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Implicit Methods for Numerical Solution of Singular Initial Value Problems

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Abstract

Various order of implicit method has been formulated for solving initial value problems having an initial singular point. The method provides better result than those obtained by used implicit formulae developed based on Euler and Runge-Kutta methods. Romberg scheme has been used for obtaining more accurate result.

Keywords: Singular Integrals, Romberg scheme, Singular initial value problems, Implicit Runge-Kutta method.
AMS 2010 codes: 35F10, 35F25, 35G10, 35G25.

1 Introduction

Recently many approximate techniques [1-6] have been developed for handling complicated physical problems. One of such complications arises for solving the singular initial value problems. The study of this topic has concerned the interest of many mathematicians and physicists. A first order singular initial value problem is encountered in ecology in the computation of avalanche run-up [7]. Several authors evaluated the singular initial value problems by both analytical and numerical techniques. Koch and Weinmuller [8] discussed the existence of an analytic solution of the first order singular initial value problems. Auzinger et al. [9] and Koch et al. [10-11] applied well-known acceleration technique Iterated Defect Correction (IDeC) based on implicit Euler method. Recently, Hasan et al. [12-16] derived some implicit formulae for solving first and second order singular initial value problems based on the integral formulae derived in [17-18]. These implicit methods give more accurate results than those obtained by the implicit Euler, second, third and fourth- order implicit Runge-Kutta (RK) methods. In this article, we develop three new (i.e. third, fourth and fifth-order) implicit formulae

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for solving singular initial value problems. Romberg scheme have been applied at the initial point for obtaining the improved results. Some suitable examples have been presented to illustrate these methods.

2 Methodology

Earlier singular integrals were evaluated by Fox [19] based on extrapolation approach and then several Romberg schemes were used to improve the results. But Huq et al. [17] derived a straightforward formula for evaluating

$$I = \int_0^1 f(x)dx \tag{1}$$

when $f(x)$ is singular at $x = x_0$. In [17] a third order Lagrange's interpolation formula

$$f(x) \cong \frac{(x-x_1)(x-x_2)f_0}{(x_0-x_1)(x_0-x_2)} + \frac{(x-x_0)(x-x_2)f_1}{(x_1-x_0)(x_1-x_2)} + \frac{(x-x_0)(x-x_1)f_2}{(x_2-x_0)(x_2-x_1)}, \tag{2}$$

was used where $x_1 = x_0 + h/3$ and $x_2 = x_0 + h$, $h > 0$ and the following formula was obtained

$$\int_{x_0}^{x_2} f(x)dx = h[3f_1 + f_2]/4; \tag{3}$$

f_1 and f_2 are the functional values of $f(x)$ respectively for $x_1 = x_0 + h/3$ and $x_2 = x_0 + h$. Formula Eq. (3) excludes f_0 and thus it does not requires extrapolation to evaluate an approximate value of $f(x)$ at $x = x_0$. Then based on formula (3), Hasan et al. [12-15] used implicit formula

$$y_{i+1} = y_i + h(3f(x_i + h/3, (y_i + (y_{i+1} - y_i)/3)) + f(x_i + h, y_{i+1}))/4, \tag{4}$$

for solving initial value problem

$$y' = f(x,y), y(x_0) = y_0, \tag{5}$$

where $f(x,y)$ becomes infinite when $x = x_0$ and $i = 0,1,2,....$
By considering fourth order Lagrange's interpolation, Hasan et al. [18] derived another formula for evaluating integral Eq. (1) as

$$\int_{x_0}^{x_0+h} f(x)dx = h[4f_1 + 5f_2 + f_3]/10; \tag{6}$$

where $x_1 = x_0 + h/6$, $x_2 = x_0 + 4h/6$ and $x_3 = x_0 + h$.

Then the following implicit formula

$$\begin{gathered} y_{i+1} = y_i + h[4f(x_i + h/6, (y_i + (y_{i+1} - y_i)/6)) + \\ 5f(x_i + 4h/6, (y_i + 4(y_{i+1} - y_i)/6)) + f(x_i + h, y_{i+1})]/10, \end{gathered} \tag{7}$$

where $i = 0,1,2,...$, was used for solving singular initial value problems (see [16] for details).

