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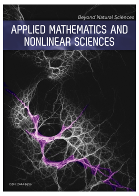

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Small C^1-smooth perturbations of skew products and the partial integrability property

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Abstract

In this paper we investigate stability of the integrability property of skew products of interval maps under small C^1-smooth perturbations satisfying some conditions. We obtain here (sufficient) conditions of the partial integrability for maps under considerations. These conditions are formulated in the terms of properties of the unperturbed skew product. We give also the example of the partially integrable map.

Keywords: skew products of interval maps, perturbation, integrability of a discrete dynamical system, partial integrability of a discrete dynamical system, curvelinear fiber.
AMS 2010 codes: 37Exx; 37Cxx.

1 Introduction

1. This paper is the direct continuation of the work [18], where stability of the Sharkovsky's order respectively small C^1-smooth perturbations of skew products of interval maps is proved. Results of [18] are announced in [19], where the part of the Author's report at the Conference "Mathematical Physics, Dynamical Systems and Infinite-Dimensional Analysis" (17-21 June 2019, Dolgoprudny, Russia) devoted to periodic orbits of C^1-smooth maps defined below is presented.

We consider a map F of a closed rectangle $I = I_1 \times I_2$ into itself, where I_1, I_2 are closed intervals of the straight line $\mathbf{R^1}$, $I_k = [a_k, b_k]$ for $k = 1, 2$, and F satisfies the equality

$$F(x, y) = (f(x) + \mu(x, y), g(x, y)) \quad \text{for any } (x, y) \in I. \tag{1}$$

Further we use the notation $g_x(y)$ for $g(x, y)$, where (x, y) is an arbitrary point of the rectangle I.

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In the paper [31] the integrable system of differential equations is constructed so that this system approximates the Lorenz system of differential equations [3]. The paper [31] generated the wave of interest to dynamical systems (1) (see, e.g., [5], [29] – [30]) for the case of the discontinuous Lorenz map f of the closed interval I_1 into itself [8], [26].

Following [18], [19] we suppose in this paper that the map (1) is C^1-smooth on I, and the map $f : I_1 \to I_1$ is so that the conditions hold:
(i_f) $f(\partial I_1) \subset \partial I_1$, where $\partial(\cdot)$ is the boundary of a set;
(ii_f) f is the Ω-stable in the space of C^1-smooth self-maps of the interval I_1 with the invariant boundary.

We suppose also that the C^1-smooth function μ (of variables x and y) satisfies the boundary conditions:
(i_μ) the equalities $\mu(x, a_2) = \mu(x, b_2) = 0$ are valid for every $x \in I_1$; and the equalities $\mu(a_1, y) = \mu(b_1, y) = 0$ are valid for every $y \in I_2$.

By the properties (i_f) and (i_μ) the set $(\{a_1\} \times I_2) \bigcup (\{b_1\} \times I_2)$ is F-invariant. If, in addition, the inclusion $g_x(\partial I_2) \subset \partial I_2$ holds for all $x \in I_1$ then the union of the horizontal intervals $I_1 \times \{a_2\}$ and $I_1 \times \{b_2\}$ is F-invariant too.

2. Give the list of the functional spaces connected with the map (1).

Let $\widetilde{C}^1_\omega(I_1)$ be the set of all C^1-smooth maps of the interval I_1 into itself satisfying conditions (i_f) – (ii_f). The standard C^1-norm $||\cdot||_{1,1}$ of the linear normalized space of C^1-smooth maps of the interval I_1 into the straight line $\mathbf{R^1}$ generates the C^1-topology in $\widetilde{C}^1_\omega(I_1)$. Denote by $\widetilde{B}^1_{1,\varepsilon}(f)$ elements of the base of the C^1-topology in $\widetilde{C}^1_\omega(I_1)$ for every $\varepsilon > 0$ and $f \in \widetilde{C}^1_\omega(I_1)$.

By the C^1- Ω-stability of the map f (see the condition (ii_f)) for any $\delta > 0$ there exists an ε-neighborhood $\widetilde{B}^1_{1,\varepsilon}(f)$ of f in the space $\widetilde{C}^1_\omega(I_1)$ such that every map from this neighborhood is Ω-conjugate with f by means of a homeomorphism which is δ-close in the C^0-topology of the uniform convergence to the identity map of the nonwandering set[a] of the map f.

Let $\widetilde{C}^1(I, I_1)$ be the set of C^1-smooth maps of the rectangle I into the interval I_1 endowed with the standard C^1-norm $||\cdot||_{1,(1,1)}$ of the linear normalized space of C^1-smooth maps of the rectangle I into the straight line $\mathbf{R^1}$ that contains the interval I_1. This norm induces the C^1-topology in the space $\widetilde{C}^1(I, I_1)$ with the base given by the set of ε-balls $\widetilde{B}^1_{(1,1),\varepsilon}(\varphi)$ for every $\varphi \in \widetilde{C}^1(I, I_1)$ and $\varepsilon > 0$.

We suppose that the function $\mu = \mu(x, y)$ satisfies the following "condition of smallness":
(ii_μ) $||\mu||_{1,(1,1)} < \varepsilon$, where ε is found for $\delta > 0$ by the property of the C^1- Ω-stability of f.

