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Dirichlet Problem for Poisson Equation on the Rectangle in Infinite Dimensional Hilbert Space

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Abstract

We study the class of finite additive shift invariant measures on the real separable Hilbert space E . For any choice of such a measure we consider the Hilbert space \mathcal{H} of complex-valued functions which are square-integrable with respect to this measure. Some analogs of Sobolev spaces of functions on the space E are introduced. The analogue of Gauss theorem is obtained for the simplest domains such as the rectangle in the space E . The correctness of the problem for Poisson equation in the rectangle with homogeneous Dirichlet condition is obtained and the variational approach of the solving of this problem is constructed.

Keywords: Shift invariant measure on Banach space, random walks, Laplas operator, Sobolev space, Dirichlet problem.

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1 Introduction

The studying of a random processes in infinite dimension Banach spaces and its description by a partial differential equation for a functions on the Banach space are the important topics of contemporary mathematics (see [4, 8, 9]). To the investigation of the above topics and to construct the quantum theory of infinite dimension

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Hamiltonian systems the analogs of the Lebesgue measure on the infinite dimension linear space are introduced in the works [1, 10, 13, 17].

To study the random walks in the real Hilbert space E we introduce a class of measures on the Hilbert space which are invariant with respect to a shift on an arbitrary vector of the space E (see [10, 14]). For any choice of such measure we construct the Hilbert space \mathcal{H} of complex-valued functions any of each is square integrable with respect to this measure. We study operators of argument shifts on the spaces \mathcal{H} .

We study the random shift operator on the vector whose distribution on Hilbert space E is given by a semigroup $\gamma_t, t \geq 0$, of Gaussian measures with respect to the convolution. We prove that the mean values of the random shift operator form the one-parametric semigroup $U(t), t \geq 0$, of self-adjoint contractions in the space \mathcal{H} . The criteria of strong continuity of this semigroup U is obtained.

We prove that if the semigroup U is strongly continuous in the space \mathcal{H} then for any $t > 0$ the image $U(t)f$ of any vector $f \in \mathcal{H}$ has the derivatives of any order in the direction of any eigenvector of covariation operator D of Gaussian measure γ_1 . Therefore the space of smooth functions is defined as the image of the space \mathcal{H} under the actions of the operators $U(t), t > 0$, of semigroup U .

For any non-negative non-degenerated trace-class operator D in the space E the Sobolev space $W_{2,D}^m(E)$ is defined as the space of functions $u \in \mathcal{H}$ such that $(\partial_k)^l u \in \mathcal{H}$ for any $l \in \{1, \dots, m\}$ and any $k \in \mathbb{N}$ and the following series converges

$$\sum_{k=1}^{\infty} d_k \|(\partial_k)^m u\|_{\mathcal{H}}^2 < +\infty.$$

Here $\{d_k\}$ is the sequence of eigenvalues of the operator D and $\{e_k\}$ is the sequence of corresponding eigenvectors. The function $u \in \mathcal{H}$ has the derivative $\partial_h u \in \mathcal{H}$ in the direction of the unite vector $h \in E$ if the following equality $\lim_{t \rightarrow 0} \|\frac{1}{t}(S_{th} - I)u - \partial_h u\|_{\mathcal{H}} = 0$ holds.

We study the connection of the random walks in the space E with the self-adjoint analogue of Laplace operator whose domain is the Sobolev space. We prove that the analogue of Laplace operator is the generator of the semigroup of self-adjoint operators arising as the mean value of random shift operators. The properties of smooth function space embedding into the Sobolev spaces are studied. The analogue of Gauss theorem is obtained for the simplest domains such as the rectangle in the space E . The correctness of the problem for Poisson equation in the rectangle with homogeneous Dirichlet condition is obtained and the variational approach of the solving of this problem is constructed.

2 A class of shift invariant measures on a Hilbert space

According to A. Weil theorem there is no Lebesgue measure on the infinite dimensional separable normed real linear space E , i.e. there is no Borel σ -additive σ -finite locally finite measure on the space E which is translation-invariant. Therefore an analogue of the Lebesgue measure is defined as an additive function on some ring of subsets of the space E which is translation-invariant. In this paper we present the analogue of the Lebesgue measure which is σ -finite and locally finite but not Borel and not σ -additive measure (see [10–12]). In the papers [1, 16, 17] the analogue of the Lebesgue measure is considered as the measure which is Borel and σ -additive but not σ -finite and not locally finite.

We study invariant measures on a real separable Hilbert space E , which are invariant with respect to any shift. In this article finite-additive analogues of the Lebesgue measure are constructed. The non-negative finite-additive translation-invariant measure λ is defined on the special ring \mathcal{R} of subsets from a space E in the work [10]. The ring \mathcal{R} contains all infinite-dimensional rectangles whose products of side lengths are absolutely convergent.

Now we describe some class of translation-invariant measures on separable Hilbert space E any of each is the restriction of measure λ from the work [10] on a ring $\mathcal{R}_{\mathcal{E}}$ depending on the chois of an orthonormal basis $\mathcal{E} = \{e_j\}$ in te space E . Let \mathcal{S} be a set of orthonormal bases in the space E . Firstly we describe a class of measures on the space E which are invariant with respect to the shift on any vector of this space.

Let us introduce the following family of the elementary sets. Rectangle in the real separable Hilbert space E is the set $\Pi \subset E$ such that there is an orthonormal basis $\{e_j\} \equiv \mathcal{E}$ in E and there is an elements $a, b \in l_\infty$ such that

$$\Pi = \{x \in E : (x, e_j) \in [a_j, b_j] \forall j \in \mathbf{N}\}. \quad (1)$$

The rectangle (1) is noted by the symbol $\Pi_{\mathcal{E}, a, b}$.

The rectangle (1) is called measurable if it either empty set, or the following condition holds

$$\sum_{j=1}^{\infty} \max\{0, \ln(b_j - a_j)\} < +\infty. \quad (2)$$

Let $\lambda(\Pi) = 0$ if $\Pi = \emptyset$, and let

$$\lambda(\Pi_{\mathcal{E}, a, b}) = \exp\left(\sum_{j=1}^{\infty} \ln(b_j - a_j)\right) \quad (3)$$

for any nonempty measurable rectangle $\Pi_{\mathcal{E}, a, b}$.

For any orthonormal basis $\mathcal{F} = \{f_k\}$ of the space E the symbol $K(\mathcal{F})$ note the set of measurable rectangles with the edges collinear to the vectors of ONB \mathcal{F} . Let the symbol $r_{\mathcal{F}}$ notes the minimal ring of subsets containing the set of measurable rectangles $K(\mathcal{F})$.

Theorem 1. [11] For any orthonormal basis $\mathcal{F} = \{f_k\}$ of the space E there exists the unique measure $\lambda_{\mathcal{F}} : r_{\mathcal{F}} \rightarrow [0, +\infty)$ such that the equality (3) holds for any rectangle $\Pi_{\mathcal{F}, a, b} \in \mathcal{K}_{\mathcal{F}}$. The measure $\lambda_{\mathcal{F}}$ has the unique completion onto the ring $\mathcal{R}_{\mathcal{F}}$ which is completion of the ring $r_{\mathcal{F}}$ by measure $\lambda_{\mathcal{F}}$.

Note 2. Here the ring $\mathcal{R}_{\mathcal{F}}$ consists on the sets $A \subset E$ such that $\overline{\lambda_{\mathcal{F}}}(A) = \lambda_{\mathcal{F}}(A) \in \mathbb{R}$ where $\overline{\lambda_{\mathcal{F}}}(A) = \inf_{B \in r_{\mathcal{F}}, B \supset A} \lambda_{\mathcal{F}}(B)$, $\underline{\lambda_{\mathcal{F}}}(A) = \sup_{B \in r_{\mathcal{F}}, B \subset A} \lambda_{\mathcal{F}}(B)$ are external and inner measure of a set A with respect to the measure $\lambda_{\mathcal{F}}$.

Note 3. Note that there are translation-invariant measures on the space E of another type which is countable additive but not σ -finite (see [17]). There are measures on infinite dimensional topological vector spaces which are translation-invariant with respect to only some subspace of acceptable vectors (see [14]).

2.1 Quadratically integrable functions

Now we define space of quadratically integrable functions with respect to $\lambda_{\mathcal{E}}$. Since we will use it very often, we define it concisely $\mathcal{H}_{\mathcal{E}} = L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$.

Let $\mathcal{S}(E, \mathcal{R}_{\mathcal{E}}, \mathbb{C})$ be the linear space hull over field \mathbb{C} of indicator functions of the sets from the ring $\mathcal{R}_{\mathcal{E}}$. Let $\beta_{\mathcal{E}}$ be the sesquilinear form on the space $\mathcal{S}(E, \mathcal{R}_{\mathcal{E}}, \mathbb{C})$ which is defined by the following conditions: $\beta_{\mathcal{E}}(\chi_A, \chi_B) = \lambda_{\mathcal{E}}(A \cap B)$ for any sets $A, B \in \mathcal{K}_{\mathcal{E}}$; for any functions $u, v \in \mathcal{S}(E, \mathcal{R}_{\mathcal{E}}, \mathbb{C})$ such that $u = \sum_{j=1}^n c_j \chi_{A_j}$, $v = \sum_{k=1}^m b_k \chi_{B_k}$ the value $\beta_{\mathcal{E}}(u, v)$ is given by the equality

$$\beta_{\mathcal{E}}(u, v) \equiv (u, v)_{\mathcal{H}(E)} = \left(\sum_{j=1}^n c_j \chi_{A_j}, \sum_{k=1}^m b_k \chi_{B_k} \right) = \sum_{k=1}^m \sum_{j=1}^n \bar{b}_k c_j (\chi_{A_j}, \chi_{B_k}). \quad (1)$$

This sesquilinear form on the space $\mathcal{S}(E, \mathcal{R}_{\mathcal{E}}, \mathbb{C})$ is Hermitian and nonnegative.

