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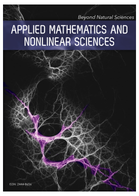


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# On 4-dimensional flows with wildly embedded invariant manifolds of a periodic orbit

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**Abstract**

In the present paper we construct an example of 4-dimensional flows on $\mathbb{S}^3 \times \mathbb{S}^1$ whose saddle periodic orbit has a wildly embedded 2-dimensional unstable manifold. We prove that such a property has every suspension under a non-trivial Pixton's diffeomorphism. Moreover we give a complete topological classification of these suspensions.

**Keywords:** Suspension, classification, Pixton diffeomorphism, wild embedding
**AMS 2010 codes:** 37C15

## 1 Introduction and statement of results

Qualitative study of dynamical systems reveals various topological constructions naturally emerged in the modern theory. For example, the Cantor set with cardinality of continuum and Lebesgue measure zero as an expanding attractor or an contracting repeller. Also, a curve in 2-torus with an irrational rotation number, which is not a topological submanifold but is an injectively immersed subset, can be found being invariant manifold of the Anosov toral diffeomorphism's fixed point.

Another example of linkage between topology and dynamics is the Fox-Artin arc [4] appeared in work by D. Pixton [9] as the closure of a saddle separatrix of a Morse-Smale diffeomorphism on the 3-sphere. A wild behaviour of the Fox-Artin arc complicates the classification of dynamical systems, there is no combinatorial description as Peixoto's graph [8] for 2-dimensional Morse-Smale flows.

It is well known that there are no wild arcs in dimension 2. They exist in dimension 3 and can be realized as invariant sets for discrete dynamics, unlike regular 3-dimensional flows, which do not possess wild invariant sets. The dimension 4 is very rich. Here appear wild objects for both discrete and continuous dynamics. Although there are no wild arcs in this dimension, there are wild objects of co-dimension 1 and 2. So, the closure of

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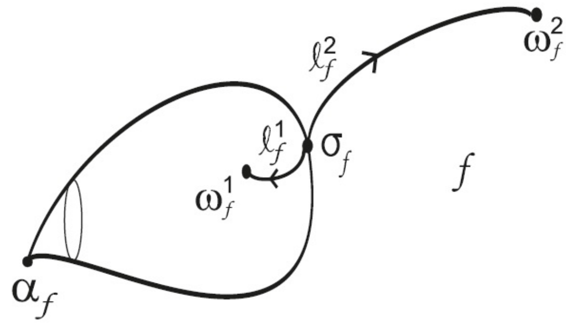


**Fig. 1** The phase portrait of a diffeomorphism of class $\mathscr{P}$

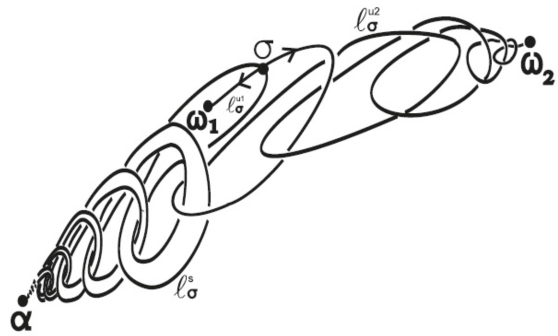


**Fig. 2** The phase portrait of a non-trivial diffeomorphism of class $\mathscr{P}$

2-dimensional saddle separatrix can be wild for 4-dimensional Morse-Smale system (a diffeomorphism or a flow). Such examples have been recently constructed by V. Medvedev and E. Zhuzoma [6]. T. Medvedev and O. Pochinka [7] have shown that the wild Fox-Artin 2-dimension sphere appears as closure of heteroclinic intersection of Morse-Smale 4-diffeomorphism.

In the present paper we prove that the suspension under a non-trivial Pixton's diffeomorphism provides a 4-flow with wildly embedded 3-dimensional invariant manifold of a periodic orbit. Moreover, we show that there are countable many different wild suspensions. In more details.

Denote by $\mathscr{P}$ the class of the Morse-Smale diffeomorphisms of 3-sphere $S^3$ whose non-wondering set consists of the fixed source $\alpha$, the fixed saddle $\sigma$ and the fixed sinks $\omega_1$, $\omega_2$. Class $\mathscr{P}$ diffeomorphism phase portrait is shown in Figure 1.

As the Pixton's example belongs to this class we call it the *Pixton class*. That example is characterized by the *wild embedding* of the stable manifold $W^s_\sigma$, namely its closure is not locally flat at $\alpha$. We call such diffeomorphism *non-trivial* (see Figure 2).

