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On classes of infinite loaded graphs with randomly deleted edges

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Abstract

For a collection of infinite loaded graphs, random perturbations of special type are considered. It is shown that some known classes of these graphs are stable with respect to small random perturbations of this type, while the rest are not.

Keywords: directed graph, infinite loaded graph, mixing transformation, classification of loaded graphs, random perturbation of loaded graph

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1 Introduction

A well-known classification of the infinite loaded graphs, or, which is the same, the infinite non-negative matrices, is very close to that of the Markov chains with infinitely many states: there are recurrent and transient loaded graphs, null recurrent and positive recurrent ones, while the latter can be stable positive and unstable positive. Distinctions between the last two classes (for details see [1]) justify their names. The aim of this note is to show by simple examples what happens if one deletes some edges of the graph, using a random mechanism. It turns out that some of the above classes are stable with respect to small random perturbations, while the others are not.

2 Basic definitions, description of the model

Consider a directed graph $G = (V, E)$, where $V = V(G)$ is the set of its vertices and $E = E(G) \subset V \times V$ the set of edges. A path of length n in G is a sequence $\gamma = e_1, \dots, e_n$ of edges such that the beginning of the edge

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e_{i+1} coincides with the end of the edge e_i , $1 \leq i < n$. Every function $\mathscr{W} : E \rightarrow (0, \infty)$ is called a weight function, and the pair (G, \mathscr{W}) is said to be a *loaded graph*. The weight $\mathscr{W}(\gamma)$ of a path γ is defined as the product of the weights over the edges forming γ . The graph G is assumed to be connected: for every pair of vertices $u, v \in V(G)$ there exists a path from u to v .

Recall a classification of the loaded graphs (see [1], [2] for details). For every vertex $v \in V(G)$, let $\Gamma_{v,v}$ denote the set of all v -cycles, i.e., the paths γ from v to v , and $\Gamma_{v,v|v}$ denote the set of those $\gamma \in \Gamma_{v,v}$ in which v is only the first and last vertex. We refer to elements of $\Gamma_{v,v|v}$ as *primitive v -cycles*.

For a fixed $v \in V(G)$, denote by $q_n = q_n(G, \mathscr{W}, v)$ the sum of the weights over all primitive v -cycles of length n , and assume that $q_n < \infty$ for all n . Introduce the generating function

$$\Phi_{v,v|v}(G, \mathscr{W}, t) := \sum_{n=1}^{\infty} q_n t^n. \quad (1)$$

Let $R_v = R(G, \mathscr{W}, v)$ be the radius of convergence of this power series; assume that $R_v > 0$.

Definition 1. A loaded graph (G, \mathscr{W}) is said to be *stable positive*, *unstable positive*, and *null recurrent* if there is a vertex $v \in V(G)$ such that

$$\Phi_{v,v|v}(G, \mathscr{W}, R_v) > 1, \quad (2)$$

$$\Phi_{v,v|v}(G, \mathscr{W}, R_v) = 1, \quad \frac{d}{dt} \Phi_{v,v|v}(G, \mathscr{W}, t)|_{t=R_v} < \infty, \quad (3)$$

and

$$\Phi_{v,v|v}(G, \mathscr{W}, R_v) = 1, \quad \frac{d}{dt} \Phi_{v,v|v}(G, \mathscr{W}, t)|_{t=R_v} = \infty, \quad (4)$$

respectively. Stable positive and unstable positive loaded graphs form the set of *positive recurrent* loaded graphs, while positive recurrent and null recurrent loaded graphs form the set of *recurrent* ones. The remaining loaded graphs with $R_v > 0$ are said to be *transient*.

Since G is connected, the following is true: if $R_v > 0$ for some v , then $R_v > 0$ for all $v \in V(G)$, and if some of the conditions (2)–(4) holds for a given v , then it does for all $v \in V(G)$.

In [1], §3 several characteristic properties of stable recurrent and unstable recurrent loaded graphs are collected and this terminology is justified.

Let us now describe the loaded graphs (G, \mathscr{W}) that will be the subject of further studies in this paper. For $V = V(G)$ we take \mathbb{N} , the set of positive integers, and for $E = E(G)$ the union $E_1 \cup E_2 \cup E_3$, where E_1 is the collection of all pairs $(i, i+1)$, $i \in \mathbb{N}$; E_2 is the collection of the pairs $(i, 1)$, $i \in \mathbb{N}$, and E_3 is formed by some pairs of the form (i, j) , where $i, j \in \mathbb{N}$, $i < j-1$. We will refer to such a graph as *linear*. Suppose the following *bounded forward jump* (BFJ) condition is satisfied: there is $d > 0$ such that $j - i \leq d$ for all $(i, j) \in E_3$.