2.1 Derivation of more accurate formulae

In this article, we have derived three implicit formulae which follow a class together with formula Eq. (3) [17]. The error of this formula is $\frac{3h^4 f^{(3)}(x_0)}{8}$. On the other hand the error of formula Eq. (6) is $\frac{h^5 f^{(4)}(x_0)}{120}$. Using fourth order Lagrange's interpolation formula a more accurate integration formula can be found whose order of error is h^6. Similarly fifth and sixth order Lagrange's interpolation formulae provide two integration formulae whose order of errors are respectively h^8 and h^{10}. Then the integral formulae can be used as implicit formulae for solving singular initial value problems (e.g., Eq. (5)). Finally Romberg technique can be applied to obtain improved results.

2.1.1 The new third order formula

By utilization of fourth order Lagrange's interpolation formula we have derived a third order integral formula as

$$\int_{x_0}^{x_0+h} f(x)dx = h[(16-\sqrt{6})f_1 + (16+\sqrt{6})f_2 + 4f_3]/36, \tag{8}$$

choosing $x_1 = x_0 + ph$, $x_2 = x_0 + qh$, $x_3 = x_0 + h$; $p = (4-\sqrt{6})/10$, $q = (4+\sqrt{6})/10$, where f_1, f_2 and f_3 are the functional values of $f(x)$ at $x_1 = x_0 + ph$, $x_2 = x_0 + qh$ and $x_3 = x_0 + h$ respectively. This formula also excludes f_0. The error of this formula has been calculated as $\frac{h^6 f^{(5)}(x_0)}{7200}$. So that formula Eq. (8) measures more accurate result than that of Eq. (6) presented in [18]. Now based on formula (8), a third order implicit formula can be proposed for solving singular initial value problems having singular point at $x = x_0$ as

$$\begin{gathered} y_{i+1} = y_i + h[(16-\sqrt{6})f(x_i+ph,(y_i+p(y_{i+1}-y_i)))+ \\ (16+\sqrt{6})f(x_i+qh,(y_i+q(y_{i+1}-y_i)))+4f(x_i+h,y_{i+1})]/36, \end{gathered} \tag{9}$$

where $i = 0,1,2,\ldots$.

2.1.2 The new fourth order formula

By utilization of fifth order Lagrange's interpolation formula we obtain the fourth order integral formula as

$$\int_{x_0}^{x_0+h} f(x)dx = h[w_1 f_1 + w_2 f_2 + w_3 f_3 + w_4 f_4], \tag{10}$$

choosing $x_1 = x_0 + ph$, $x_2 = x_0 + qh$, $x_3 = x_0 + rh$, $x_4 = x_0 + h$; $p = 0.08858795951270394$, $q = 0.4094668644407347, r = 0.787659461760847; w_1 = 0.220462211176768$, $w_2 = 0.3881934688317074, w_3 = 0.32884431998006125$ and $w_4 = 0.0625000000000022$. The error of this formula is $2.02462\times10^{-8}h^8 f^{(7)}(x_0)$.
Based on formula (10), the fourth order implicit method has been proposed as

$$\begin{aligned} y_{i+1} = y_i + h(w_1 f(x_i+ph,(y_i+p(y_{i+1}-y_i)))+w_2 f(x_i+qh,(y_i+q(y_{i+1}-y_i)))+ \\ w_3 f(x_i+rh,(y_i+r(y_{i+1}-y_i)))+w_4 f(x_i+h,y_{i+1})), \end{aligned} \tag{11}$$

where $i = 0,1,2,\ldots$.