Let $\mathscr{P}^t$ be a set of flows which are suspensions on Pixton's diffeomorphisms. By the construction the ambient manifold for every such flow $f^t$ is diffeomorphic to $S^3 \times S^1$ and the non-wandering set consists of exactly four periodic orbits $\mathscr{O}_\alpha$, $\mathscr{O}_\sigma$, $\mathscr{O}_{\omega_1}$, $\mathscr{O}_{\omega_2}$. Let $W^s_{\mathscr{O}_\sigma}$ denote stable manifold of the saddle orbit. In the present paper we prove the following theorems.

**Theorem 1.** *If $W^s_\sigma$ is a wild for $f \in \mathscr{P}$ then $W^s_{\mathscr{O}_\sigma}$ is a wild for $f^t \in \mathscr{P}^t$.*

**Corollary 2.** *(Existence theorem) There is a flow $f^t$ with saddle orbit $\mathscr{O}_\sigma$ such that $cl(W^s_{\mathscr{O}_\sigma})$ is wild.*

**Theorem 3.** *Two flows $f^t, f'^t \in \mathscr{P}^t$ are topologically equivalent iff the diffeomorphisms $f, f' \in \mathscr{P}$ are topologically conjugated.*

The complete classification of diffeomorphisms from the class $\mathscr{P}$ has been done by Ch. Bonatti and V. Grines [1]. They proved that a complete invariant for Pixton's diffeomorphism is an equivalent class of the embedding of a knot in $S^2 \times S^1$. In section 4 we briefly give another idea to classify such systems. It was described in [5] and led to complete classification on Morse-Smale 3-diffeomorphisms in [2].

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## 2 Auxiliary facts

### 2.1 Dynamical concepts

Diffeomorphism $f\colon M^n \to M^n$ of smooth closed connected orientable $n$-manifold $(n \geq 1)M^n$ is called *Morse-Smale diffeomorphism* $(f \in MS(M^n))$ if:

1. Non-wandering set $\Omega_f$ is finite and hyperbolic;
2. Stable and unstable manifolds $W^s_p$, $W^u_q$ intersect transversally for any periodic points $p, q$.

Two diffeomorphisms $f, f'$ are called *topologically conjugated* if there exists a homeomorphism $h : M^n \to M^n$ such that $fh = hf'$.

Let $f : M^n \to M^n$ be a diffeomorphism. Let $\varphi^t$ be a flow on the manifold $M^n \times \mathbb{R}$ generated by the unite vector field parallel to $\mathbb{R}$ and directed to $+\infty$, that is

$$\varphi^t(x,r) = (x, r+t).$$

Let $g : M^n \times \mathbb{R} \to M^n \times \mathbb{R}$ be a diffeomorphism given by the formula $g(x,r) = (f(x), r-1)$. Let $G = \{g^k,\ k \in \mathbb{Z}\}$ and $W = (M^n \times \mathbb{R})/G$. Denote $p_W : M^n \times \mathbb{R} \to W$ the natural projections. It is verified directly that $g\varphi^t = \varphi^t g$. Then the map $f^t : W \to W$ given by the formula

$$f^t(x) = p_W(\varphi^t(p_W^{-1}(x)))$$

is a well-defined flow on $W$ which is called *the suspension of* $f$.

When $f \in MS(M^n)$ the non-wandering set of the suspension $f^t$ consist of a finite number of periodic orbits composed by $p_W(\Omega_f \times \mathbb{R})$. The obtained flow is so-called *non-singular*, what means it has no singular points.

Two flows $f^t, f'^t$ are called *topologically equivalent* if exists a homeomorphism $h : W \to W$ which maps the trajectories of $f^t$ to trajectories of $f'^t$ and preservs orientation on the trajectories.

### 2.2 Topological concepts

A closed subset $X$ of a PL-manifold $N$ is said to be *tame* if there is a homeomorphism $h : N \to N$ such that $h(X)$ is a subpolyhedron; the other are called *wild*.

For example, Fox-Artin arc is wild (see [4]).

Let $A$ be a closed subset of a metric space $X$. $A$ is called *locally k-co-connected* in $X$ at $a \in A$ (*k-LCC* at a) if each neighbourhood $U$ of $a$ in $X$ contains a smaller neighbourhood $V$ of $a$ such that each map $\partial I^{k+1} \to V \setminus A$ extends to a map $I^{k+1} \to U \setminus A$.

We say that $A$ is *locally k-co-connected* ($k$-LCC in $X$) if $A$ is $k$-LCC at $a$ for each $a \in A$.

For example, Fox-Artin 2-sphere is not 1-LCC (see Exercise 2.8.1 [3]).