Notice that in [4], Example 5.6, a close class of loaded graphs was introduced in other terms and for another purpose.

Given an arbitrary linear graph G , one can choose, for each of the above classes of loaded graphs, a weight function \mathscr{W} in such a way that the loaded graphs (G, \mathscr{W}) will fall into this class.

For a fixed loaded graph (G, \mathscr{W}) we consider a random loaded graph (G, \mathscr{W}, ξ) that differs from (G, \mathscr{W}) in that the weight of every edge of the form $(i, 1) \in E_2$ is multiplied by a random variable ξ_i with values in $\{0, 1\}$, i.e., remains intact or disappears under the action of some stochastic mechanism. Let us assume that this mechanism is described by a stationary random sequence $\xi := (\xi_i, i \in \mathbb{Z})$ with decay of long-range dependence.

3 Classes of random linear loaded graphs

We begin with the following notion from ergodic theory.

An invertible measure preserving transformation T (automorphism) of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is said to be *lightly mixing* (see [5]) if

$$\liminf_{n \rightarrow \infty} \mathbf{P}(A \cap T^{-n}B) > 0$$

for all $A, B \in \mathcal{F}$ with $P(A) > 0$, $P(B) > 0$. The next lemma is due to Blum, Christiansen and Quiring [6], who used another terminology.

Lemma 1. *If T is a lightly mixing automorphism of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then for every infinite increasing sequence of positive integers n_k , $k = 0, 1, \dots$, and every set $F \in \mathcal{F}$ with $\mathbf{P}(F) > 0$, we have*

$$\mathbf{P}(\cup_{k=0}^{\infty} T^{-n_k} F) = 1. \quad (5)$$

Remark 1. (see [6]) The automorphisms for which (5) holds are sometimes called sequence mixing or sweeping out. This property is known to be equivalent to the lightly mixing one.

If the shift in the sample space of a stationary process is lightly mixing, we say that the process itself is lightly mixing.

Corollary 2. *If a stationary random process $\xi = \{\xi_i, i \in \mathbb{Z}_+\}$ with values 0 and 1 is lightly mixing, then for every increasing infinite sequence of positive integers r_i and every $r \in \mathbb{N}$, there exists with probability 1 an i such that $r_i > r$ and $\xi_{r_i} = 1$.*

Proof. Let Ω be the sample space of the process ξ (Ω consists of the double-sided infinite sequences of zeroes and ones), T the one step left shift in Ω , \mathcal{F} the cylinder σ -algebra in Ω and \mathbf{P} the probability measure on (Ω, \mathcal{F}) induced by the process ξ . Let us assume the process ξ is defined on $(\Omega, \mathcal{F}, \mathbf{P})$ as follows: if $\omega = (\omega_i, i \in \mathbb{Z}) \in \Omega$, then $\xi_n(\omega) = \omega_n$, $n \in \mathbb{Z}$. Put $F := \{\omega \in \Omega : \omega_0 = 1\}$. Suppose that the automorphism T of the space $(\Omega, \mathcal{F}, \mathbf{P})$ is lightly mixing. Then the desired statement follows from Lemma 1 applied to F and the numbers $n_k := r_{i_0+k}$, where $i_0 = \min\{i : r_i > r\}$.

We now come back to the random linear loaded graph described in Section 2. For such a graph we will write $\Phi_{v,v|v}^{(\xi)}(G, \mathcal{W})$, $R_v^{(\xi)}$, and $q_n(\xi)$ instead of $\Phi_{v,v|v}(G, \mathcal{W})$, R_v and q_n , respectively. Below it is assumed that $v = 1$.