The function $u \in \mathcal{S}(E, \mathcal{R}_{\mathcal{E}}, \mathbb{C})$ is called equivalent to the function $v \in \mathcal{S}(E, \mathcal{R}_{\mathcal{E}}, \mathbb{C})$ iff $\beta_{\mathcal{E}}(u - v, u - v) = 0$. The linear space $\mathcal{S}_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ of the equivalence classes of functions of the space $\mathcal{S}(E, \mathcal{R}_{\mathcal{E}}, \mathbb{C})$ endowing with the sesquilinear form $\beta_{\mathcal{E}}$ is the pre-Hilbert space. The Hilbert space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C}) \equiv \mathcal{H}_{\mathcal{E}}$ is defined as the completion of the space $\mathcal{S}_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$.

Thus for any ONB \mathcal{E} in the space E there are the ring of subsets $\mathcal{R}_{\mathcal{E}}$ of $\lambda_{\mathcal{E}}$ -measurable sets, the measure $\lambda_{\mathcal{E}} : \mathcal{R}_{\mathcal{E}} \rightarrow [0, +\infty)$ and the Hilbert space $\mathcal{H}_{\mathcal{E}}$ of complex valued $\lambda_{\mathcal{E}}$ -measurable square integrable functions on the space E . Since the pre-Hilbert space $\mathcal{S}_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ of a simple functions is dense linear manifold in the space $\mathcal{H}_{\mathcal{E}}$ then the Hilbert space $\mathcal{H}_{\mathcal{E}}$ is the space of continuous linear functionals on the pre-Hilbert space $\mathcal{S}_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$.

Lemma 4. [10] *The space $\mathcal{H}_{\mathcal{E}}$ is not separable.*

2.2 The products of the spaces with finite additive measures

Let $\mathcal{E} = \{e_1, e_2, \dots\}$ be an ONB in the space E . Let E_1 be the Hilbert space with the ONB $\mathcal{E}_1 = \{e_2, e_3, \dots\}$. Then $E = \mathbb{R} \oplus E_1$, $E \ni x = (x_1, \xi) \in \mathbb{R} \oplus E_1$, where $x_1 \in \mathbb{R}$; $\xi \in E_1$.

Let \mathcal{J} be the isomorphism of the Hilbert space E onto the Hilbert space E_1 such that $\mathcal{E}_1 = \mathcal{J}(\mathcal{E})$. Let $\lambda_{\mathcal{E}}$ be a complete translation invariant measure on the space E such that the measure $\lambda_{\mathcal{E}}$ is defined on the ring $\mathcal{R}_{\mathcal{E}}$ by the theorem 1. Let $\mathcal{R}_{\mathcal{E}_1} = \mathcal{J}(\mathcal{R}_{\mathcal{E}})$ and $\lambda_{\mathcal{E}_1}$ is the measure on the space $(E_1, \mathcal{R}_{\mathcal{E}_1})$ such that $\lambda_{\mathcal{E}_1}(A) = \lambda_{\mathcal{E}}(\mathcal{J}^{-1}(A)) \forall A \in \mathcal{R}_{\mathcal{E}_1}$.

Let $\mathbf{I}_{\mathbb{R}}$ be the Jordan measure on the real line \mathbb{R} . Let $r(\mathbb{R})$ be a ring of measurable by Jordan subsets of real line \mathbb{R} . Remind that $K_{\mathcal{E}}$ and $K_{\mathcal{E}_1}$ are the collections of measurable rectangles in the spaces E and E_1 whose edges are collinear to the vectors of ONB \mathcal{E} and \mathcal{E}_1 respectively; $r_{\mathcal{E}}$ and $r_{\mathcal{E}_1}$ are the minimal rings containing the collections of sets $K_{\mathcal{E}}$ and $K_{\mathcal{E}_1}$ respectively.

We will use the following notations $\Pi = \Pi' \times \Pi'' \subset \mathbb{R} \times E_1$ where Π' the finite segment of real line and Π, Π'' are the measurable rectangles in the spaces E, E_1 respectively.

Lemma 5. ([2], lemma 3.3) *The inner measure of the set $X \subset E$ is defined by the equality*

$$\underline{\lambda}_{\mathcal{E}}(X) = \sup_{\bigcup_{k=1}^n Q_k \subset X, Q_k \in K_{\mathcal{E}}} \lambda_{\mathcal{E}} \left(\bigcup_{k=1}^n Q_k \right)$$

where supremum is defined over the set of finite union of measurable rectangles but not on the whole ring $r_{\mathcal{E}}$.

Lemma 6. ([2], lemma 3.4). *Let $A = g \times \Pi$, where $\Pi \in \mathcal{K}_{\mathcal{E}_1}$ and $\lambda_{\mathcal{E}_1}(\Pi) \neq 0$. Then $A \in \mathcal{R}_{\mathcal{E}}$ iff $g \in r(\mathbb{R})$. In this case the following equality $\lambda_{\mathcal{E}}(A) = \mathbf{I}_{\mathbb{R}}(g)\lambda_{\mathcal{E}_1}(\Pi)$ holds.*

The collection $\mathcal{K}_{\mathcal{E}}$ of measurable rectangles is the part of the following collection of sets $\{A_0 \times A_1, A_0 \in r(\mathbb{R}), A_1 \in \mathcal{R}_{\mathcal{E}_1}\}$; the last collection of sets is the part of the ring $\mathcal{R}_{\mathcal{E}}$. Since the ring $r_{\mathcal{E}}$ is the minimal ring containing the collection of sets $\mathcal{K}_{\mathcal{E}}$ and the ring $\mathcal{R}_{\mathcal{E}}$ is the completion of the ring $r_{\mathcal{E}}$ by the measure $\lambda_{\mathcal{E}}$, then the ring $\mathcal{R}_{\mathcal{E}}$ is the completion by measure $\lambda_{\mathcal{E}}$ of the minimal ring, containing the collection of the sets $\{A_0 \times A_1, A_0 \in r(\mathbb{R}), A_1 \in \mathcal{R}_{\mathcal{E}_1}\}$. Hence the following statement holds.

Lemma 7. *The space with finite additive measure $(E, r_{\mathcal{E}}, \lambda_{\mathcal{E}})$ is the product of the spaces with finite additive measures $(\mathbb{R}, r(\mathbb{R}), \mathbf{I}_{\mathbb{R}})$ and $(E_1, r_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1})$*

Proof. According to the lemma 1 [5] (page 222) the space with finite additive measure $(E, r_{\mathcal{E}}, \lambda_{\mathcal{E}})$ is the product of the spaces with the finite additive measures $(\mathbb{R}, r(\mathbb{R}), \mathbf{I}_{\mathbb{R}})$ and $(E_1, r_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1})$. In fact, since the rings $\mathcal{R}_{\mathcal{E}}$ and $\mathcal{R}_{\mathcal{E}_1}$ is obtained by using of completions by measures $\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{E}_1}$ procedure from the rings $r_{\mathcal{E}}$ and $r_{\mathcal{E}_1}$ respectively then the measure $\lambda_{\mathcal{E}}$ be a unique finite additive measure which is defined on the ring $\mathcal{R}_{\mathcal{E}}$ and satisfies the conditions $\lambda_{\mathcal{E}}(A_0 \times A_1) = \mathbf{I}_{\mathbb{R}}(A_0)\lambda_{\mathcal{E}_1}(A_1) \forall A_0 \in r(\mathbb{R}), A_1 \in \mathcal{R}_{\mathcal{E}_1}$.

Definition 1. A tensor product of the finite additive measures $\mu = \mu_1 \otimes \mu_2$ on the space $X = X_1 \times X_2$ is the measure μ on the space X which the completion of the measure $\mu_1 \times \mu_2$. Here $\mu_1 \times \mu_2$ is the measure which satisfies following two conditions:

- 1) it is defined on the minimal ring containing the collection of sets $\{A_1 \times A_2, A_1 \in R_1, A_2 \in R_2\}$,
- 2) it satisfies the equality $\mu_1 \times \mu_2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \forall A_1 \in R_1, A_2 \in R_2$.

Lemma 8. *The following equality $\lambda_{\mathcal{E}} = \mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}$ holds in the sense of definition 1.*

Proof. In fact, the procedures of definition of the measure $\lambda_{\mathcal{E}}$ and the measure $\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}$ have the following common constructions.

The definition the measure $\lambda_{\mathcal{E}}$ consists of three parts:

1. The function of a set is defined firstly on the collection of measurable rectangles $\mathcal{K}_{\mathcal{E}}$;
2. the function $\mathcal{K}_{\mathcal{E}} \rightarrow \mathbb{R}$ is extended onto the measure λ on the minimal ring $r_{\mathcal{E}}$ containing the collection of sets $\mathcal{K}_{\mathcal{E}}$;
3. the measure $\lambda : r_{\mathcal{E}} \rightarrow \mathbb{R}$ is extended onto the ring $\mathcal{R}_{\mathcal{E}}$ which is completion of the ring $r_{\mathcal{E}}$ by measure λ .