Let $e\colon M^m \to N^n$ be a topological embedding of $m$-dimensional manifold $M^m$ with a boundary in $n$-manifold $N^n \quad (n \geq m)$. $e$ is called *locally flat* at $x \in M^m$ (and $e(M^m)$ is *locally flat* at $e(x)$) if there exist a neighbourhood $U$ of $e(x) \in N^n$ and a homeomorphism $h$ of $U$ onto $\mathbb{R}^n$ such that:

(1) $h(U \cap e(M^m)) = \mathbb{R}^m \subset \mathbb{R}^n$ when $x \in int\, M^m$ or

(2) $h(U \cap e(M^m)) = \mathbb{R}^m_+ \subset \mathbb{R}^n$ when $x \in \partial M^m$.

Since tameness implies local flatness for embeddings of manifolds in all co-dimensionals except two, we will say that $e\colon M^m \to N^n, m \neq n-2$ is *wild* at $e(x)$ when $e(M^m)$ is fails to be locally flat at $e(x)$.

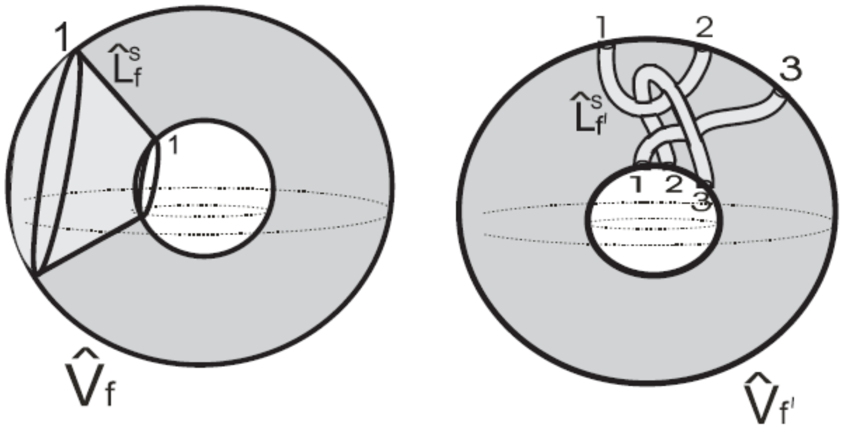


**Fig. 3** The complete invariants for trivial and non-trivial Pixton's diffeomorphisms

**Proposition 4** (Proposition 1.3.1 [3])**.** *Suppose the manifold $M^{n-1}$ is locally flatly embedded in the n-manifold $N^n$. Then $M^{n-1}$ is k-LCC in $N^n$ for all $k \geq 1$.*

**Proposition 5** (Proposition 1.3.6 [3])**.** *Suppose $Y$ is a locally contractible space and $A \subset X$. Then $A$ is k-LCC in $X$ iff $A \times Y$ is k-LCC in $X \times Y$.*

Notice that any manifold is a locally contractible space.

## 3 Wildness of the stable manifold of the saddle periodic orbit for the suspension

**Proof of Theorem 1.**

Let $f$ be a non-trivial Pixton's diffeomorphism. Then the closure of the stable manifold $W^s_\sigma$ of the saddle point $\sigma$ is a wild 2-sphere in $S^3$ and it is not 1-LCC at a source $\alpha$. By the construction the circle $\sigma \times S^1$ in $S^3 \times S^1$ coincides with the saddle periodic orbit $\mathscr{O}_\sigma$ for the suspension $f^t$ of the diffeomorphism $f$. Moreover, the closure of stable separatrice $W^s_{\mathscr{O}_\sigma}$ of $\mathscr{O}_\sigma$ coincides with $cl(W^s_\sigma) \times S^1$ and it is a 3-manifold homeomorphic to $S^2 \times S^1$. Due to Proposition 5 the set $cl(W^s_{\mathscr{O}_\sigma})$ is not 1-LCC in $S^3 \times S^1$. Thus, by Proposition 4, $W^s_{\mathscr{O}_\sigma}$ is wild.

## 4 Topological classification of suspensions

Firstly we give a brief idea of the topological classification of diffeomorphisms from class $\mathscr{P}$.

### 4.1 Classification of diffeomorphisms from $\mathscr{P}$

Let $f \in \mathscr{P}$ and $V_f = W^u_\alpha \setminus \alpha$. Denote by $\hat{V}_f$ the orbit space with respect to $f$ in $V_f$ and by $p_f : V_f \to \hat{V}_f$ the natural projection. According to [5], the space $\hat{V}_f$ is diffeomorphic to $S^2 \times S^1$ and the projection $p_f$ is a covering map which induces an epimorphism $\eta_f : \pi_1(\hat{V}_f) \to \mathbb{Z}$. Let $\hat{L}^s_f = p_f(W^s_\sigma \setminus \sigma)$. According to [5], $\hat{L}^s_f$ is a homotopically non-trivial 2-dimensional torus in $\hat{V}_f$ (see Figure 4).