The following inequality connects norms $||\cdot||_{1,1}$ for every $y \in I_2$ and $||\cdot||_{1,(1,1)}$:

$$||\mu||_{1,1} < ||\mu||_{1,(1,1)}. \tag{2}$$

Every function f of one variable can be considered as the function of two variables of the type $f \circ pr_1$, where $pr_1 : I \to I_1$ is the natural projection of I on I_1. Hence, by the condition (ii_μ) and the inequality (2) we have $(f + \mu) \in \widetilde{B}^1_{(1,1),\varepsilon}(f \circ pr_1)$, and the belonging

$$(f + \mu) \in \widetilde{B}^1_{1,\varepsilon}(f) \tag{3}$$

holds for every $y \in I_2$.

Denote by $C^1_\omega(I)$ the set of C^1-smooth maps (1) such that the function f satisfies conditions (i_f) – (ii_f), and μ satisfies conditions (i_μ) – (ii_μ). Endow $C^1_\omega(I)$ with the standard C^1-norm $||\cdot||_1$ of the linear normalized space of C^1-smooth maps of the rectangle I into the plane $\mathbf{R^2}$. The base of the C^1-topology generated by this norm, is given by the system of ε-balls

$$B^1_\varepsilon(F) = \{G \in C^1_\omega(I) : ||G - F||_1 < \varepsilon\}$$

[a] A point $x \in I_1$ ($(x, y) \in I$) is said to be f-nonwandering (F-nonwandering) point if for every its neighborhood $U_1(x)$ ($U((x, y)) = U_1(x) \times U_2(y)$) there is a natural number n such that the inequality $U_1(x) \bigcap f^n(U_1(x)) \neq \emptyset$ ($U((x, y)) \bigcap F^n(U((x, y))) \neq \emptyset$) holds. The set of all f-nonwandering (F-nonwandering) points is said to be the nonwandering set of f (F) [22]. We use the notation $\Omega(f)$ ($\Omega(F)$) for this set.

with the center F for every $\varepsilon > 0$ and $F \in C^1_\omega(I)$.

3. The map (1) from the space $C^1_\omega(I)$ is obtained by small C^1-smooth perturbations (satisfying conditions (i_μ) – (ii_μ)) of the C^1-smooth skew product of interval maps of the type

$$\Phi_0(x, y) = (f(x), g_x(y)). \tag{4}$$

In [18] it is shown also that the autonomous discrete dynamical system (1) is connected with the nonautonomous discrete dynamical system generated by skew products of interval maps. So, one can represent the value of n-th $(n \geq 1)$ iteration of the map F in every initial point $(x^0, y^0) \in I$ as the composition of values of various skew products of interval maps in the corresponding points:

$$F^n(x^0, y^0) = (x^n, y^n) = \Phi_{y^{n-1}} \circ \ldots \circ \Phi_{y^0}(x^0, y^0), \tag{5}$$

where a skew product $\Phi_{y^i} : I \to I$ for every i $(0 \leq i \leq n-1)$ is presented in the form

$$\Phi_{y^i}(x, y) = (\varphi_{y^i}(x), g_x(y)). \tag{6}$$

Here

$$\varphi_{y^i}(x) = f(x) + \mu_{y^i}(x), \text{ and } \mu_{y^i}(x) = \mu(x, y^i). \tag{7}$$

To write the equality (5) in the coordinate form we set

$$\varphi_{y^0,n}(x^0) = \varphi_{y^{n-1}} \circ \ldots \circ \varphi_{y^0}(x^0); \;\; g_{(x^0,y^0),n}(y^0) = g_{\varphi_{y_{n-1}}(x_{n-1})} \circ \ldots \circ g_{x^0}(y^0). \tag{8}$$

Then we have:

$$F^n(x^0, y^0) = (\varphi_{y^0,n}(x^0), \; g_{(x^0,y^0),n}(y^0)). \tag{9}$$

All previous information of the item 3 means that an important role in this paper belongs to skew products of interval maps.

Denote by $T^1_{\omega,0}(I)$ the space of C^1-smooth skew products of interval maps with quotients satisfying conditions (i_f) – (ii_f). Endow this space with the C^1-topology generated by the standard C^1-norm $||\cdot||_1$. The structure of the functional space $T^1_{\omega,0}(I)$ and dynamical properties of skew products from this space are studied in [9] – [16].

We use also the space $\widetilde{T}^1_{\omega,0}(I)$ of skew products of interval maps respectively which the boundary ∂I of the rectangle I is invariant. Then the inclusions are valid:

$$\widetilde{T}^1_{\omega,0}(I) \subset T^1_{\omega,0}(I) \subset C^1_\omega(I).$$

The base of the C^1-topology in $\widetilde{T}^1_{\omega,0}(I)$ is given by the set of ε-balls $\widetilde{B}^1_\varepsilon(\Phi)$ with the center Φ for all $\varepsilon > 0$ and $\Phi \in \widetilde{T}^1_{\omega,0}(I)$.

Note that by the formula (1) and the condition (i_μ) every map $F \in C^1_\omega(I)$ obtained from the skew products of interval maps $\Phi_0 \in \widetilde{T}^1_{\omega,0}(I)$ possesses the property:

$$F(\partial I) \subset \partial I.$$

4. There is a vast literature devoted to different integrability aspects of dynamical systems both with continuous time (see, e.g., [7], [23] – [24]), and with discrete time (see, e.g., [1] – [2], [35] – [36]). Originally, the concept of integrability of dynamical systems with discrete time was introduced for systems obtained by digitization of known differential equations [1] – [2], [35] – [36]). But there are discrete dynamical systems that do not belong to this class. We consider here precisely this case.

Remind the following Birkhoff's thought: "If we try to formulate the exact definition of integrability then we see that many definitions are possible, and every of them is of the specific theoretical interest" [6].