In the case of the measure $\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}$ the collection of a sets $\{A_1 \times A_2 \mid A_1 \in r(\mathbb{R}), A_2 \in \mathcal{K}_{\mathcal{E}_1}\}$ is used instead of the collection $\mathcal{K}_{\mathcal{E}}$ on the first step. According to definition the equality $\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}(I_1 \times \Pi_1) = \lambda_{\mathcal{E}}(I_1 \times \Pi_1)$ holds for any measurable rectangle $I_1 \times \Pi_1$ where $I_1 \in r(\mathbb{R})$, $\Pi_1 \in \mathcal{K}_{\mathcal{E}_1}$. Therefore the inequalities $\lambda_{\mathcal{E}}(A) \leq \mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}(A) \leq \overline{\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}}(A) \leq \overline{\lambda_{\mathcal{E}}}(A)$ hold for any set $A \subset E$. Hence if a set $A \subset E$ is measurable with respect to the measure $\lambda_{\mathcal{E}}$ then it is measurable with respect measure $\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}$ and the extensions of the measures $\lambda_{\mathcal{E}}$ and $\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}$ coincides on the ring $\mathcal{R}_{\mathcal{E}}$.

On the other hand any set of type $\{A_1 \times A_2 \mid A_1 \in r(\mathbb{R}), A_2 \in \mathcal{K}_{\mathcal{E}_1}\}$ belongs to the ring $\mathcal{R}_{\mathcal{E}}$ and in this case the equality $\lambda_{\mathcal{E}}(A_1 \times A_2) = (\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1})(A_1 \times A_2)$ holds. Since the measure $\lambda_{\mathcal{E}} : \mathcal{R}_{\mathcal{E}} \rightarrow \mathbb{R}$ is complete then the continuation of the function of a set $(\mathbf{I}_{\mathbb{R}} \otimes \lambda_{\mathcal{E}_1}) : \{A_1 \times A_2 \mid A_1 \in r(\mathbb{R}), A_2 \in \mathcal{K}_{\mathcal{E}_1}\} \rightarrow \mathbb{R}$ by means of continuation on the minimal ring (steps 2)) and completion (step 3)) coincides with the measure $\lambda_{\mathcal{E}}$.

Lemma 9. *The linear space $\text{span}(\{\chi_{\Pi}, \Pi \in \mathcal{K}_{\mathcal{E}}\})$ is dense in the Hilbert space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$.*

Proof. In fact, according to definition the Hilbert space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ is the closure of the linear space $\text{span}(\{\chi_A, A \in \mathcal{R}_{\mathcal{E}}\})$ endowed with the norm of the space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$. Since the ring $\mathcal{R}_{\mathcal{E}}$ is the completion of the ring $r_{\mathcal{E}}$ by the measure $\lambda_{\mathcal{E}}$ then the following equality $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C}) = \overline{\text{span}(\{\chi_A, A \in r_{\mathcal{E}}\})}$ holds. Note that any set $A \in r_{\mathcal{E}}$ is the finite union of disjoint sets $A = \bigcup_{k=1}^N B_k$ where B_k is the complement of a measurable rectangle to finite union of measurable rectangles: $\forall k \in \overline{1, N} \ B_k = \Pi_{k,0} \setminus \bigcup_{j=1}^{m_k} \Pi_{k,j}$, $\Pi_{k,i} \in \mathcal{K}_{\mathcal{E}} \ \forall i \in \overline{0, m_k}$. So, $\chi_A \in \text{span}(\{\chi_{\Pi}, \Pi \in \mathcal{K}_{\mathcal{E}}\})$ for any $A \in r_{\mathcal{E}}$. Consequently, $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C}) = \overline{\text{span}(\{\chi_{\Pi}, \Pi \in \mathcal{K}_{\mathcal{E}}\})}$.

Theorem 10. *Morphism \mathcal{I} mapping element of $L_2(\mathbb{R}) \otimes L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$, which is limit of fundamental sequence $f_k \otimes v_k$ into limit of sequence $f_k \otimes v_k$ in space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ provides us with canonical isomorphism between these two space.*

Proof. Space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ according to lemma 9 is a completion of $\text{span}(\{\chi_A, A \in \mathcal{K}_{\mathcal{E}}\})$ with respect to norm $\|\cdot\|_{L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})}$, defined by sesquilinear form $\beta_{\mathcal{E}}$, see (1). So, $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ is completion of the space $\text{span}(\{\chi_{A_0 \times A_1}, A_0 \in \mathcal{K}_{\mathbb{R}}, A_1 \in \mathcal{K}_{\mathcal{E}_1}\})$ with respect to norm $\|\cdot\|_{L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})}$, defined by sesquilinear form $\beta_{\mathcal{E}}$. (Here we define by $\mathcal{K}_{\mathbb{R}}$ a set of all bounded intervals of \mathbb{R}).

Space $L_2(\mathbb{R}) \otimes L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$ is completion of linear span \mathcal{L} of elements $f \otimes v$, where $f \in L_2(\mathbb{R})$, $v \in L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$ with respect to norm $\|\cdot\|_{\otimes}$, defined by sesquilinear form β_{\otimes} on \mathcal{L} with restriction $\beta_{\otimes}(f_1 \otimes v_1, f_2 \otimes v_2) = (f_1, f_2)_{L_2(\mathbb{R})} (v_1, v_2)_{L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})}$. Note that in space $L_2(\mathbb{R})$ linear span \mathcal{L}_0 of set of characteristic functions from $\mathcal{K}_{\mathbb{R}}$ is dense linear submanifold, and in space $L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$ according to lemma 9 linear span \mathcal{L}_1 of set of characteristic functions of set from $\mathcal{K}_{\mathcal{E}_1}$ is also a dense linear submanifold. That's why space $L_2(\mathbb{R}) \otimes L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$ is exactly a completion of linear span $\text{span}\{\chi_{A_0} \otimes \chi_{A_1}, A_0 \in \mathcal{K}_{\mathbb{R}}, A_1 \in \mathcal{K}_{\mathcal{E}_1}\}$ with respect to norm $\|\cdot\|_{\otimes}$.

Since for any interval $\Delta \in \mathcal{K}_{\mathbb{R}}$ and any measurable rectangle $\Pi' \in \mathcal{K}_{\mathcal{E}_1}$ holds $\|\chi_{\Delta} \otimes \chi_{\Pi'}\|_{\otimes} = \|\chi_{\Pi}\|_{L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})}$, where $\Pi = \Delta \times \Pi'$, then for any sets $A_0 \in \mathcal{K}_{\mathbb{R}}, A_1 \in \mathcal{K}_{\mathcal{E}_1}$ holds

$$\|\chi_{A_0} \otimes \chi_{A_1}\|_{\otimes} = \|\chi_{A_0 \times A_1}\|_{L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})}. \quad (2)$$

Since linear span $\text{span}(\{\chi_{A_0 \times A_1}, A_0 \in \mathcal{K}_{\mathbb{R}}, A_1 \in \mathcal{K}_{\mathcal{E}_1}\})$ is dense in space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$, and linear span $\text{span}(\{\chi_{A_0} \otimes \chi_{A_1}, A_0 \in \mathcal{K}_{\mathbb{R}}, A_1 \in \mathcal{K}_{\mathcal{E}_1}\})$ is dense in space $L_2(\mathbb{R}) \otimes L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$, then (see equation 2) spaces $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ and $L_2(\mathbb{R}) \otimes L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$ are isometrically isomorphic and $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C}) = \mathcal{I}(L_2(\mathbb{R}) \otimes L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C}))$.

2.3 Partial Fourier transforms

. Fourier transform of the the space $L_2(\mathbb{R})$ is unitary mapping of the space $L_2(\mathbb{R})$ into itself. Therefore the partial Fourier transform \mathcal{F}_1 with respect to the first coordinate is defined on the space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C}) = L_2(\mathbb{R}) \otimes L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$. The partial Fourier transform \mathcal{F}_1 is defined on the linear hull of the elements of type $u_1(x_1)u_2(\xi)$, $u_1 \in L_2(\mathbb{R})$, $u_2 \in L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$ by the equality

$$\mathcal{F}_1 \left(\sum_{k=1}^n u_{1,k} u_{2,k} \right) = \sum_{k=1}^n \hat{u}_{1,k} u_2,$$

where $\hat{u}_{1,k} \in L_2(\mathbb{R})$ is Fourier transform of the function $u_{1,k} \in L_2(\mathbb{R})$. Since the linear hull of the elements of type $u_1(x_1)u_2(\xi)$, $u_1 \in L_2(\mathbb{R})$, $u_2 \in L_2(E_1, \mathcal{R}_{\mathcal{E}_1}, \lambda_{\mathcal{E}_1}, \mathbb{C})$ is dense in the space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ then the partial Fourier transform \mathcal{F}_1 has the unique continuation up to the unitary transform of the space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ into itself.