2.1.3 The new fifth order formula

Similarly, by using sixth order Lagrange's interpolation formula we have derived a fifth order integral formula as

$$\int_{x_0}^{x_0+h} f(x)dx = h[w_1 f_1 + w_2 f_2 + w_3 f_3 + w_4 f_4 + w_5 f_5], \tag{12}$$

choosing $x_1 = x_0 + ph$, $x_2 = x_0 + qh$, $x_3 = x_0 + rh$, $x_4 = x_0 + sh$, $x_5 = x_0 + h$; p=0.05710419612460134, q=0.27684301367379077, r=0.5835904324091858, s=0.8602401356748276, $w_1 = 0.14371356081477416$, $w_2 = 0.28135601516952136$, $w_3 = 0.3118265229640613$,
$w_4 = 0.2231039010576481$ and $w_5 = 0.03999999999402437$.
The error of this formula is $1.73578\times10^{-11}h^{10} f^{(9)}(x_0)$.
Based on formula (12), the fifth order implicit method has been proposed as

$$\begin{aligned} y_{i+1} = y_i + h(w_1 f(x_i+ph,(y_i+p(y_{i+1}-y_i)))+w_2 f(x_i+qh,(y_i+q(y_{i+1}-y_i)))+ \\ w_3 f(x_i+rh,(y_i+r(y_{i+1}-y_i)))+w_4 f(x_i+sh,(y_i+s(y_{i+1}-y_i)))+w_5 f(x_i+h,y_{i+1})), \end{aligned} \tag{13}$$

where $i = 0,1,2,\ldots$.

3 The Romberg scheme of new third, fourth and fifth-order formula

According to [19], the first Romberg's formula for evaluating $I = \int_0^1 \frac{1}{\sqrt{x}}dx$ is expressed as

$$I(h,\frac{h}{2}) = \frac{\sqrt{2}I(\frac{h}{2}) - I(h)}{(\sqrt{2}-1)} \tag{14}$$

Using Eq. (14), the Romberg scheme of the third order integral formula (i.e. Eq. (8)) can be written as

$$\begin{aligned} I(h,\frac{h}{2}) = (1+\sqrt{2})h((16-\sqrt{6})(-2f_1+\sqrt{2}(f_4+f_7))+ \\ (16+\sqrt{6})(-2f_2+\sqrt{2}(f_5+f_8))+4(-2f_3+\sqrt{2}(f_6+f_9))/72, \end{aligned} \tag{15}$$

where

$$I(h) = h[(16-\sqrt{6})f_1 + (16+\sqrt{6})f_2 + 4f_3]/36, \tag{16}$$

and

$$I(\frac{h}{2}) = h[(16-\sqrt{6})(f_4+f_7) + (16+\sqrt{6})(f_5+f_8) + 4(f_6+f_9)]/72. \tag{17}$$

Romberg formula was originally used to evaluate integration. But it can be easily used to solve initial value problems especially when $f(x,y)$ becomes infinite for $x = x_0$. The first Romberg formula for our third order implicit formula Eq. (9) can be written as

$$\begin{aligned} y = y_0 + (1+\sqrt{2})h((16-\sqrt{6})(-2f_1+\sqrt{2}(f_4+f_7))+ \\ (16+\sqrt{6})(-2f_2+\sqrt{2}(f_5+f_8))+4(-2f_3+\sqrt{2}(f_6+f_9))/72, \end{aligned} \tag{18}$$

where

$$f_{i+1} = \begin{cases} f(x_0+jh, y_0+j(y-y_0); i=0,1,2 \text{ and } j=p,q,1. \\ f(x_0+jh, y_0+j(y-y_0)); i=3,4,5 \text{ and } j=\frac{p}{2},\frac{q}{2},\frac{1}{2}. \\ f(x_0+jh, y_0+j(y-y_0)); i=6,7,8 \text{ and } j=\frac{(1+p)}{2},\frac{(1+q)}{2},1. \end{cases}$$

Similarly, the proposed Romberg scheme for the fourth order implicit formula becomes

$$\begin{aligned} y = y_0 + h(-0.532242860213573f_1 - 0.9371819373057972f_2 - 0.7939004172053609f_3 - \\ 0.15088834764836398f_4 + 0.37635253569514127f_5 + 0.6626877030744751f_6 + \\ 0.5613723685927399f_7 + 0.10669417382419143f_8 + 0.37635253569514127f_9 + \\ 0.6626877030744751f_{10} + 0.5613723685927399f_{11} + 0.10669417382419143f_{12}), \end{aligned} \tag{19}$$

where

$$f_{i+1} = \begin{cases} f(x_0+jh, y_0+j(y-y_0); i=0,1,2,3 \text{ and } j=p,q,r,1. \\ f(x_0+jh, y_0+j(y-y_0)); i=4,5,6,7 \text{ and } j=\frac{p}{2},\frac{q}{2},\frac{r}{2},\frac{1}{2}. \\ f(x_0+jh, y_0+j(y-y_0)); i=8,9,10,11 \text{ and } j=\frac{(1+p)}{2},\frac{(1+q)}{2},\frac{(1+s)}{2},1. \end{cases}$$