Our definition of integrability of dynamical systems with discrete time given in [4] (see also [16]) follows the paper [20] and generalizes the definition from [20] given for polynomials and rational maps, on the case of arbitrary maps. (The last set contains maps that can not be obtained by the procedure of digitization of differential equations.)

Definition 1. [4] We say that a map G of some (open or closed) domain $\Pi \subset \mathbf{R^2}$ to itself is *integrable* if there exists a self-map ψ of an interval J of the real line $\mathbf{R^1}$ such that G is semiconjugate with ψ by means of a continuous surjection $\widetilde{H} : \Pi \to J$, so that

$$\widetilde{H} \circ G = \psi \circ \widetilde{H}.$$

Remark 1. In the framework of the suggested approach in the paper [17] the definition of integrability is introduced for some multifunctions.

Remark 2. As it follows from Definition 1 skew products of interval maps are integrable maps. Here $\widetilde{H} = pr_1$. Moreover, every integrable map satisfying some natural conditions can be reduced to a skew product[b].

Theorem 1. *[4]. Let Π be a convex connected compact subset of $\mathbf{R^2}$ such that the section of Π by an arbitrary line $y = const$ (if it is non-empty) is a non-degenerate interval, and let $G : \Pi \to \Pi$ be a continuous map. Then G is integrable in the sense of Definition 1 by means of a continuous surjection $\widetilde{H} : \Pi \to J$ that is one-to-one with respect to x (here J is a closed interval of $\mathbf{R^1}$) if and only if some homeomorphism reduces G to a skew product of interval maps defined in a compact planar rectangle.*

Remark 3. Definition 1 distinguishes such feature of integrable dynamical systems satisfying conditions of Theorem 1, as the existence of an invariant foliation. This property is the key point of the proof of the integrability property of a dynamical system.

Remark 4. Point out that the existence of a continuous invariant foliation for Lorenz type maps is proven in [3], and existence of a C^1-smooth invariant foliation (with C^2-smooth fibers) for these maps is proven in [34].

In different problems of dynamical systems theory only existence of an invariant lamination (but not an invariant foliation!) can be proved (see, e.g., [4], [18]). Therefore, it is naturally to introduce the following concept of the partial integrability for discrete dynamical systems.

Definition 2. We say that a map G defined on some (open or closed) domain Π of the plane $\mathbf{R^2}$ with values in Π is *partially integrable* if there exist a closed invariant set $A \subset \Pi$ ($A \neq \Pi$), a self-map ψ of an interval J of the real line $\mathbf{R^1}$ and a closed invariant set $B \subset J$ ($B \neq J$) such that $G_{|A}$ is semiconjugate with $\psi_{|B}$ by means of a continuous surjection $\widetilde{H} : A \to B$, i. e. the equality holds

$$\widetilde{H} \circ G_{|A} = \psi_{|B} \circ \widetilde{H}. \tag{10}$$

5. In this paper we investigate stability of the integrability property of skew products of interval maps under small C^1-smooth perturbations satisfying conditions $(i_\mu) - (ii_\mu)$. We obtain here (sufficient) conditions of partial integrability for maps from the space $C^1_\omega(I)$ (§3). These conditions are formulated in the terms of properties of the unperturbed skew product $\Phi_0 \in T^1_{\omega,0}(I)$ (see the formula (4)). We give also the example of the partially integrable map (1) (§3).

2 Preliminaries

This section contains the relevant definitions and results on dynamics both of continuous maps and C^1-smooth Ω-stable maps of a closed interval.

1. We begin from the famous Sharkovsky's Theorem [32].

[b] The reducibility problem of integrable maps to skew products has been formulated by Grigorchuk to the Author during our verbal discussions (the formulation of the problem is not published) in the framework of the Conference devoted to the 70-th birthday of Professor V.M. Alexeev (Moscow, Russia, 2002).

Theorem 2. *[32] If a continuous map $f : I_1 \to I_1$ contains a periodic orbit of a (least) period $m > 1$ then it contains also periodic orbits of every (least) period n, where n precedes m ($n \prec m$) in the Sharkovsky's order:*

$$\begin{gathered} 1 \prec 2 \prec 2^2 \prec 2^3 \prec \ldots \prec \ldots \prec 2^2 \cdot 9 \prec 2^2 \cdot 7 \prec 2^2 \cdot 5 \prec 2^2 \cdot 3 \prec \ldots \\ \prec 2 \cdot 9 \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \ldots \prec 9 \prec 7 \prec 5 \prec 3. \end{gathered} \tag{11}$$

In accordance with the Sharkovsky's Theorem, the space of continuous self-maps of a closed interval can be presented as the union of three subspaces (see, e.g., [33]): the first of which consists of the maps of type $\prec 2^\infty$, that is, the maps that have periodic points with (least) periods $\{1, 2, 2^2, \ldots, 2^\nu\}$, where $0 \leq \nu < +\infty$; the second subspace consists of the maps of type 2^∞, that is, maps whose periodic points have (least) periods $\{1, 2, 2^2, \ldots, 2^i, 2^{i+1}, \ldots\}$; the third subspace consists of the maps of type $\succ 2^\infty$, that is, maps with periodic points that possess (least) periods outside the set $\{2^i\}_{i \geq 0}$.

In this paper we consider maps from the space $C^1_\omega(I)$ such that f satisfies the following additional condition: (iii_f) f has type $\succ 2^\infty$.