Theorem 3. *Let 1) a linear graph G satisfies the BFJ condition, 2) a weight function \mathcal{W} is bounded, and 3) a stationary random process ξ with values 0, 1 induces a lightly mixing shift. Then for the random linear loaded graph (G, \mathcal{W}, ξ) we have $\mathbf{P}(R_v^{(\xi)} = R_v) = 1$.*

Proof. Let us first consider the linear loaded graph (G, \mathcal{W}) . By the Cauchy-Hadamard formula there is a sequence of positive integers n_k such that

$$1/R_v = \limsup_{n \rightarrow \infty} (q_n)^{1/n} = \lim_{k \rightarrow \infty} (q_{n_k})^{1/q_{n_k}}. \quad (6)$$

The last expression will not change if we take an arbitrary infinite subsequence of $\{n_k\}$ instead of $\{n_k\}$ itself. By Corollary 2 there exists with probability one an infinite subsequence $\{n'_k\}$ such that $\xi_{n'_k} = 1$ for every k . Hence, by (6), with probability one

$$1/R_v^{(\xi)} = \limsup_{k \rightarrow \infty} (q_{n'_k}(\xi))^{1/n'_k} \geq 1/R_v.$$

But it is clear that the converse inequality holds, since $q_n(\xi) \leq q_n$ for all n . Hence $R_v^{(\xi)} = R_v$ with probability 1, which proves the theorem.

Let us now find out how $\Phi_{v,v|v}^{(\xi)}(G, \mathcal{W}, R_v^{(\xi)})$ behaves when the process ξ can be treated as a small random perturbation of the process that identically equals one. Consider a sequence of stationary random processes $\xi^{(k)} = \{\xi_i^{(k)}, i \in \mathbb{Z}\}$ defined on the above-described sequence space (Ω, \mathcal{F}) , and for $k = 1, 2, \dots$, introduce the

probability measure $\mathbf{P}^{(k)}$, corresponding to $\xi^{(k)}$. Assume that there exists an increasing sequence of positive integers r_k such that

$$\lim_{k \rightarrow \infty} \mathbf{P}^{(k)}(\xi_1^{(k)} = \dots = \xi_{r_k}^{(k)} = 1) = 1. \quad (7)$$

This will be referred to as the *dominating 1* (D 1) condition. If $\xi_i^{(k)}$, for every k , are independent random variables such that $\xi_i^{(k)} = 1$ with probability p_k and $\xi_i^{(k)} = 0$ with probability $1 - p_k$, then the D 1 condition is satisfied when $\lim_{k \rightarrow \infty} p_k = 1$. It is easy to define a sequence of stationary Markov chains for which it is also obeyed. But there are Markov chains, even mixing ones, without this property. On the other hand, it is obeyed by some non-Markovian stationary processes.

Theorem 4. Let (G, \mathscr{W}) be a stable positive loaded graph, where G is a linear graph defined in § 2, $\{\xi^{(n)}\}$ a sequence of lightly mixing stationary random processes with values 0 and 1 satisfying the D 1 condition, and $\{(G, \mathscr{W}, \xi^{(n)})\}$ the corresponding sequence of random loaded graphs. Then the random loaded graph $(G, \mathscr{W}, \xi^{(n)})$ is stable positive with probability one as n is large enough.

Proof. In what follows we mean that all relations between random variables hold with probability one. Since (G, \mathscr{W}) is stable positive, we have $\sum_{n=1}^{\infty} q_n \cdot (R_v)^n > 1$ (see (1) and Definition 1), and one can choose an $n_0 \in \mathbb{N}$ such that $\sum_{n=1}^{n_0} q_n \cdot (R_v)^n > 1$. The D 1 condition enables one to find k with $r_k \geq n_0$ (see (7)). Then $q_n(\xi^{(k)}) = q_n$ for all $n \leq n_0$, and hence $\sum_{n=1}^{n_0} q_n(\xi^{(k)}) \cdot (R_v)^n > 1$. From Theorem 3 we know that $R_v^{(\xi^{(k)})} = R_v$. Therefore,

$$\Phi_{v,v|v}(G, \mathscr{W}, R_v^{(\xi^{(k)})}) = \sum_{n=1}^{\infty} q_n(\xi^{(k)}) \cdot (R_v^{(\xi^{(k)})})^n > 1,$$

i.e., the random loaded graph $(G, \mathscr{W}, \xi^{(k)})$ is stable positive with probability one.

Remark 2. It is easy to see that if (G, \mathscr{W}) is unstable positive, null recurrent, or transient, while $\xi = (\xi_i, i \in \mathbb{Z})$ is a lightly mixing stationary random process with values 0 and 1 such that $\mathbf{P}(\xi_i = 0) > 0$, then the random loaded graph (G, \mathscr{W}, ξ) is transient with probability one (in view of Theorem 3). Thus among the linear loaded graphs we defined in Section 2 only the stable positive and transient classes are stable with respect to the small random perturbations of the above type.

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