And the Romberg scheme for the fifth order implicit formula Eq. (13) can be proposed as

$$\begin{aligned} y = y_0 + h(-0.34695522761595826f_1 - 0.6792535076775085f_2 - 0.752815820847482f_3 - \\ 0.538620463751719f_4 + 0.09656854248049733f_5 + 0.24533439421536624f_6 + \\ 0.4803047614235149f_7 + 0.5323211719057717f_8 + 0.38086218240468356f_9 + \\ 0.06828427123726084f_{10} + 0.24533439421536624f_{11} + 0.4803047614235149f_{12} + \\ 0.5323211719057717f_{13} + 0.38086218240468356f_{14} + 0.06828427123726084f_{15}), \end{aligned} \tag{20}$$

where

$$f_{i+1} = \begin{cases} f(x_0+jh, y_0+j(y-y_0);\ i=0,1,2,3.4 \text{ and } j=p,q,r,s,1. \\ f(x_0+jh, y_0+j(y-y_0));\ i=5,6,7,8,9 \text{ and } j=\frac{p}{2},\frac{q}{2},\frac{r}{2},\frac{s}{2},\frac{1}{2}. \\ f(x_0+jh, y_0+j(y-y_0));\ i=10,11,12,13,14 \text{ and } j=\frac{(1+p)}{2},\frac{(1+q)}{2},\frac{(1+r)}{2},\frac{(1+s)}{2},1. \end{cases}$$

Theoretically, second, third... order Romberg formulae can be derived; but in practical it is too much a laborious task.

4 Examples

In this section, the method has been illustrated with the following examples.

4.1 Let us consider a first order linear initial value problem in the form as

$$y' = -\frac{y(x)}{\sqrt{x}}, 0 < x \leq 1, y(0) = 1. \tag{21}$$

First we have solved this equation by our third order (i.e. Eq. (9)), fourth order (i.e. Eq. (11)) and fifth order ((i.e. Eq. (13)) formulae, and then Hasan's [13,16] formulae. All the absolute errors have been calculated and presented respectively in Table 4.1(a) and Table 4.1(b). Finally we have obtained the solution of the Eq. (21) using the first Romberg's scheme of third order (i.e. Eq. (18)), fourth order (i.e. Eq. (19)) and fifth order ((i.e. Eq. (20)) formulae (together with Hasan's formulae) at initial point and then solved the rest again by third, fourth and fifth order formulae. The absolute errors of all the solutions have been presented in Tables 4.1(c), Tables 4.1(d) and Table 4.1(e). The exact solution of Eq. (21) is $e^{(-2\sqrt{x})}$.

Table 4.1 (a): Absolute errors of rhe solution of Eq. (21) at $x = 0.01$ by Hasan's [13, 16] methods and new methods.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-3}	1.16741×10^{-2}	7.722254×10^{-3}	7.36938×10^{-3}	5.33270×10^{-3}	4.12995×10^{-3}
10^{-4}	3.70766×10^{-3}	2.49450×10^{-3}	2.38698×10^{-3}	1.76377×10^{-3}	1.39456×10^{-3}
10^{-5}	1.17390×10^{-3}	7.94790×10^{-4}	7.60955×10^{-4}	5.65208×10^{-4}	4.49539×10^{-4}
10^{-6}	3.71357×10^{-4}	2.51870×10^{-4}	2.41183×10^{-4}	1.79468×10^{-4}	1.42998×10^{-4}