The above condition (iii_f) means that f demonstrates a chaotic behavior (see, e.g., [33]).

2. Formulate the properties of C^1-smooth Ω-stable maps of a closed interval, give the definition of Σ-stability and remind the properties of the Σ-stable maps of a closed interval.

Lemma 3. *[21], [27] Let $f \in \widetilde{C}^1_\omega(I_1)$ and satisfy the condition (iii_f). Then*
(3.1) the nonwandering set $\Omega(f)$ is the union of a finite number of hyperbolic periodic points (that form the rarefied set $\Omega_r(f)$) and a finite number of locally maximal (i.e., maximal quasiminimal [c] sets in some their neighborhood) hyperbolic perfect nowhere dense sets (that form the perfect set $\Omega_p(f)$));
(3.2) periodic points are everywhere dense in the set $\Omega_p(f)$; moreover, for every natural number $m \geq 2$ periodic points with multiple m (least) periods are everywhere dense in $\Omega_p(f)$;
(3.3) there are numbers $\alpha = \alpha(f) > 0$ and $c = c(f) > 1$ so that for every $x \in \Omega_p(f)$ and $n \geq 1$ the inequality $|(f^n(x))'| > \alpha c^n$ holds (that is, $\Omega_p(f)$ is the repelling hyperbolic set);
(3.4) the subspace $\widetilde{C}^1_\omega(I_1)$ of maps satisfying condition (iii_f) is open and everywhere dense in the containing it space of C^1-smooth self-maps of the closed interval I_1 of type $\succ 2^\infty$ with the invariant boundary.

Corollary 4. *[21], [27] Let $f \in \widetilde{C}^1_\omega(I_1)$ and satisfy the condition (iii_f). Then the equality holds:*

$$\Omega(f) = \Omega^s(f) \bigcup \Omega^u(f).$$

Here $\Omega^s(f)$ is the nonempty finite invariant set of all f-sinks. The set $\Omega^u(f)$ is invariant and equals the union of $\Omega_p(f)$ with a finite (possibly empty) set that consists of isolated sources in the set of f-periodic points.

In the set $I_1 \setminus \Omega(f)$ (just as in the set $I_1 \setminus \Omega_p(f)$) the points of attraction domains of f-sinks are everywhere dense.

Point out that for the map $f \in \widetilde{C}^1_\omega(I_1)$ the inclusion $f^{-1}(\Omega(f)) \subset \Omega(f)$ can be false (although the equality $f(\Omega(f)) = \Omega(f)$ holds). Here $f^{-1}(\cdot)$ means the first complete preimage of a set.

Following [21] we construct the set which is invariant both with respect to f, and with respect to f^{-1}, and contains the set $\Omega^u(f)$ as its subset. For this goal we need the attraction domain of all f-sinks:

$$\Delta(f) = \bigcup_{Orb(x,f) \subset \Omega^s(f)} \bigcup_{i=0}^{+\infty} D^{-i}(Orb(x, f)),$$

where $Orb(x, f)$ is the periodic orbit of the sink $x \in \Omega^s(f)$, $D(Orb(x, f))$ is the immediate attraction domain of the periodic orbit $Orb(x, f)$, $D^{-i}(Orb(x, f))$ is i-th complete primage of the immediate attraction domain of the periodic orbit $Orb(x, f)$.

[c] A quasiminimal set is the closure of an infinite recurrent trajectory (see [28]).

Immediate attraction domain $D(Orb(x,f))$ of the periodic orbit $Orb(x,f)$ consists of m (m is the (least) period of x) pairwise disjoint open (in the topology of the segment I_1) f^m-invariant intervals $D_{f^j(x)}$ $(0 \le j \le m-1)$ such that every of these intervals contains the point $f^j(x)$:

$$D(Orb(x,f)) = \bigcup_{j=0}^{m-1} D_{f^j(x)}.$$

Complete f-invariance of the immediate attraction domain[d] $D(Orb(x,f))$ implies correctness of the definition of preimages $D^{-i}(Orb(x,f))$ $(D^{-i}(Orb(x,f)) \ne \emptyset)$ for every $i \ge 0$.

By the condition (iii_f) the set $\Delta(f)$ is a countable union (see the claim (3.1) of Lemma 3) of pairwise disjoint intervals (open in the topology of the closed interval I_1); $\Delta(f)$ is invariant both with respect to f, and with respect to f^{-1}.

We suppose further that the set $Cr(f)$ of f-critical points satisfies the condition
(iv_f) $Cr(f) \subset \Delta^\circ(f)$, where $\Delta^\circ(f)$ is the interior of the set $\Delta(f)$.

Lemma 5. *[33] Let $f \in \widetilde{C}^1_\omega(I_1)$ satisfy conditions (iii_f) – (iv_f). Then one of the following three cases is realized for the boundary $\partial(D_{f^j(x)})$ of every interval $D_{f^j(x)}$ $(0 \le j \le m-1)$:*
(5.1) *$\partial(D_{f^j(x)})$ consists of two f^m-fixed points;*
(5.2) *points of $\partial(D_{f^j(x)})$ form a periodic orbit of (least) period 2 with respect to f^m;*
(5.3) *one of the points of $\partial(D_{f^j(x)})$ is f^m-fixed point source, and the other is its preimage with respect to f^m.*

Define the closed set that is invariant with respect to f and f^{-1}:

$$\Sigma(f) = I_1 \setminus \Delta(f). \tag{12}$$

By Lemma 5 and formula (12) we have: $\Omega^u(f) \subset \Sigma(f)$, and $\Omega^s(f) \bigcap \Sigma(f) = \emptyset$.