Analogously, partial Fourier transform \mathcal{F}_n with respect to first n coordinates is defined on the space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$. According to the properties of Fourier transform of the space $L_2(\mathbb{R}^n)$ with some $n \in \mathbb{N}$ the following statement holds: partial Fourier transform of the space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ with respect to first n coordinates is unitary mapping of the space $L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ into itself for any $n \in \mathbb{N}$.

Partial Fourier transform will be useful in the studying of the operators of multiplication on coordinate and momentum operator with respect to direction of a vector e_j of the basis \mathcal{E} . It also be used further in the studying of generators of diffusion semigroups and its fraction powers.

3 Sobolev spaces and spaces of smooth functions

3.1 Averaging of random shifts and space of smooth functions

Let $\mathbf{D} \in B(E)$ be a nonnegative trace class operator with the orthonormal basis \mathcal{E} of eigenvectors. Any operator \mathbf{D} of the above class defines the centered countable additive Gaussian measure $\nu_{\mathbf{D}}$ on the space E such that the measure $\nu_{\mathbf{D}}$ has the covariance operator \mathbf{D} and zero mean value.

Shift operator on the vector $h \in E$ is defined on the space $\mathcal{H}_{\mathcal{E}} = L_2(E, \mathcal{R}_{\mathcal{E}}, \lambda_{\mathcal{E}}, \mathbb{C})$ by the equality

$$\mathbf{S}_h u(x) = u(x - h).$$

It is obvious that for any $h \in H$ operator \mathbf{S}_h belongs to the Banach space $B(\mathcal{H}_{\mathcal{E}})$ of bounded linear operators in the space $\mathcal{H}_{\mathcal{E}}$ endowed with the operator norm; moreover \mathbf{S}_h is the unitary operator in the space $\mathcal{H}_{\mathcal{E}}$. Let h be a random vector of the space E whose distribution is given by the measure ν . Then the mean value $\mathbf{U} \in B(\mathcal{H}_{\mathcal{E}})$ of random shift operator \mathbf{S}_h is given by the Pettis integral

$$\int_E \mathbf{S}_h d\nu(h) = \mathbf{U} \quad \Leftrightarrow \quad (\mathbf{U}f, g) = \int_E (\mathbf{S}_h f, g) d\nu(h) \quad \forall f, g \in \mathcal{H}_{\mathcal{E}}.$$

According to the paper [12] (see also [15]) the following statement holds.

Theorem 11. *Let $\mathbf{D} \in B(E)$ be a nonnegative trace class operator with the orthonormal basis \mathcal{E} of eigenvectors. Then one-parametric family of operators*

$$\mathbf{U}_t = \int_E \mathbf{S}_h d\nu_{t\mathbf{D}}(h), \quad t \geq 0,$$

is a one-parametric semigroup of self-adjoint contractions in the space $\mathcal{H}_{\mathcal{E}}$. The semigroup $\mathbf{U}_t, t \geq 0$, is strong continuous in the space $\mathcal{H}_{\mathcal{E}}$ if and only if $\mathbf{D}^{\frac{1}{2}}$ is trace class operator.

Definition 2. A function $u'_j \in \mathcal{H}_{\mathcal{E}}$ is called the derivative of the function $u \in \mathcal{H}_{\mathcal{E}}$ in the direction of a unite vector e_j if the following equality holds $\lim_{s \rightarrow 0} \|\frac{1}{s}(\mathbf{S}_{se_j}u - u) - u'_j\|_{\mathcal{H}_{\mathcal{E}}} = 0$.

Lemma 12. [see [11], lemmas 7.1, 7.2]. Let $\{e_j\} \equiv \mathcal{E}$ be the orthonormal basis of eigenvectors of positive trace class operator \mathbf{D} . If $\nu_{\mathbf{D}}$ be a probability Gaussian measure on the space E with covariance operator \mathbf{D} and $\mathcal{U}_{\mathbf{D}}(t) = \int_E \mathbf{S}_{\sqrt{th}} d\nu_{\mathbf{D}}(h), t \geq 0, u \in \mathcal{H}_{\mathcal{E}}$ then for any $l \in \mathbb{N}$ there is the number $c_l > 0$ such that for any $u \in \mathcal{H}_{\mathcal{E}}, j \in \mathbb{N}, t > 0$ there is the derivative of the power l $\partial_j^l \mathcal{U}_{\mathbf{D}}(t)u \in \mathcal{H}_{\mathcal{E}}$ and the following estimates take place

$$\|\partial_j^l \mathcal{U}_{\mathbf{D}}(t)u\|_{\mathcal{H}_{\mathcal{E}}} \leq \frac{c_l}{(\sqrt{td_j})^l} \|u\|_{\mathcal{H}_{\mathcal{E}}}. \quad (3)$$

Let \mathbf{D} be a positive trace class operator in the space E . Let $C_{\mathbf{D}}^{\infty}(E)$ be a linear hull of the following system of elements $\{\mathcal{U}_{\mathbf{D}}(t)u, t > 0, u \in \mathcal{H}_{\mathcal{E}}\}$. The linear manifold $C_{\mathbf{D}}^{\infty}(E)$ is called the space of smooth functions according to lemma 12.

3.2 Sobolev spaces and embedding theorem

For any $a > 0$ the symbol $W_{2,\mathbf{D}^a}^1(E)$ notes the linear space of elements u of the space $\mathcal{H}_{\mathcal{E}}$ such that the following two condition hold

1) for any $j \in \mathbb{N}$ there is the derivative $u_j \equiv \frac{\partial}{\partial x_j} u \in \mathcal{H}_{\mathcal{E}}$ with respect to the direction of eigenvector e_j of operator \mathbf{D} ;

$$2) \quad \sum_{j=1}^n d_j^a \|u_j\|_{\mathcal{H}_{\mathcal{E}}}^2 < +\infty. \quad (4)$$

The space $W_{2,\mathbf{D}^a}^1(E)$ endowed with the norm $\|u\|_{W_{2,\mathbf{D}^a}^1(E)} = (\|u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \sum_{j=1}^n d_j^a \|u_j\|_{\mathcal{H}_{\mathcal{E}}}^2)^{\frac{1}{2}}$ is the Hilbert space which is continuously embedded into the space $\mathcal{H}_{\mathcal{E}}$ (see [3, 11]).

For any numbers $a > 0$ and any $l \in \mathbb{N}$ the symbol $W_{2,\mathbf{D}^a}^l(E)$ notes the linear space of elements u of the space $\mathcal{H}_{\mathcal{E}}$ such that the following two conditions hold:

1) for any $j \in \mathbb{N}$ there is the l -order derivative $\frac{\partial^l}{\partial e_j^l} u \in \mathcal{H}_{\mathcal{E}}$;

$$2) \quad \sum_{j=1}^n d_j^a \left\| \frac{\partial^l}{\partial e_j^l} u \right\|_{\mathcal{H}_{\mathcal{E}}}^2 < +\infty. \quad (5)$$

Analogously, the space $W_{2,\mathbf{D}^a}^l(E)$ endowed with the norm

$$\|u\|_{W_{2,\mathbf{D}^a}^l(E)} = (\|u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \sum_{j=1}^n d_j^a \left\| \frac{\partial^l}{\partial e_j^l} u \right\|_{\mathcal{H}_{\mathcal{E}}}^2)^{\frac{1}{2}}$$

is the Hilbert space which is continuously embedded into the space $\mathcal{H}_{\mathcal{E}}$ (see [3, 11]).

Theorem 13. [see [11], lemmas 7.1, 7.2]. Let $u \in \mathcal{H}_{\mathcal{E}}$. Let \mathbf{D} be a positive trace class operator in the space E such that $\mathbf{D}^{\frac{1}{2}}$ is the trace class operator. Then for any $t > 0$ the inclusion $\mathcal{U}_{\mathbf{D}^{\frac{1}{2}}}(t)u \in W_{2,\mathbf{D}}^1(E)$ holds.

Theorem 14. Let \mathbf{D} be a positive trace class operator in Hilbert space E such that \mathbf{D}^{γ} is trace class operator with some $\gamma > 0$. Let $l \in \mathbb{N}$. If $b \geq l\alpha + \gamma$ with some $\alpha \in [\gamma, +\infty)$ then $C_{\mathbf{D}^a}^{\infty}(E) \subset W_{2,\mathbf{D}^b}^l(E)$.

If, moreover, $\alpha \geq 2\gamma$ then the linear manifold $C_{\mathbf{D}^a}^{\infty}(E)$ is dense in the space $W_{2,\mathbf{D}^b}^l(E)$.

The proof of this theorem is published in the work [3].

3.3 The traces of a functions on the codimension 1 hyperplane

Let \mathcal{E} be an orthonormal basis in the space E . Let $e_1 \in \mathcal{E}$ and $\mathcal{E}_1 = \mathcal{E} \setminus \{e_1\}$. Let $E_1 = (\text{span}(e_1))^\perp$ and $\mathcal{H}_{\mathcal{E}_1} = L_2(E_1, \mathcal{R}_1, \lambda_{\mathcal{E}_1}, \mathbb{C})$. Let $\mathcal{R}(\mathbb{R})$ be a ring of Lebesgue integrable subsets of the space \mathbb{R} with finite Lebesgue measure. Let λ_L be the Lebesgue measure on the space \mathbb{R} .