Table 4.1 (b): Absolute errors of rhe solution of Eq. (21) at $x = 0.0001$ by Hasan's [13, 16] methods and new methods.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-5}	1.40542×10^{-3}	9.51680×10^{-4}	9.11070×10^{-4}	6.76719×10^{-4}	5.38238×10^{-4}
10^{-6}	4.44595×10^{-4}	3.01520×10^{-4}	2.88749×10^{-4}	2.14863×10^{-4}	1.71200×10^{-4}
10^{-7}	1.40609×10^{-4}	9.54200×10^{-5}	9.13754×10^{-5}	6.80328×10^{-5}	5.42385×10^{-5}
10^{-8}	4.44661×10^{-5}	3.01800×10^{-5}	2.89019×10^{-5}	2.15226×10^{-5}	1.71617×10^{-5}

Table 4.1 (c): Absolute errors of the solution of Eq. (21) at $x = 0.0001$ by Hasan's [13, 16] methods and new methods with Romberg scheme.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-5}	1.70708×10^{-5}	6.05027×10^{-6}	6.49296×10^{-6}	6.94702×10^{-6}	6.93944×10^{-6}
10^{-6}	6.96009×10^{-6}	3.94596×10^{-7}	5.37472×10^{-7}	6.93500×10^{-7}	6.95451×10^{-7}
10^{-7}	2.35661×10^{-6}	2.76102×10^{-8}	1.78571×10^{-8}	6.84388×10^{-8}	6.94899×10^{-8}
10^{-8}	7.60778×10^{-7}	2.39618×10^{-8}	9.55524×10^{-9}	6.56421×10^{-9}	6.93995×10^{-9}

Table 4.1 (d): Absolute errors of the solution of Eq.(21) at $x = 0.000001$ by Hasan's [13, 16] methods and new methods with Romberg scheme.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-7}	9.50211×10^{-5}	2.85167×10^{-8}	1.77693×10^{-8}	6.92689×10^{-8}	3.69929×10^{-8}
10^{-8}	9.18055×10^{-6}	2.44011×10^{-8}	9.73290×10^{-9}	6.67930×10^{-9}	7.06188×10^{-9}
10^{-9}	2.46532×10^{-7}	9.26524×10^{-9}	6.62384×10^{-9}	5.78785×10^{-10}	7.04187×10^{-10}
10^{-10}	7.81185×10^{-8}	3.08492×10^{-9}	1.61689×10^{-9}	2.95883×10^{-11}	6.96841×10^{-11}

Table 4.1 (e): Absolute errors of the solution of Eq. (21) at $x = 0.0000000001$ by Hasan's [13, 16] methods and new methods with Romberg scheme.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-10}	2.42985×10^{-8}	9.92961×10^{-10}	5.27850×10^{-10}	6.03890×10^{-12}	6.70960×10^{-12}
10^{-11}	7.83390×10^{-9}	3.15568×10^{-10}	1.68461×10^{-10}	3.43440×10^{-12}	6.01400×10^{-13}
10^{-12}	2.47747×10^{-9}	9.99449×10^{-11}	5.34257×10^{-11}	1.23850×10^{-12}	3.83000×10^{-14}
10^{-13}	7.83460×10^{-10}	3.16220×10^{-11}	1.69114×10^{-11}	4.08100×10^{-13}	4.70000×10^{-15}

4.2 Now we consider a first order non-nonlinear initial value problem in the form

$$y' = -\frac{y^2(x)}{\sqrt{x}}, 0 < x \leq 1, y(0) = 1. \tag{22}$$

with exact solution $\frac{1}{(1+2\sqrt{x})}$. The absolute error of the solution of the Eq. (22) obtain by the Hasan [13, 16] (i.e., Eq. (4)) and Eq. (7)) methods, the new third order (i.e. Eq. (9)), fourth order (i.e. Eq. (11)) and fifth order ((i.e. Eq. (13)) methods are tabulated in Table 4.2(a). Also the absolute errors of the solution of the Eq. (22) obtained by these methods with Romberg scheme at initial point are tabulated in Tables 4.2 (b) and Tables 4.2 (c).