**Proposition 6** (Theorem 4.5 [5])**.** *Diffeomorphisms $f, f' \in \mathscr{P}$ are topologically conjugated iff the tori $\hat{L}^s_f, \hat{L}^s_{f'}$ are equivalent (that is there is a homeomorphism $\hat{h} : \hat{V}_f \to \hat{V}_{f'}$ such that $\hat{h}(\hat{L}^s_f) = \hat{L}^s_{f'}$ and $\eta_f = \eta_{f'}\hat{h}_*$).*

### 4.2 Proof of the sufficiency of Theorem 3

Let $f, f' \in \mathscr{P}$. Recall the notion of the suspensions of $f, f'$.

Let $\varphi^t$ be a flow on the manifold $S^3\times\mathbb{R}$ generated by the unite vector field parallel to $\mathbb{R}$ and directed to $+\infty$, that is

$$\varphi^t(x,r)=(x,r+t).$$

Let $g,g':S^3\times\mathbb{R}\to S^3\times\mathbb{R}$ be diffeomorphisms given by the formulas $g(x,r)=(f(x),r-1)$, $g'(x,r)=(f'(x),r-1)$. Let $G=\{g^k,\ k\in\mathbb{Z}\}$, $G'=\{g'^k,\ k\in\mathbb{Z}\}$ and $W=(S^3\times\mathbb{R})/G$, $W'=(S^3\times\mathbb{R})/G'$. Since $f,f'$ preserve orientation of $S^3$, $W,W'$ are diffeomorphic to $S^3\times S^1$. Denote $p_{_W}:S^3\times\mathbb{R}\to W$, $p_{_{W'}}:S^3\times\mathbb{R}\to W'$ the natural projections. It is verified directly that $g\varphi^t=\varphi^t g$, $g'\varphi^t=\varphi^t g'$. Then maps $f^t:W\to W$, $f'^t:W'\to W'$ given by the formulas $f^t(x)=p_{_W}(\varphi^t(p_{_W}^{-1}(x)))$, $f'^t(x)=p_{_{W'}}(\varphi^t(p_{_{W'}}^{-1}(x)))$ are well-defined flows on $W,W'$ which are called the suspensions of $f,f'$, respectively, that is $f^t,f'^t\in\mathscr{P}^t$.

Now let $f,f'\in\mathscr{P}$ be topologically conjugate by the homeomorphism $h:S^3\to S^3$. Define a homeomorphism $\tilde H:S^3\times\mathbb{R}\to S^3\times\mathbb{R}$ by the formula

$$\tilde H(x,r)=(h(x),r),\ (x,r)\in S^3\times\mathbb{R}.$$

Directly verifies that $\tilde Hg=g'\tilde H$, then $\tilde H$ can be projected as a homeomorphism $H:W\to W'$ by the formula

$$H=p_{_{W'}}\tilde H p_{_W}^{-1}.$$

Since $\tilde H\varphi^t=\varphi'^t\tilde H$, then $Hf^t=f'^tH$. Thus $H$ is a required homeomorphism which realizes an equivalency of the suspensions $f^t$ and $f'^t$.

### 4.3 Proof of necessity of Theorem 3

Let suspensions $f^t,f'^t$ be topologically equivalent by means of a homeomorphism $H:S^3\times S^1\to S^3\times S^1$. Let us prove that then the diffeomorphisms $f,f'$ are topologically conjugate.

For this aim recall that the diffeomorphisms $f,f'$ in the basins of sources $\alpha,\alpha'$ are topologically conjugate by homeomorphisms $h_\alpha:W^u_\alpha\to\mathbb{R}^3$, $h_{\alpha'}:W^u_{\alpha'}\to\mathbb{R}^3$ with the linear extension $a\colon\mathbb{R}^3\to\mathbb{R}^3$ given by the formula

$$a(x_1,x_2,x_3)=(2x_1,2x_2,2x_3).$$

Let $S^r=\{(x_1,x_2,x_3)\in\mathbb{R}^3:x_1^2+x_2^2+x_3^2=2^r,r\in\mathbb{R}\}$, $S^r_\alpha=h_\alpha^{-1}(S^r)$ and $S^r_{\alpha'}=h_{\alpha'}^{-1}(S^r)$. Define cylinders $\tilde\Sigma,\tilde\Sigma'\subset S^3\times\mathbb{R}$ by the formulas

$$\tilde\Sigma=\{(x,r)\in S^3\times\mathbb{R}:x\in S^r_\alpha,r\in\mathbb{R}\},\tilde\Sigma'=\{(x,r)\in S^3\times\mathbb{R}:x\in S^r_{\alpha'},r\in\mathbb{R}\}.$$