Definition 3. [21] The map $f \in \widetilde{C}^1_\omega(I_1)$ is said to be Σ-*stable (in the C^1-topology)* if for every $\delta > 0$ there exists an ε-neighborhood $\widetilde{B}^1_{1,\varepsilon}(f)$ of the map $f \in \widetilde{C}^1_\omega(I_1)$ such that every map $\varphi \in \widetilde{B}^1_{1,\varepsilon}(f)$ is Σ-conjugate to f, that is, the equality

$$h \circ f_{|\Sigma(f)} = \varphi_{|\Sigma(\varphi)} \circ h$$

holds for some homeomorphism $h : \Sigma(f) \to \Sigma(\varphi)$. Here h is δ-close to the identity map on $\Sigma(f)$ in the C^0-topology.

The following claim is the direct corollary of Definition 3.

Lemma 6. *Let $f \in \widetilde{C}^1_\omega(I_1)$ satisfy conditions (iii_f) – (iv_f). Then f is Σ-stable in the C^1-topology.*

Remark 5. By [21] the set of maps from $\widetilde{C}^1_\omega(I_1)$ satisfying conditions (iii_f) – (iv_f) contains the open everywhere dense subset of C^2-smooth maps with nondegenerate critical points.

3 Sufficient conditions of partial integrability of the map (1)

In this section we prove the main result of the paper.

Theorem 7. *Let the quotient f of the skew product of interval maps $\Phi_0 \in \widetilde{T}^1_{\omega,0}(I)$ satisfy conditions (iii_f) – (iv_f). Then for any $\delta > 0$ there exists an ε-neighborhood $B^1_\varepsilon(\Phi_0)$ of the map Φ_0 in the space $C^1_\omega(I)$ such that every map $F \in B^1_\varepsilon(\Phi_0)$ obtained from Φ_0 by means of the C^1-perturbation $\mu = \mu(x,y)$, where μ satisfies*

[d] It means correctness of the equality $f(D(Orb(x,f))) = D(Orb(x,f))$.

conditions (i_μ) – (ii_μ), is partially integrable on the closed invariant set $£(F)$ that consists of pairwise disjoint curvelinear fibers. These fibers start from the points of the set $\Sigma^*(f) \times \{a_2\}$ (where $\Sigma^*(f) = \Sigma(f) \bigcup \Omega^s(f)$) and are graphs of C^1-smooth functions $x = x(y)$ defined on the interval I_2. The function $\widetilde{H} : £(F) \to \Sigma^*(f)$ that realizes the partial integrability property, is C^1-smooth surjection, ε-close in the C^1-norm to the natural projection $pr_1 : \Sigma^*(f) \times I_2 \to \Sigma^*(f)$.

The following statement proved in [18], is the first step of the proof of Theorem 7.

Proposition 8. *[18] Let $\Phi_0 \in \widetilde{T}^1_{\omega,0}(I)$. Then for any $\delta > 0$ there exists an ε-neighborhood $B^1_\varepsilon(\Phi_0)$ of the map Φ_0 in the space $C^1_\omega(I)$ such that every map $F \in B^1_\varepsilon(\Phi_0)$ obtained from Φ_0 by means of the C^1-perturbation $\mu = \mu(x, y)$, where μ satisfies conditions (i_μ) – (ii_μ), has the invariant closed set $£^0(F)$ that consists of pairwise disjoint curvelinear fibers. These fibers start from the points of the set $\Omega(f) \times \{a_2\}$, and are the graphs of continuous functions $x = x(y)$ defined on the interval I_2; moreover, fibers starting from the points (x, a_2), where x is f-periodic point, are C^1-smooth. Every curvelinear fiber is δ-close in the C^0-norm to the vertical closed interval that starts from the same initial point of the set $\Omega(f) \times \{a_2\}$ just as the curvelinear fiber.*

Prove C^1-smoothness of all fibers from the set $£^0(F)$.

Proposition 9. *Let $\Phi_0 \in \widetilde{T}^1_{\omega,0}(I)$. Then for any $\delta > 0$ there exists an ε-neighborhood $B^1_\varepsilon(\Phi_0)$ of the map Φ_0 in the space $C^1_\omega(I)$ such that the closed invariant set $£^0(F)$ of every map $F \in B^1_\varepsilon(\Phi_0)$ obtained from Φ_0 by means of the C^1-perturbation $\mu = \mu(x, y)$, where μ satisfies conditions (i_μ) – (ii_μ), consists of graphs of C^1-smooth functions $x = x(y)$ defined on the interval I_2. In addition, every curvelinear fiber of the set $£^0(F)$ is ε-close in the C^1-norm to the vertical closed interval that starts from the same initial point of the set $\Omega(f) \times \{a_2\}$ just as the curvelinear fiber.*

Proof. 1. Fix a number $\delta > 0$. We find a positive number $\varepsilon > 0$ for δ using the C^1- Ω-stability property of the map $f \in \widetilde{C}^1_\omega(I_1)$. The neighborhood $\widetilde{B}^1_{1,\varepsilon}(f)$ of the map f consists of maps such that every map is Ω-conjugate with f by means of the homeomorphism that is δ-close to the identity map of the set $\Omega(f)$.