Table 4.2 (a): Absolute errors of the solution of Eq. (22) at x=0.0001 by Hasan's [13, 16] methods and new methods.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-5}	1.37739×10^{-3}	9.30410×10^{-4}	8.90420×10^{-4}	6.59620×10^{-4}	5.23240×10^{-4}
10^{-6}	4.35887×10^{-4}	2.95420×10^{-4}	2.82850×10^{-4}	2.10290×10^{-4}	1.67420×10^{-4}
10^{-7}	1.37870×10^{-4}	9.35400×10^{-5}	8.95700×10^{-5}	6.66700×10^{-5}	5.31400×10^{-5}
10^{-8}	4.36000×10^{-5}	2.95900×10^{-5}	2.83400×10^{-5}	2.11000×10^{-5}	1.68200×10^{-5}

Table 4.2 (b): Absolute errors of the solution of Eq. (22) at $x = 0.0001$ by Hasan's [13, 16] methods and new methods with Romberg scheme.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-5}	9.67336×10^{-6}	1.28488×10^{-5}	1.32698×10^{-5}	1.36583×10^{-5}	5.29373×10^{-6}
10^{-6}	6.11857×10^{-6}	1.07858×10^{-6}	1.21738×10^{-6}	1.36474×10^{-6}	5.29644×10^{-7}
10^{-7}	2.24030×10^{-6}	4.20101×10^{-8}	8.64651×10^{-8}	1.35502×10^{-7}	5.27899×10^{-8}
10^{-8}	7.38957×10^{-7}	1.65915×10^{-8}	2.47758×10^{-9}	1.32726×10^{-8}	5.26552×10^{-9}

Table 4.2 (c): Absolute errors of the solution of Eq. (22) at x=0.0000000001 by Hasan's [13, 16] methods and new methods with Romberg scheme.

h	Errors				
	Hasan 2^{nd}	Hasan 3^{rd}	New 3^{rd}	New 4^{th}	New 5^{th}
10^{-10}	2.47256×10^{-8}	9.85800×10^{-10}	5.20713×10^{-10}	1.02940×10^{-12}	5.06390×10^{-12}
10^{-11}	7.83301×10^{-9}	3.14843×10^{-10}	1.67741×10^{-10}	2.72390×10^{-12}	4.40500×10^{-13}
10^{-12}	2.47734×10^{-9}	9.98726×10^{-11}	5.33553×10^{-11}	1.16940×10^{-12}	2.0100×10^{-14}
10^{-13}	7.83433×10^{-10}	3.16100×10^{-11}	1.68998×10^{-11}	3.96500×10^{-13}	1.40000×10^{-15}

5 Result and Discussions

Three implicit formulae and their Romberg's scheme have been presented for solving singular initial value problems. The approximate solutions of some first order linear and non-linear equations have been compared with their exact solutions. For the linear equation, the approximate solution of Eq. (21) has been obtained by the new formulae and the errors have been presented in Tables 4.1(a) and 4.1(b) together with corresponding errors of Hasan's methods. Tables 4.1(a) and 4.1(b) show that, the new methods give better results than those obtained by Hasan's methods. For Romberg scheme, when $x = 0.0001$, Table 4.1(c) shows that Hasan's [16] third and our new third order formulae are calculating better results than others. When decreasing i.e. at $x = 0.000001$, Table 4.1(d) shows that the results obtain by the third order formulae are not presenting significant change whereas fourth order formula is giving better results. In Table 4.1(e), we observed that, when x is very close to singular point i. e. $x = 0.0000000001$, the results by third and fourth order formulae are not significantly changed where fifth order formula is giving significant better result. For the non-linear case Eq. (22), similar results hold as linear case which has been shown in Table 4.2(a), 4.2(b) and 4.2(c).

6 Conclusion

From the above observation it may conclude that, our third-, fourth- and fifth order formulae provide better results than other existing formulae for both linear and non-linear equations. The utilization of Romberg scheme at initial stage provides more and more accurate results. However, the fifth order formula provides the best results among all the formulae presented in this article. It is clear that the six order formula must provide better results than our presented fifth order formulae since its order of error is h^{12}; but the derivation of this formula is a tremendously difficult.

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