Then according to the theorem 10 $\mathcal{H}_{\mathcal{E}} = L_2(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ where $L_2(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ is the space of integrable in Bohnner sense with respect to Lebesgue measure λ_L mappings $\mathbb{R} \rightarrow \mathcal{H}_{\mathcal{E}_1}$.

We define a linear space $W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1}) = \{u \in \mathcal{H}_{\mathcal{E}} : \frac{\partial}{\partial e_1} u \in \mathcal{H}_{\mathcal{E}}\}$ endowed with the Sobolev norm

$$\|u\|_{W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})} = (\|u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\frac{\partial}{\partial e_1} u\|_{\mathcal{H}_{\mathcal{E}}}^2)^{\frac{1}{2}}. \quad (6)$$

According to the definition of partial Fourier transform

$$\|u\|_{W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})}^2 = \int_{\mathbb{R}} (1 + \xi^2) \|\hat{u}_1(\xi)\|_{\mathcal{H}_{\mathcal{E}_1}}^2 d\xi$$

for any $u \in W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ where $\hat{u} = \mathcal{F}_1(u)$. Hence the space $W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ endowed with the norm (6) is the Hilbert space.

Theorem 15. *If $u \in W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ then the equivalence class u contains the continuous function $\tilde{u} \in C(\mathbb{R}, \mathcal{H}_{\mathcal{E}_1})$. Moreover, there is the constant $C > 0$ such that*

$$\|\tilde{u}\|_{C(\mathbb{R}, \mathcal{H}_{\mathcal{E}_1})} \leq C \|u\|_{W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})}. \quad (7)$$

Proof. In the case of separable space $\mathcal{H}_{\mathcal{E}_1}$ the proof of this theorem is given in the monograph [6]. In the case under consideration the space $\mathcal{H}_{\mathcal{E}_1}$ is not separable. But according to the condition $u \in W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ there is the separable subspace of the space $\mathcal{H}_{\mathcal{E}_1}$ containing the values of the mapping $u : \mathbb{R} \rightarrow \mathcal{H}_{\mathcal{E}_1}$. Therefore the proof of the theorem 3.1 by [6] can be applied to the obtaining of the statement of theorem 15.

If $u \in W_{2,D}^1(E)$ then the function u can be considered as the function of the space $W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$. Therefore the function $u \in W_{2,D}^1(E)$ can be considered as the continuous mapping $\tilde{u} : \mathbb{R} \rightarrow \mathcal{H}_{\mathcal{E}_1}$ according to theorem 15. Hence we can use the following definition of the trace of function.

Definition 3. The trace of the function $u \in W_{2,D}^1(E)$ at the hyperplane $x_1 = t_0$, $t_0 \in \mathbb{R}$, is the value of function $\tilde{u} \in C(\mathbb{R}, \mathcal{H}_{\mathcal{E}_1})$ at the point t_0 .

Corollary 16. *If $u \in W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ then for any $t_0 \in \mathbb{R}$ the estimate $\|\tilde{u}(t_0)\|_{\mathcal{H}_{\mathcal{E}_1}} \leq C \|u\|_{W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})}$ holds.*

Corollary 17. *If $u \in W_2^1(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$ and $\frac{d}{ds} \mathbf{S}_{se_1} u = v \in L_2(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_1})$, then for any $t_1, t_2 \in \mathbb{R}$ the equality holds*

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} v(s) ds.$$

4 The analog of Gauss theorem for rectangle

Let \mathcal{E} be the ONB of positive trace-class covariation operator \mathbf{D} of Gaussian measure γ . Let $\Pi_{a,b} \in \mathcal{K}(\mathcal{E})$ be a measurable rectangle. For any $j \in \mathbb{N}$ the equality $E = \mathbb{R} \oplus E_j$ where $\mathbb{R} = \text{span}(e_j)$ and $E_j = (\text{span}(e_j))^\perp$. Let $\mathcal{E}_j = \{e_1, \dots, e_{j-1}, e_{j+1}, \dots\}$. Therefore the following equality $\Pi_{a,b} = [a_j, b_j] \times \hat{\Pi}_{\hat{a}_j, \hat{b}_j}$ holds where $\hat{\Pi}_{\hat{a}_j, \hat{b}_j} \in \mathcal{K}(\mathcal{E}_j)$ is the measurable rectangle in the space E_j .

Let $u \in W_{2,\mathbf{D}}^2(E)$. Then for any $j \in \mathbb{N}$ and any $a \in R$ there is the trace $u|_{x_j=a} \in \mathcal{H}_{\mathcal{E}_j}$ of the function u according to the theorem 3.4. Since for any $j \in \mathbb{N}$ the function $\partial_j u$ has the derivative $\partial_j^2 u \in \mathcal{H}_{\mathcal{E}}$ in the direction e_j , then according to the theorem 3.4 there is the trace $(\partial_j u)|_{x_j=a} \in \mathcal{H}_{\mathcal{E}_j}$.

Theorem 18. Let \mathbf{D} be a positive trace class operator in Hilbert space E . Let $u \in W_{2,\mathbf{D}}^2(E)$, let $\Pi_{a,b} \in \mathcal{K}(\mathcal{E})$ be a measurable rectangle. Then the equality

$$\int_{\Pi_{a,b}} \Delta_{\mathbf{D}} u \phi d\lambda_{\mathcal{E}} = - \int_{\Pi_{a,b}} (\nabla u, \mathbf{D} \nabla \phi)_E d\lambda_{\mathcal{E}} + \int_{\partial \Pi_{a,b}} (\mathbf{n}, \mathbf{D} \nabla u) \phi ds, \quad (8)$$

holds for any function $\phi \in W_{2,\mathbf{D}}^1(E)$. Here

$$\int_{\partial \Pi_{a,b}} (\mathbf{n}, \mathbf{D} \nabla u) \phi ds = \sum_{j=1}^{\infty} d_j \int_{\hat{\Pi}_{\hat{a}_j, \hat{b}_j}} [(\phi \partial_j u)|_{x_j=b_j} - (\phi \partial_j u)|_{x_j=a_j}] d\lambda_{\mathcal{E}_j}. \quad (9)$$

Proof. Since $\phi \in W_{2,\mathbf{D}}^1(E)$ then the condition $\phi|_{x_j=a} \in \mathcal{H}_{\mathcal{E}_j}$ holds for any $j \in \mathbb{N}$ and any $a \in R$ according to the theorem 15, moreover

$$\|\phi|_{x_j=a}\|_{\mathcal{H}_{\mathcal{E}_j}} \leq C(\|\phi\|_{L_2(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_j})}^2 + \|\partial_j \phi\|_{L_2(\mathbb{R}, \mathcal{R}(\mathbb{R}), \lambda_L, \mathcal{H}_{\mathcal{E}_j})}^2)^{\frac{1}{2}}$$

according to theorem 15 and corollary (16). Therefore

$$\begin{aligned} \left| \int_{\hat{\Pi}_{\hat{a}_j, \hat{b}_j}} [(\phi \partial_j u)|_{x_j=b} - (\phi \partial_j u)|_{x_j=a}] d\lambda_j \right| &= |(\phi|_{x=b}, \partial_j u|_{x=b})_{\mathcal{H}_{\mathcal{E}_j}} - (\phi|_{x=a}, \partial_j u|_{x=a})_{\mathcal{H}_{\mathcal{E}_j}}| \leq \\ &\leq 2\|\partial_j u\|_{C(\mathbb{R}, \mathcal{H}_{\mathcal{E}_j})} \|\phi\|_{C(\mathbb{R}, \mathcal{H}_{\mathcal{E}_j})} \end{aligned}$$

for any $j \in \mathbb{N}$. Hence the estimates

$$\begin{aligned} \left| \int_{\hat{\Pi}_{\hat{a}_j, \hat{b}_j}} [(\phi \partial_j u)|_{x_j=b} - (\phi \partial_j u)|_{x_j=a}] d\lambda_j \right| &\leq \\ &\leq 2C^2(\|\phi\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j \phi\|_{\mathcal{H}_{\mathcal{E}}}^2)^{\frac{1}{2}} (\|\partial_j u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j^2 u\|_{\mathcal{H}_{\mathcal{E}}}^2)^{\frac{1}{2}} \leq \\ &\leq C^2(\|\phi\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j \phi\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j^2 u\|_{\mathcal{H}_{\mathcal{E}}}^2) \end{aligned}$$

hold for any $j \in \mathbb{N}$. Let us note that

$$\sum_{j=1}^{\infty} d_j (\|\phi\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j \phi\|_{\mathcal{H}_{\mathcal{E}}}^2) = (\text{Tr} \mathbf{D} - 1) \|\phi\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\phi\|_{W_{2,\mathbf{D}}^1(E)}^2.$$

Since partial Fourier transform with respect to coordinate x_j is unitary operator in the space $\mathcal{H}_{\mathcal{E}}$ and according to the inequality $k^2 \leq 1 + k^4$, $k \in \mathbb{R}$, we obtain

$$\sum_{j=1}^{\infty} d_j \|\partial_j u\|_{\mathcal{H}_{\mathcal{E}}}^2 \leq \sum_{j=1}^{\infty} d_j (\|u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j^2 u\|_{\mathcal{H}_{\mathcal{E}}}^2)$$

Hence

$$\sum_{j=1}^{\infty} d_j (\|\partial_j u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \|\partial_j^2 u\|_{\mathcal{H}_{\mathcal{E}}}^2) \leq 2\|u\|_{W_{2,\mathbf{D}}^2(E)}^2 + (\text{Tr} \mathbf{D} - 2) \|u\|_{\mathcal{H}_{\mathcal{E}}}^2.$$

Therefore the series in the right hand side (9) converges.