We use also the ε-neighborhood $B^1_\varepsilon(\Phi_0)$ of the map Φ_0 in the space $C^1_\omega(I)$. Then by formulas (1), (4) and by the property (ii_μ) the inequality

$$||F - \Phi_0||_1 = ||\mu||_{1,(1,1)} < \varepsilon$$

is valid for any map $F \in B^1_\varepsilon(\Phi_0)$. It implies, in particular, correctness of the belonging (3) for any $y \in I_2$ and means also that the map $(f + \mu)$ is Ω-conjugate with f for every $y \in I_2$ by means of the homeomorphism that is δ-close to the identity map of the set $\Omega(f)$.

We need ε_m-neighborhoods $B^1_{\varepsilon_m}(\Phi^m_0)$ of iterations Φ^m_0 for any $m > 1$ in the space $C^1_\omega(I)$ that correspond the chosen ε-neighborhood $B^1_\varepsilon(\Phi_0)$ of the map Φ_0. Since $F \in B^1_\varepsilon(\Phi_0)$ then $F^m \in B^1_{\varepsilon_m}(\Phi^m_0)$. Using formulas (8) – (9) we obtain the inequalities

$$||\varphi_{y,m} - f^m||_{1,(1,1)} \le ||F^m - \Phi^m_0||_1 < \varepsilon_m;$$

moreover, for every $y \in I_2$ the inequality holds:

$$||\varphi_{y,m} - f^m||_{1,1} < ||\varphi_{y,m} - f^m||_{1,(1,1)}.$$

Therefore,

$$\varphi_{y,m} \in \widetilde{B}^1_{(1,1),\varepsilon_m}(f^m \circ pr_1) \text{ and, the more so, } \varphi_{y,m} \in \widetilde{B}^1_{1,\varepsilon_m}(f^m) \tag{13}$$

for every $y \in I_2$. Here $\widetilde{B}^1_{(1,1),\varepsilon_m}(f^m \circ pr_1)$ is the ε_m-neighborhood of the map $(f^m \circ pr_1)$ in the space $\widetilde{C}^1(I, I_1)$, and $\widetilde{B}^1_{1,\varepsilon_m}(f^m)$ is the ε_m-neighborhood of the map f^m in the space $\widetilde{C}^1_\omega(I_1)$. The neighborhood $\widetilde{B}^1_{1,\varepsilon_m}(f^m)$ consists of the maps, which are Ω-conjugate to f^m by means of the homeomorphisms δ-close to the identity map on the nonwandering set $\Omega(f^m) = \Omega(f)$ (see Lemma 3).

2. Let f satisfy the condition (iii_f). Denote by $\pounds^u(F)$ ($\pounds^u(F) \subset \pounds^0(F)$) the set of curvelinear fibers that start from all points of $\Omega^u(f)$.

Prove that there is the universal natural number n_* such that the equality

$$\inf_{(x,y)\in \pounds^u(F)} \left|\frac{\partial}{\partial x}\varphi_{y,n_*}(x)\right| = M_*, \text{ where } M_* > 1, \tag{14}$$

holds for every $y \in I_2$.

In fact, using the definition of the set $\pounds^u(F)$ and Lemma 3 (see claims (3.1) – (3.3)) for every $y \in I_2$ we point out the least natural number $n_*(y)$ satisfying

$$\inf_{x\in(\pounds^u(F))(y)} \left|\frac{\partial}{\partial x}\varphi_{y,n_*(y)}(x)\right| > 1. \tag{15}$$

By the claim (3.3) of Lemma 3 the inequality

$$\inf_{x\in(\pounds^u(F))(y)} \left|\frac{\partial}{\partial x}\varphi_{y,n}(x)\right| > 1$$

is valid for every $n \geq n_*(y)$.

Since the partial derivative $\frac{\partial}{\partial x}\varphi_{y,n_*(y)}(x)$ is uniformly continuous on the compact

$$(\pounds^u(F))(y) \times \{y\}$$

then by the inequality (15) there exists a $\theta(y)$-neighborhood $U_{\theta(y)}\big((\pounds^u(F))(y)\times\{y\}\big)$ of the set $(\pounds^u(F))(y)\times\{y\}$ in I such that

$$\inf_{(x,y')\in U_{\theta(y)}\big((\pounds^u(F))(y)\times\{y\}\big)} \left|\frac{\partial}{\partial x}\varphi_{y',n_*(y)}(x)\right| > 1. \tag{16}$$

Moreover, by the formula (13) $n_*(y)$ is the least natural number for which the inequality (16) holds.

Let 2^{I_1} be the topological space of all closed subsets of the closed interval I_1 with the exponential topology. By compactness of the closed intervals I_1, I_2 the set $2^{I_1}\times I_2$ is the compact [24]. Then the closed set

$$\pounds^u(F) = \bigcup_{y\in I_2}\big((\pounds^u(F))(y)\times\{y\}\big)$$

is the compact in $2^{I_1}\times I_2$. Using compactness of the set $\pounds^u(F)$ in $2^{I_1}\times I_2$ we distinguish from its infinite open cover $\{U_{\theta(y)}\big((\pounds^u(F))(y)\times\{y\}\big)\}_{y\in I_2}$ the finite subcover. Let neighborhoods $\{U_{\theta(y_j)}\big((\pounds^u(F))(y_j)\times\{y_j\}\big)\}_{1\leq j\leq q}$ of the sets $\{(\pounds^u(F))(y_j)\times\{y_j\}\}_{1\leq j\leq q}$ form this finite subcover.