It follows from the definition of suspension that $\tilde\Sigma,\tilde\Sigma'$ are sections for trajectories of $\varphi^t$, $\varphi'^t$ passing through $V_{\varphi^t},V_{\varphi'^t}$, where $V_{\varphi^t}=W^u_{O_\alpha}\setminus O_\alpha$ and $V_{\varphi'^t}=W^u_{O_{\alpha'}}\setminus O_{\alpha'}$ and $W^u_{O_\alpha},W^u_{O_{\alpha'}}$ are unstable manifolds of orbits $O_\alpha,O_{\alpha'}$ of flows $\varphi^t$, $\varphi'^t$ respectively. Let $V_{f^t}=W^u_{\mathscr{O}_\alpha},V_{f'^t}=W^u_{\mathscr{O}_{\alpha'}}$. Then $\Sigma=p_{_W}(\tilde\Sigma)$, $\Sigma'=p_{_{W'}}(\tilde\Sigma')$ are homeomorphic to $S^2\times S^1$ and are sections for trajectories of flows $f^t,f'^t$ in $V_{f^t},V_{f'^t}$, respectively.

Since $H$ realizes an equivalence of the flows $f^t,f'^t$ then $H(\Sigma)$ is also a section for trajectories of the flows $f'^t$ in $V_{f'^t}$. Thus we can get $\Sigma'$ from $H(\Sigma)$ by a continuous shift along the trajectories, that is there is a homeomorphism $\psi:V_{f'^t}\to V_{f'^t}$ which preserves the trajectories of $f'^t$ in $V_{f'^t}$ and such that $\psi(H(\Sigma))=\Sigma'$. Let $h_\Sigma=\psi H|_\Sigma:\Sigma\to\Sigma'$.

Then the homeomorphism $h_\Sigma$ has a lift $h_{\tilde\Sigma}:\tilde\Sigma\to\tilde\Sigma'$ which is a homeomorphism such that $h_\Sigma=p_{_{W'}}h_{\tilde\Sigma}p_{_W}^{-1}$. Let us introduce the canonical projection $q:S^3\times\mathbb{R}\to S^3$ by the formula $q(x,r)=x$ and define a homeomorphism $h:V_f\to V_{f'}$ by the formula

$$hq|_{\tilde\Sigma}=qh_{\tilde\Sigma}.$$

By the construction the homeomorphism $h$ conjugates $f|_{V_f}$ with $f'|_{V_{f'}}$. Since $H(W^s_{\mathscr{O}_\sigma})=W^s_{\mathscr{O}_{\sigma'}}$ then $h(W^s_\sigma\setminus\sigma)=W^s_{\sigma'}\setminus\sigma'$. Let us define a homeomorphism $\hat h:\hat V_f\to\hat V_{f'}$ by the formula

$$\hat h p_{_f}=p_{_{f'}}h.$$

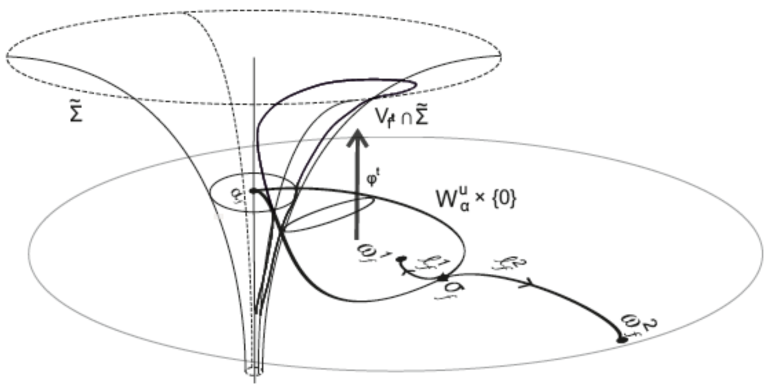


**Fig. 4** The vector field and $\tilde{\Sigma}$ on $\mathbb{S}^3 \times \mathbb{R}$

Then $\hat{h}(\hat{L}^s_f) = \hat{L}^s_{f'}$ and $\eta_f = \eta_{f'}\hat{h}_*$. Thus, by Proposition 6, the diffeomorphisms $f, f'$ are topologically conjugated.

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