Since

$$\partial_j(\phi d_j \partial_j u) = d_j \partial_j \phi \partial_j u + d_j \partial_j^2 u \phi,$$

then according to the corollary 17 the equality

$$\int_{\partial \Pi_{a,b}} (\mathbf{n}, \mathbf{e}_j d_j \partial_j u) \phi ds = \int_{\Pi_{a,b}} d_j \partial_j^2 u \phi d\lambda + \int_{\Pi_{a,b}} d_j \partial_j u \partial_j \phi d\lambda. \quad (10)$$

holds for any $j \in \mathbb{N}$. As it shown above, under the summation of the equalities (10) by $j \in \mathbb{N}$ the series of left hand side absolutely converges. The series of first components in right hand side absolutely converges by definition of the space $W_{2,\mathbf{D}}^2(E)$. Hence the series in right hand side of (10) absolutely converges and the equality (8) takes place.

5 Dirichlet problem for Poisson equation

Let us consider the unique rectangle $\Pi_{0,1} \in \mathcal{H}_\mathcal{E}$. Let symbol $L_2(\Pi_{0,1})$ notes the subspace of the space $\mathcal{H}_\mathcal{E}$ with the support in the set $\Pi_{0,1}$. Let us note that $f = \chi_{\Pi_{0,1}} f$ for any $f \in L_2(\Pi_{0,1})$.

Let symbol $\dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ notes the space of function $W_{2,\mathbf{D}}^1(E) \cap L_2(\Pi_{0,1})$ with the support in the set $\Pi_{0,1}$. It should be note that for any $u \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ the following statement holds:

$$u|_{\partial \Pi_{0,1}} = 0; \quad \partial_{x_k} u \in L_2(\Pi_{0,1}) \quad \forall k \in \mathbb{N}.$$

We also introduce the space $\dot{W}_{2,\mathbf{D}}^2(\Pi_{0,1})$ of functions $u \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ such that

$$\exists g_{jk} \in L_2(\Pi_{0,1}) : (\mathbf{D} \partial_k u, \partial_j \phi) = -(g_{jk}, \phi) \quad \forall \phi \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1}), \quad \forall j, k \in \mathbb{N};$$

$$\sum_{j=1}^{\infty} \|g_{jj}\|_{\mathcal{H}_\mathcal{E}}^2 < +\infty.$$

We pose the following problem. For a given function $f \in L_2(\Pi_{0,1})$ and a given number $a \geq 0$ we should find a function $u \in \dot{W}_{2,\mathbf{D}}^2(\Pi_{0,1})$ such that

$$\Delta_{\mathbf{D}} u = au + f, \quad (11)$$

$$u|_{\partial \Pi_{0,1}} = 0. \quad (12)$$

To investigate the above problem we apply the variation approach (see [7]). At first we study the space $\dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$.

Let us introduce the trapezoid-like function $\psi_\delta : \mathbb{R} \rightarrow \mathbb{R}$ which is given by the equalities $\psi_\delta(x) = 0, x \in (-\infty, 0] \cup [1, +\infty)$; $\psi_\delta(x) = 1, x \in [\delta, 1 - \delta]$; $\psi_\delta(x) = \frac{1}{\delta}, x \in (0, \delta)$; $\psi_\delta(x) = -\frac{1}{\delta}(x - 1), x \in (1 - \delta, 1)$ for any $\delta \in (0, \frac{1}{2})$. Then $\psi_\delta \in \dot{W}_2^1([0, 1])$, $\|\psi_\delta\|_{L_2([0, 1])} \in [1 - 2\delta, 1]$, $\|\partial_x \psi\|_{L_2([0, 1])} = \sqrt{2/\delta}$.

Let \mathbf{D} a the nonnegative trace-class operator such that $\mathbf{D}^{\frac{1}{2}}$ is trace-class operator. Let $\{e_j\}$ be the ONB of eigenvectors of the operator \mathbf{D} and $\{d_j\}$ be the corresponding sequence of eigenvalues. For any sequence $\{\delta_k\} : \mathbb{N} \rightarrow (0, \frac{1}{2})$ the function $\Psi_{\{\delta\}}(x) = \Pi_{j=1}^{\infty} \psi_{\delta_j}(x_j)$ is defined.

Lemma 19. *If $\{\delta_k\} : \mathbb{N} \rightarrow (0, \frac{1}{2})$ and $\sum_{j=1}^{\infty} \delta_j < +\infty$ then $\Psi_{\{\delta\}} \in \mathcal{H}_\mathcal{E}$ and the estimates $\Pi_{k=1}^{\infty} (1 - 2\delta_k) \leq \|\Psi_\delta\|_{L_2} \leq 1$ hold.*

Proof. The inclusion $\Psi_{\{\delta\}} \in \mathcal{H}_{\mathcal{E}}$ for the nonnegative function $\Psi_{\{\delta\}}$ is equivalent to the condition

$$\forall \varepsilon > 0 \exists g, G \in \mathcal{H}_{\mathcal{E}} : g \leq \Psi_{\{\delta\}} \leq G \quad \text{and} \quad \|G - g\|_{\mathcal{H}_{\mathcal{E}}} < \varepsilon.$$

Let us define

$$G_n(x) = \Pi_{k=1}^n \psi_{\delta_k}(x_k) \Pi_{k=n+1}^{\infty} \chi_{[0,1]}(x_k)$$

and $g_n(x) = \Pi_{k=1}^n \psi_{\delta_k}(x_k) \Pi_{k=n+1}^{\infty} \chi_{[\delta_k, 1-\delta_k]}(x_k)$ for some $n \in \mathbb{N}$. Then $g_n \leq \Psi_{\{\delta\}} \leq G_n$ and $\|G_n - g_n\|_{\mathcal{H}} < 1 - \exp(-\sum_{k=n+1}^{\infty} \ln(1-2\delta_k))$ for any $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} \delta_k = 0$ then $\lim_{n \rightarrow \infty} \|G_n - g_n\|_{\mathcal{H}} = 0$. Therefore $\Psi_{\{\delta\}} \in \mathcal{H}$. \square

Let us note that $0 \leq \chi_{\Pi_{\delta,1-\delta}} \leq \Psi_{\delta} \leq \chi_{\Pi_{0,1}}$. Hence $\lambda_{\mathcal{E}}(\Pi_{\delta,1-\delta}) \leq \|\Psi_{\delta}\|_{\mathcal{H}}^2 \leq \lambda_{\mathcal{E}}(\Pi_{0,1})$. Therefore the statements of the 19 are proved. \square

Note that $\lambda(\Pi_{\delta,1-\delta}) = \exp(\sum_{k=1}^{\infty} \ln(1-2\delta_k))$. Since $\delta_k \in (0, \frac{1}{2})$ for any $k \in \mathbb{N}$ then the series $\sum_{k=1}^{\infty} \ln(1-2\delta_k)$ converges if and only if the series $\sum_{k=1}^{\infty} \delta_k$ converges. In this case $\lambda(\Pi_{\delta,1-\delta}) > 0$. In the other case $\lambda(\Pi_{\delta,1-\delta}) = 0$.

Lemma 20. The condition $\sum_{k=1}^{\infty} \frac{d_k}{\delta_k} < +\infty$ is necessary and sufficient to the inclusion $\Psi_{\{\delta\}} \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$.

Proof. We should proof that $\partial_j \Psi_{\{\delta\}} \in \mathcal{H}_{\mathcal{E}}$ for any $j \in \mathbb{N}$ and the series

$$\sum_{j=1}^{\infty} d_j \|\partial_j \Psi_{\{\delta\}}\|_{\mathcal{H}_{\mathcal{E}}}^2$$

converges. Note that $\Psi_{\{\delta\}}(x) = \psi_{\delta_j}(x_j) \Psi_{\{\hat{\delta}\}}(\hat{x})$, $x \in E$, for any $j \in \mathbb{N}$ where $\hat{x} = \{x_1, \dots, x_{j-1}, x_{j+1}, \dots\}$ and $\hat{\delta} = \{\delta_1, \dots, \delta_{j-1}, \delta_{j+1}, \dots\}$. Therefore for any $j \in \mathbb{N}$ the following equality holds

$$\partial_j \Psi_{\{\delta\}}(x) = \partial_j \psi_{\delta_j}(x_j) \Psi_{\{\hat{\delta}\}}(\hat{x}), x \in E.$$

Hence $\partial_j \Psi_{\{\delta\}} \in \mathcal{H}_{\mathcal{E}}$ for any $j \in \mathbb{N}$ and

$$\|\partial_j \Psi_{\{\delta\}}\|_{\mathcal{H}_{\mathcal{E}}} = \|\partial_j \psi_{\delta_j}\|_{L_2(R)} \|\Psi_{\{\hat{\delta}\}}\|_{\mathcal{H}_{E_j}} \leq (2/\delta_j)^{\frac{1}{2}}.$$

Therefore the series $\sum_{j=1}^{\infty} d_j \|\partial_j \Psi_{\{\delta\}}\|_{\mathcal{H}}^2$ converges if and only if the series $\sum_{j=1}^{\infty} \frac{d_j}{\delta_j}$ converges. \square

Note 21. Let \mathbf{D} be a positive operator in the space E such that $\sqrt{\mathbf{D}}$ is trace class operator. Then there is the sequence $\{\delta_k\} : \mathbb{N} \rightarrow (0, \frac{1}{2})$ such that $\sum_{j=1}^{\infty} \delta_j < +\infty$ and the condition $\sum_{k=1}^{\infty} \frac{d_k}{\delta_k} < +\infty$ satisfies. For example, $\delta_k = \sqrt{d_k}$, $k \in \mathbb{N}$.