The set $\pounds^u(F)$ consists of continuous fibers (see Proposition 8). Therefore, for every j $(1\leq j\leq q)$ there exists j' ($j'\neq j$, $1\leq j'\leq q$) such that

$$U_{\theta(y_j)}\big((\pounds^u(F))(y_j)\times\{y_j\}\big)\bigcap U_{\theta(y_{j'})}\big((\pounds^u(F))(y_{j'})\times\{y_{j'}\}\big)\neq\emptyset.$$

Thus, using the above considerations of this item 2 we obtain from here that the equalities hold

$$n_*(y_1) = \ldots = n_*(y_j) = \ldots = n_*(y_q).$$

Set $n_* = n_*(y_j)$ $(1\leq j\leq q)$. Using continuity of the partial derivative $\frac{\partial}{\partial x}\varphi_{y,n_*}(x)$ on I we verify that the equality (14) holds.

Hence, without loss of generality we will suppose further that $n_* = 1$. In fact, if $n_*\neq 1$ then we get over consideration of the map F^{n_*} and use the claim (3.2) of Lemma 3.

3. Prove that every curvelinear fiber is ε-close in the C^1-norm to the vertical closed interval that starts from the same initial point (x, a_2) of the set $\Omega(f) \times \{a_2\}$ just as the curvelinear fiber.

3.1. Begin from the curvelinear fibers γ_{x^0} that start from all points (x^0, a_2), where $x^0 \in Per(f) \bigcap \Omega^u(f)$. By Proposition 8 every this fiber is the graph of the C^1-smooth implicit function $x = x(y)$ defined on the interval I_2. Moreover, $x = x(y)$ satisfies the initial conditions

$$x(a_2) = x(b_2) = x^0$$

and the equation

$$\varphi_{y,m}(x) = x \ \ [18], \tag{17}$$

where m is the least period of the initial f-periodic point x^0. By the theorem about the C^1-smooth implicit function and the equation (17) we have

$$\frac{d}{dy}x(y) = -\frac{\frac{\partial}{\partial y}\varphi_{y,m}(x)}{\frac{\partial}{\partial x}\varphi_{y,m}(x) - 1} \tag{18}$$

in any point $(x, y) \in \gamma_{x^0}$.

Note that by the item 2 the sequence $\{\varepsilon_m\}_{m\geq 1}$ is increasing. Therefore, we construct the special δ-trajectory for the real trajectory $\{F^j(x, y)\}_{j\geq 0}$. Denote by l_x the vertical closed interval that starts from the point (x, a_2).

Choose the points

$$z_0 = (x^0, y), z_1 = (x^1, y^1), \dots, z_i = (x^i, y^i), \dots, z_{m-1} = (x^{m-1}, y^{m-1}) \dots,$$

$$z_{rm+i} = (x^{rm+i}, y^{rm+i}) \ \ (r \geq 0, 0 \leq i \leq m-1),$$

where

$$x^{rm+i} = f^{rm+i}(x^0) = f^i(x^0),$$

$$y^{rm+i} = g_{(x,y),rm+i}(y) \ \ (\text{here } 1 \leq i \leq m-1, \ m \geq 2).$$

Then $(x^{rm+i}, y^{rm+i}) \in l_{x^i}$. Since

$$F^{rm+i}(x, y) = (\varphi_{y,rm+i}(x), \ g_{(x,y),rm+i}(y)), \text{and } F^{rm+i}(x, y) \in \gamma_{x^i},$$

then $\{z_{rm+i}\}_{r\geq 0, 0\leq i\leq m-1}$ is δ-trajectory for the real trajectory of the point (x, y). Moreover,

$$|f^j(x^0) - \varphi_{y,j}(x)| < \delta.$$

It implies, in particular, correctness of the inequality

$$|\frac{\partial}{\partial y}\varphi_{y,j}(x)| < \varepsilon$$

for every $j \geq 1$. Therefore, using (18) we have

$$|\frac{d}{dy}x(y)| < \frac{\varepsilon}{M_*^m - 1}. \tag{19}$$

Since $M_* > 1$ then the equality holds:

$$\lim_{m\to+\infty} M_*^m = +\infty.$$

Then there exists $m^0 \geq 1$ such that the inequality

$$|\frac{d}{dy}x(y)| < \varepsilon \ (\text{see the inequality } (19)) \tag{20}$$

is valid for every $m \geq m^0$.

3.2. As it follows from the item 3.1, the set of C^1-smooth functions $\{x = x(y)\}$ that start from the points $\{(x^0, a_2)\}$, where $x^0 \in Per(f) \bigcap \Omega_p(f)$, $m(x^0) \geq m^0$ (here $m(x^0)$ is the (least) period of x^0), is dense in itself in the C^1-topology. It implies, in particular, that for every C^1-smooth function $x = x(y)$ that starts from a point (x_0, a_2) for $x_0 \in Per(f) \bigcap \Omega_p(f)$, $m(x_0) < m^0$, the inequality holds

$$|\frac{d}{dy}x(y)| \leq \varepsilon. \tag{21}$$

In addition, for every nonperiodic point $x^0 \in \Omega(f)$ there exists the unique C^1-smooth function $x = x(y) : I_2 \to I_1$ with the graph that starts from the point (x^0, a_2) and with the derivative satisfying the inequality (21) (see the inequality (20)). Proposition 9 is proved.

Extend the lamination $\pounds^0(F)$ up to the lamination $\pounds(F)$, where $\pounds(F)$ consists of C^1-smooth fibers that start from the points of the set $\Sigma^*(f) \times \{a_2\}$.

Use conditions (iii_f) – (iv_f), definition of the set $\Delta(f)$, Lemma 5 and Proposition 9. Then we obtain the following statement.

