$x_n = t_n z_{k_n} + m(1-t_n) z_{l_n}, \quad n \in \mathbb{N}.$$

Hence, by the m-convexity of f, we have

$$f(x_n) = f(t_n z_{k_n} + m(1-t_n) z_{l_n}) \le t_n f(z_{k_n}) + m(1-t_n) f(z_{l_n})$$

for every $n \in \mathbb{N}$. Letting here $n \to \infty$ we get

$$\limsup_{n\to\infty} f(x_n) \le 0,$$

which proves that

$$\limsup_{x\to 0+} f(x) \le 0.$$

Since $m > 1$, we can choose $t \in (0, 1)$ such that the numbers

$$\alpha := t, \quad \beta := m(1 - t),$$

fulfill the inequalities

$$0 < \alpha < 1 < \alpha + \beta,$$

and f satisfies the linear functional inequality

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y), \quad x, y \in (0, +\infty). \tag{5.1}$$

Since $\limsup_{x\to 0+} f(x) \le 0$, the result follows from [11, Theorem 1] (see also [9, 10, 14]). $\square$

REMARK 5.2. If, in the proposition above, $f : [0, +\infty) \to \mathbb{R}$ is m-convex and $f(0) = 0$, one can also apply a simple direct reasoning.

First, let us observe that, by m-convexity where $m > 1$, there exist real numbers α, β such that $0 < \alpha < 1 < \alpha + \beta$, $\frac{\log \beta}{\log \alpha}$ is an irrational number and (5.1) holds. Taking $y = x$ in (5.1), we have

$$f((\alpha + \beta)x) \le (\alpha + \beta)f(x), \quad x \in (0, +\infty),$$

whence, by induction,

$$f((\alpha + \beta)^k x) \le (\alpha + \beta)^k f(x), \quad x \in (0, +\infty), \ k \in \mathbb{N}.$$

Choose $k \in \mathbb{N}$ such that $\bar{\beta} := \beta(\alpha + \beta)^k > 1$. Hence, by (5.1), for all $x, y \in (0, +\infty)$,

$$\begin{aligned} f(\alpha x + \bar{\beta} y) &= f(\alpha x + \beta(\alpha + \beta)^k y) \\ &\le \alpha f(x) + \beta f((\alpha + \beta)^k y) \\ &\le \alpha f(x) + \beta(\alpha + \beta)^k f(y) = \alpha f(x) + \bar{\beta} f(y). \end{aligned}$$

So, if $\beta < 1$ we can replace it by $\bar{\beta}$.

Setting $y = 0$ and then $x = 0$, yields

$$f(\alpha x) \le \alpha f(x), \quad f(\beta x) \le \beta f(x), \quad x \in (0, +\infty),$$

that is, f satisfies the simultaneous system of two inequalities. Hence, by induction, we obtain

$$f(\alpha^k x) \le \alpha^k f(x), \quad f(\beta^n x) \le \beta^n f(x), \quad x \in (0,+\infty),\ k,n \in \mathbb{N},$$

whence

$$f(\alpha^k \beta^n x) \le \alpha^k \beta^n f(x), \quad x \in (0,+\infty),\ k,n \in \mathbb{N}.$$

Now, by the continuity of f in $(0,+\infty)$ (see Remark 3.1 (i)) and the Kronecker theorem on the density of the set $\{\alpha^k\beta^n : k,n \in \mathbb{N}\}$, one gets

$$f(rx) \le rf(x), \quad r,x > 0.$$

Replacing here x by $\frac{x}{r}$ we hence get $\frac{1}{r}f(x) \le f(\frac{1}{r}x)$ for all $r,x > 0$, whence

$$rf(x) \le f(rx), \quad r,x > 0,$$

and, consequently,

$$f(rx) = rf(x), \quad r,x > 0.$$

Taking here $x = 1$ we get $f(r) = f(1)r$ for all $r > 0$, which completes the proof.

From Proposition 5.1 we immediately get the following

Corollary 5.3. *Let $\alpha\colon [0,1] \to [0,1]$ and m (in Definition 2.2) be such that for some $t \in (0,1)$,*

$$\min\{t, m\alpha(t)\} < 1 < t + m\alpha(t).$$

If $f\colon (0,+\infty) \to \mathbb{R}$ is m-convex wrt α, and $\liminf_{x\to 0+} f(x) \le 0$, then $f(x) = f(1)x$ for all $x > 0$.

Remark 5.4. In this corollary we need not to assume that a function α is continuous as we do in Proposition 2.1 (ii).

It follows that considering the functions which are m-convex wrt α, it is rational to assume that either $t + \alpha(t) \le 1$ for all $t \in [0,1]$ or $t + \alpha(t) \ge 1$ for all $t \in [0,1]$.

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Teodoro Lara
Departamento de Física y Matemáticas
Universidad de los Andes
N. U. Rafael Rangel
Trujillo
Venezuela
e-mail: tlara@ula.ve

Janusz Matkowski
Faculty of Mathematics,
Computer Science and Econometrics
University of Zielona Góra
Szafrana 4A
65-516 Zielona Góra
Poland
e-mail: J.Matkowski@wmie.uz.zgora.pl

Nelson Merentes
Escuela de Matemáticas
Universidad Central de Venezuela
Caracas
Venezuela
e-mail: nmerucv@gmail.com

Roy Quintero
Departamento de Física y Matemáticas
Universidad de los Andes
N. U. Rafael Rangel
Trujillo
Venezuela
e-mail: rquinter@ula.ve

Małgorzata Wróbel
Institute of Mathematics
Czestochowa University of Technology
al. Armii Krajowej 21
42-200 Częstochowa
Poland
e-mail: malgorzata.wrobel@im.pcz.pl