Lemma 22. Let $f \in W_{2,\mathbf{D}}^1(E)$. Let the sequence $\{\delta_k\} : \mathbb{N} \rightarrow (0, \frac{1}{2})$ satisfies the condition $\sum_{k=1}^{\infty} \frac{d_k}{\delta_k} < +\infty$. Then $\Psi_{\{\delta\}} f \in \dot{W}_2^1(\Pi_{0,1})$.

Proof. In fact, $\|\Psi_{\{\delta\}} f\|_{\mathcal{H}_{\mathcal{E}}} \leq \|f\|_{\mathcal{H}_{\mathcal{E}}}$ according to Cauchy inequality and lemma 1. Since $\partial_j(\Psi_{\{\delta\}} f) = \partial_j \Psi_{\{\delta\}} f + \Psi_{\{\delta\}} \partial_j f$ then

$$\|\partial_j(\Psi_{\{\delta\}} f)\|_{\mathcal{H}_{\mathcal{E}}}^2 \leq 2 \sup_{x \in E} |\Psi_{\{\delta\}}(x)| \|f\|_{W_{2,\mathbf{D}}^1}^2 + \frac{1}{\delta_j} \left(\int_0^{\delta_j} + 2 \int_{1-\delta_j}^1 \right) \int_{\hat{E}_j} |f(x)|^2 d\lambda_{\mathcal{E}_j}(\hat{x}_j) dx_j \leq$$

$$\leq 2\|f\|_{W_{2,\mathbf{D}}^1}^2 + \frac{2}{\delta_j}\|f\|_{\mathcal{H}_\varepsilon}^2.$$

Therefore

$$\sum_{j=1}^{\infty} d_j \|\partial_j(\Psi_{\{\delta\}}f)\|_{\mathcal{H}_\varepsilon}^2 \leq 2[\text{Tr}(\mathbf{D})\|f\|_{W_{2,\mathbf{D}}^1}^2 + \sum_{j=1}^{\infty} \frac{d_j}{\delta_j}\|f\|_{\mathcal{H}_\varepsilon}^2].$$

Thus we obtain that $\Psi_{\{\delta\}}f \in W_{2,\mathbf{D}}^1(E)$. Since $(\Psi_{\{\delta\}}f)|_{\partial\Pi_{0,1}} = \Psi_{\{\delta\}}|_{\partial\Pi_{0,1}}f|_{\partial\Pi_{0,1}}$ then $(\Psi_{\{\delta\}}f)|_{\partial\Pi_{0,1}} = 0$ and $\Psi_{\{\delta\}}f \in \dot{W}_{2,\mathbf{D}}^1(E)$.

Lemma 23. Let $f \in W_{2,\mathbf{D}}^1(E)$ and $\sum_{k=1}^{\infty} \frac{d_k}{\delta_k} < +\infty$. Then $\lim_{\varepsilon \rightarrow +0} \|f - \Psi_{\varepsilon\{\delta\}}f\|_{L_2(\Pi_{0,1})} = 0$.

Proof. In fact,

$$\|f(1 - \Psi_{\varepsilon\{\delta\}})\|_{L_2(\Pi_{0,1})}^2 = \int_{\Pi_{0,1}} |f(x)|^2 (1 - \Psi_{\varepsilon\{\delta\}})^2 d\lambda_\varepsilon \leq \int_{\Pi_{0,1} \setminus \Pi_{\varepsilon\{\delta\}, 1-\varepsilon\{\delta\}}} |f(x)|^2 d\lambda_\varepsilon.$$

Since $\{\delta\} \in l_1$ then $\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon(\Pi_{0,1} \setminus \Pi_{\varepsilon\{\delta\}, 1-\varepsilon\{\delta\}}) = \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon(\Pi_{0,1} \setminus \Pi_{0,1-2\varepsilon\{\delta\}}) = 0$ according to the theorem ? in [10]. Since $f \in \mathcal{H}$ then for any $\sigma > 0$ there is the simple function $g \in S_2(E, \mathcal{R}_\varepsilon, \lambda_\varepsilon, \mathbf{C})$ such that $\|f - g\|_{\mathcal{H}} < \sigma$. Since the function g has the finite number of values then $M = \sup_{x \in E} |g(x)| \in [0, +\infty)$. Therefore

$$\begin{aligned} & \int_{\Pi_{0,1} \setminus \Pi_{\varepsilon\{\delta\}, 1-\varepsilon\{\delta\}}} |f(x)|^2 d\lambda_\varepsilon \leq \\ & \|f - g\|_{\mathcal{H}_\varepsilon}^2 + \int_{\Pi_{0,1} \setminus \Pi_{\varepsilon\{\delta\}, 1-\varepsilon\{\delta\}}} |g(x)|^2 d\lambda_\varepsilon \leq \|f - g\|_{\mathcal{H}_\varepsilon}^2 + M^2 \lambda_\varepsilon(\Pi_{0,1} \setminus \Pi_{\varepsilon\{\delta\}, 1-\varepsilon\{\delta\}}). \end{aligned}$$

Hence for any $\sigma > 0$ there is a number $\varepsilon_0 > 0$ such that $\int_{\Pi_{0,1} \setminus \Pi_{\varepsilon\{\delta\}, 1-\varepsilon\{\delta\}}} |f(x)|^2 d\lambda_\varepsilon \leq 2\sigma$ for any $\varepsilon \in (0, \varepsilon_0)$.

Therefore $\lim_{\varepsilon \rightarrow +0} \|f - \Psi_{\varepsilon\{\delta\}}f\|_{L_2(\Pi_{0,1})} = 0$. \square

The consequence of the lemma 4 is the following statement.

Theorem 24. Let \mathbf{D} be a nonnegative operator in the space E such that $\sqrt{\mathbf{D}}$ is trace class operator. Then the set of functions

$$S_1 = \{\Psi_{\varepsilon\{\delta\}}f, f \in W_{2,\mathbf{D}}^1(E), \delta \in (0, \frac{1}{2}), \{\delta\} : \sum_{k=1}^{\infty} \frac{d_k}{\delta_k} < +\infty\}$$

is dense in the space $L_2(\Pi_{0,1})$.

Let $a \geq 0$ and $f \in L_2(\Pi_{0,1})$. Let the functional $J_f : \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1}) \rightarrow \mathbb{R}$ be defined by the equality

$$J_{a,f}(u) = \frac{1}{2} \int_{\Pi_{0,1}} [(\nabla \bar{u}, \mathbf{D} \nabla u)_E + a|u|^2 + \bar{u}f + u\bar{f}] d\lambda, \quad u \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1}). \quad (13)$$

Theorem 25. Let $a \geq 0$ and $f \in L_2(\Pi_{0,1})$. Let $u \in \dot{W}_{2,\mathbf{D}}^2(\Pi_{0,1})$ be a stationary point of the functional $J_{a,f}$. Then u is the solution of Dirichlet problem (11), (12).

Proof. Let $u \in \dot{W}_{2,\mathbf{D}}^2(\Pi_{0,1})$ be a stationary point of the functional (13). Then the function $J_{a,f}(u + t\phi)$, $t \in \mathbb{R}$ satisfies the equality $\frac{d}{dt}J_{a,f}(u + t\phi) = 0$ for any $\phi \in S_1$. Therefore

$$\int_{\Pi_{0,1}} [(\nabla \bar{\phi}, \mathbf{D}\nabla u)_E + a\bar{\phi}u + \bar{\phi}f]d\lambda + \int_{\Pi_{0,1}} [(\nabla \bar{u}, \mathbf{D}\nabla \phi)_E + a\bar{u}\phi + \bar{f}\phi]d\lambda = 0$$

for any $\phi \in S_1$. Hence $\int_{\Pi_{0,1}} [\bar{\phi}(\Delta_{\mathbf{D}}u - f - au)]d\lambda_{\mathcal{E}} = 0$ for any $\phi \in S_1$ according to the theorem 24. Since the set

S_1 is dense in the space \mathcal{H} then the function u satisfies Poisson equation (11). Since $u \in \dot{W}_{2,\mathbf{D}}^2(\Pi_{0,1})$ then the equality (12) is satisfied. \square

Theorem 26. Let $u \in \dot{W}_{2,\mathbf{D}}^2(\Pi_{0,1})$ be the solution of Dirichlet problem (11), (12). Then it is the critical point of the functional (13).