Corollary 10. *Let the quotient f of the skew product of interval maps $\Phi_0 \in \widetilde{T}^1_{\omega,0}(I)$ satisfy conditions (iii_f) – (iv_f). Then for any $\delta > 0$ there exists an ε-neighborhood $B^1_\varepsilon(\Phi_0)$ of the map Φ_0 in the space $C^1_\omega(I)$ such that every map $F \in B^1_\varepsilon(\Phi_0)$ obtained from Φ_0 by means of the C^1-perturbation $\mu = \mu(x, y)$, where μ satisfies conditions (i_μ) – (ii_μ), has the closed invariant set $\pounds(F)$ that consists of pairwise disjoint curvelinear fibers. These fibers start from the points of the set $\Sigma^*(f) \times \{a_2\}$, and are graphs of C^1-smooth functions $x = x(y)$ defined on the interval I_2. Every curvelinear fiber of the set $\pounds(F)$ is ε-close in the C^1-norm to the vertical closed interval that starts from the same initial point of the set $\Sigma^*(f) \times \{a_2\}$ just as the curvelinear fiber.*

Remark 6. The set $\pounds(F)$ constructed in the Corollary 10 is the C^1-smooth lamination, i.e. the set

$$\pounds(F) = \{x^0, x, y\}, \text{ where } x^0 \in \Sigma^*(f), \, x = x(y),$$

depends C^1-smoothly on the variables x^0, x, y.

Let conditions of Theorem 7 be fulfilled, and $F \in B^1_\varepsilon(\Phi_0)$ be given by the formula (1), where μ depends on x and y. Let γ_{x^0} be the curvelinear fiber from $\pounds(F)$ that starts from an arbitrary point (x^0, a_2), where $x^0 \in \Sigma^*(f)$. We set

$$\widetilde{H}(x, y) = x^0 \tag{22}$$

for any point $(x, y) \in \gamma_{x^0}$, that is $\widetilde{H}(\gamma_{x^0}) = x^0$, and $\widetilde{H}(\pounds(F)) = \Sigma^*(f)$. It means that the curvelinear projection $\widetilde{H}$ is the surjection of the set $\pounds(F)$ on the set $\Sigma^*(f)$.

By the equality (22) we have

$$\widetilde{H}(x, y) = pr_1(x, y) - x_{x^0}(y) + x^0, \tag{23}$$

where $x = x_{x^0}(y)$ is the function with the graph γ_{x^0}.

Remark 7. As it follows from the equality (23) and Corollary 10, the curvelinear projection $\widetilde{H}$ is C^1-smooth on the lamination $\pounds(F)$. Moreover, surjection $\widetilde{H} : \pounds(F) \to \Sigma^*(f)$ is ε-close in the C^1-norm to the natural projection $pr_1 : \Sigma^*(f) \times I_2 \to \Sigma^*(f)$.

Remark 8. F-invariance of the lamination $\pounds(F)$ and the formula (22) imply the equality

$$\widetilde{H} \circ F_{|\pounds(F)} = f_{|\Sigma^*(f)} \circ \widetilde{H}. \tag{24}$$

Comparison of the equalities (24) and (10) shoes that F is partially integrable map (see Definition 2). It completes the proof of Theorem 7.

In the end of the paper we give the example of the partially integrable map.

Example 11. *Let $F(x, y) = (f(x) + \lambda x(1-x)y(1-y), g(x, y))$, where $F \in C^1_\omega([0, 1]^2)$, $\lambda > 0$.*

The condition "$F \in C^1_\omega([0, 1]^2)$" implies "$f \in \widetilde{C}^1_\omega([0, 1])$". We use here the model C^1-smooth Ω-stable map f of type $\succ 2^\infty$ from the paper [16]:

$$f(x) = \begin{cases} \widetilde{h}(x), & \text{if } x \in [0, \frac{1}{4}); \\ 9(\frac{1}{4} - x)(x - \frac{3}{4}) + \frac{1}{4}, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}); \\ \widetilde{h}(1-x), & \text{if } x \in [\frac{3}{4}, 1]; \end{cases}$$

where $\widetilde{h}$ is so that $f(0) = f(1) = 0$, $f : [0, 1/4] \to [0, 1/4]$ is increasing bijection, $f : [3/4, 1] \to [0, 1/4]$ is decreasing bijection. Let $M = \sup\{f(x)\}$ satisfy the inequality $3/4 < M < 1$, and the point x_M such that the equality $f(x_M) = M$ holds, be the unique. Then the equality is valid:

$$\Omega(f) = \{0\} \bigcup K(f),$$

where $K(f)$ is the unique locally maximal quasiminimal set of f, $K(f) = \Omega_p(f) \subset [\frac{1}{4}, \frac{3}{4}]$, and $x_M \in \Delta^\circ(f)$.

Let λ be so small that the function $\mu(x, y) = \lambda x(1-x)y(1-y)$ satisfies the condition (ii_μ). We have also $\mu(0, y) = \mu(1, y) = \mu(x, 0) = \mu(x, 1) = 0$ for all $x, y \in [0, 1]$. Hence, the condition (i_μ) is valid. It means that conditions of Theorem 7 are fulfilled, and there exists the invariant lamination $\pounds(F)$ over the points of the set $\Omega(f) = \{0\} \bigcup K(f)$. It implies the semiconjugacy of $F_{|\pounds(F)}$ and $f_{|\Omega(f)}$. Therefore, F is the partially integrable map (see Definition 2).

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