Proof. Let $\phi \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$. Then the function $J_{a,f}(u + t\phi)$, $t \in \mathbb{R}$, has the derivative

$$\frac{dJ_{a,f}(u + t\phi)}{dt}\bigg|_{t=0} = \int_{\Pi_{0,1}} [(\nabla \bar{\phi}, \mathbf{D}\nabla u)_E + a\bar{\phi}u + \bar{\phi}f]d\lambda + \int_{\Pi_{0,1}} [(\nabla \bar{u}, \mathbf{D}\nabla \phi)_E + a\bar{u}\phi + \bar{f}\phi]d\lambda. \quad (14)$$

Then according to the theorem 18

$$\frac{d}{dt}J_{a,f}(u + t\phi)\bigg|_{t=0} = - \int_{\Pi_{0,1}} [\bar{\phi}(\Delta_{\mathbf{D}}u - au - f)]d\lambda - \int_{\Pi_{0,1}} [\phi(\Delta_{\mathbf{D}}\bar{u} - a\bar{u} - \bar{f})]d\lambda.$$

Since $u \in \dot{W}_{2,\mathbf{D}}^2(\Pi_{0,1})$ be the solution of Dirichlet problem (11), (12) then the equality $\frac{d}{dt}J_{a,f}(u + t\phi)\big|_{t=0} = 0$ holds for any $\phi \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ and the function u is the stationary point of the functional $J_{a,f}$. \square

The inequality $\|u\|_{\mathcal{H}} \leq \|u\|_{\dot{W}_2^1}$ holds according to the definition of the space \dot{W}_2^1 . Let $f \in H$. Then for any $u \in \dot{W}_2^1$ the inequality $|(f, u)| \leq c\|u\|_{\dot{W}_2^1}$ take place where $c \leq \|f\|_{\mathcal{H}}$. Then according to R theorem there is the element $v \in \dot{W}_2^1$ such that $(f, u)_{\mathcal{H}} = (v, u)_{\dot{W}_2^1} \forall u \in \dot{W}_2^1$.

Let us endow the space $\dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ with the equivalent norm

$$\|u\|_{\dot{W}_{2,\mathbf{D},a}^1} = (a\|u\|_{\mathcal{H}_{\mathcal{E}}}^2 + \sum_{k=1}^{\infty} d_k \|\partial_k u\|_{\mathcal{H}_{\mathcal{E}}}^2)^{\frac{1}{2}} \quad (15)$$

for arbitrary $a > 0$. The space $\dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ endowed with the equivalent norm (15) is noted by $\dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$. The inequalities

$$\frac{1}{1+a}\|u\|_{\dot{W}_{2,\mathbf{D},a}^1}^2 \leq \|u\|_{\dot{W}_{2,\mathbf{D}}^1}^2 \leq (1 + \frac{1}{a})\|u\|_{\dot{W}_{2,\mathbf{D},a}^1}^2$$

holds according to the definition of the space $\dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$. Therefore the space $\dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$ is the Hilbert space.

Theorem 27. Let $a > 0$ and $f \in L_2(\Pi_{0,1})$. Then the functional (13) has the unique point of the minimum in the space $\dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$.

Proof. Let $f \in \mathcal{H}_{\mathcal{E}}$. Then for any $u \in \dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$ the inequality $|(f, u)| \leq c\|u\|_{\dot{W}_{2,\mathbf{D},a}^1}$ take place where $c \leq \|f\|_{\mathcal{H}_{\mathcal{E}}}$. Then according to Riesz theorem for any $a > 0$ there is the element $v_a \in \dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$ such that $(f, u)_{\mathcal{H}} = (v, u)_{\dot{W}_{2,\mathbf{D},a}^1} \forall u \in \dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$.

Therefore for any $u \in \dot{W}_{2,\mathbf{D}}^1(E)$ the following equality holds

$$J_{a,f}(u) = \frac{1}{2}(u, u)_{\dot{W}_{2,\mathbf{D},a}^1} - \frac{1}{2}(v, u)_{\dot{W}_{2,\mathbf{D},a}^1} - \frac{1}{2}(u, v)_{\dot{W}_{2,\mathbf{D},a}^1} = \frac{1}{2}(u - v, u - v)_{\dot{W}_{2,\mathbf{D},a}^1} - \frac{1}{2}(v, v)_{\dot{W}_{2,\mathbf{D},a}^1}.$$

Therefore the functional $J_{a,f}$ has the unique point of the minimum in the space $\dot{W}_{2,\mathbf{D}}^1$ which coincides with the element $v \in \dot{W}_{2,\mathbf{D},a}^1$. \square

Definition 4. The function $v \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ is called the generalized solution of the equation (11) with the Dirichlet condition (12) if the equality

$$(v, \phi)_{\dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})} + (a + f, \phi)_{L_2(\Pi_{0,1})} = 0 \quad (16)$$

satisfies for any $\phi \in \dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$.

Theorem 28. Let $a > 0$ and $f \in L_2(\Pi_{0,1})$. Then the function $u \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ is point of minimum of the functional (13) if and only if it is the generalized solution of Dirichlet problem (11), (12).

Proof. If the function $u \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ is point of minimum of the functional (13) then $\frac{d}{dt} J_{a,f}(u + t\phi)|_{t=0} = 0$ for any $\phi \in \dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$. Hence the equality (16) satisfies for any $\phi \in \dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$ according to the expression (14). Therefore u is the generalized solution of Dirichlet problem (11), (12).

Let u is the generalized solution of Dirichlet problem (11), (12). Then the right hand side of the expression (14) is equal to zero. Therefore for any $\phi \in \dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})$ the following equality holds

$$J_{a,f}(u + \phi) - J(u) = \frac{1}{2} \|\phi\|_{\dot{W}_{2,\mathbf{D},a}^1(\Pi_{0,1})}^2.$$

Hence the function $u \in \dot{W}_{2,\mathbf{D}}^1(\Pi_{0,1})$ is point of strong minimum of the functional (13). \square

6 Conclusions

In this paper we show that the theory of Sobolev spaces and its application to partial differential equation can be constructed for the function on domains in infinite dimension Hilbert space endowing with finite additive shift invariant measures. We study the class of finite additive shift invariant measures on the real separable Hilbert space E . For any choice of such a measure we consider the Hilbert space \mathcal{H} of complex-valued functions which are square-integrable with respect to this measure. Some analogs of Sobolev spaces of functions on the space E are introduced. The analogue of Gauss theorem is obtained for the simplest domains such as the rectangle in the space E . The correctness of the problem for Poisson equation in the rectangle with homogeneous Dirichlet condition is obtained and the variational approach of the solving of this problem is constructed.

References

- [1] Baker R. "Lebesgue measure" on R^∞ . Proceedings of the AMS. 113 (1991), no. 4., 1023–1029.
- [2] V.M. Busovikov. *Properties of one finite additive measure on l_p invariant to shifts*. Proceedings of MIPT. 10 (2018) no. 2. 163–172.
- [3] V.M. Busovikov, V.Zh. Sakbaev. *Sobolev spaces of functions on Hilbert space with shift-invariant measure and approximation of semigroups*. Izvestiya RAN. Ser. Mathematics. (2020) no. 4.
- [4] Ya. A. Butko. Chernoff approximation of subordinate semigroups. Stoch. Dyn. 1850021 (2017), 19 p., DOI: 10.1142/S0219493718500211.
- [5] N. Dunford, J. Schwartz. *Linear operators. General Theory*. Moscow, 2004.
- [6] J.L. Lions, E. Magenes. *Problems aux limites non homogenes et applications*. Dunod, Paris, 1968.
- [7] O.A. Oleynik. *Lectures on the partial differential equations*. Lomonosov MSU, Moscow, 2015.
- [8] I.D. Remizov, *Formulas that represent Cauchy problem solution for momentum and position Schrodinger equation*. Potential Anal (2018). <https://doi.org/10.1007/s11118-018-9735-1>
- [9] I.D. Remizov, *Explicit formula for evolution semigroup for diffusion in Hilbert space*. Infinite Dimensional Analysis Quantum Probability and Related Topics (2018) Vol. 21, No. 04, 1850025.
- [10] V.Zh. Sakbaev, *Averaging of random walks and shift-invariant measures on a Hilbert space*. Theoret. and Math. Phys. 191 (2017), no. 3., 886–909.
- [11] V.Zh. Sakbaev, *Random walks and measures on Hilbert space that are invariant with respect to shifts and rotations*. Differential equations. Mathematical physics. Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz. VINITI. 140 (2017), 88–118.
- [12] V.Zh. Sakbaev, *Semigroups of operators in the space of function square integrable with respect to traslatory invariant measure on Banach space*. Quantum probability. Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz. VINITI. 151 (2018), 73–90.

- [13] N.N. Shamarov, O.G. Smolyanov, *Hamiltonian Feynman measures, Kolmogorov integral, and infinite-dimensional pseudodifferential operators* Doklady Mathematics **100** (2019) no. 2., 445–449.
- [14] A.M. Vershik. *Does There Exist a Lebesgue Measure in the Infinite-Dimensional Space?* Proc. Steklov Inst. Math. 259 (2007), 248–272.
- [15] D. V. Zavadsky, V. Zh. Sakbaev, *Diffusion on a Hilbert Space Equipped with a Shift- and Rotation-Invariant Measure*. Proc. Steklov Inst. Math., **306** (2019), 102–119.
- [16] D.V. Zavadsky. *Shift-invariant measures on sequence spaces*// Proceedings of MIPT. 9 (2017), no. 4., 142–148.
- [17] D.V. Zavadsky, *Analogs of Lebesgue measure on the sequences spaces and the classes of integrable functions*. Quantum probability. Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz. VINITI. 151 (2018), 37–44.

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