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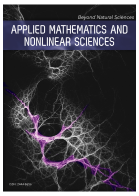

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Asymptotic behavior of solutions to barotropic vorticity equation on a sphere

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Abstract

The behavior of a viscous incompressible fluid on a rotating sphere is described by the nonlinear barotropic vorticity equation (BVE). Conditions for the existence of a bounded set that attracts all BVE solutions are given. In addition, sufficient conditions are obtained for a BVE solution to be a global attractor. It is shown that, in contrast to the stationary forcing, the dimension of the global BVE attractor under quasiperiodic forcing is not limited from above by the generalized Grashof number.

Keywords: Barotropic vorticity equation on a sphere, global attractive set, global asymptotic stability, dimension of global attractor.
AMS 2010 codes: 76D17, 76E20, 76U05, 39A14.

1 Introduction

Let us denote by $\mathbb{C}_0^\infty(S)$ the space of infinitely differentiable functions on the unit sphere $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ which are orthogonal to any constant, and by

$$\langle f,g\rangle = \int_S f(x)\overline{g(x)}dS \quad \text{and} \quad \|f\| = \langle f,f\rangle^{1/2} \tag{1}$$

the inner product and the norm of functions of $\mathbb{C}_0^\infty(S)$, respectively. Denote by $Y_n^m(\lambda,\mu)$ the spherical harmonics $(n \geq 1, \ \ |m| \leq n)$ that form the orthonormal system in $\mathbb{C}_0^\infty(S)$: $\langle Y_n^m, Y_l^k\rangle = \delta_{mk}\delta_{nl}$ where δ_{mk} is the Kronecker delta. Each spherical harmonic Y_n^m is the eigenfunction of the eigenvalue problem

$$-\Delta Y_n^m = \chi_n Y_n^m\,, \quad |m| \leq n$$

for symmetric and positive definite spherical Laplace operator, where

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$$\chi_n = n(n+1) \tag{2}$$

Lemma 1. *[10] Let n be a natural, and $K_n = \left[\frac{2n+1}{4\pi}\right]^{1/2}$. Then*

$$\sum_{m=-n}^{n} |Y_n^m(x)|^2 = K_n^2 \quad \text{and} \quad \sum_{m=-n}^{n} |\nabla Y_n^m(x)|^2 = \chi_n K_n^2 \ .$$

Let $n \geq 1$. Denote by

$$\mathbf{H}_n = \{\psi : -\Delta\psi = \chi_n \psi\}$$

the $(2n+1)$-dimensional eigensubspace of homogeneous spherical polynomials of degree n, which corresponds to the eigenvalue (2) [8]. The set $2n+1$ of spherical harmonics $Y_n^m(\lambda,\mu)$ $(-n < m \leq n)$ form the orthonormal basis in $\mathbf{H}_n$. The Hilbert space $\mathbb{L}_0^2(S) = \oplus_{n=1}^{\infty}\mathbf{H}_n$ being the direct orthogonal sum of the subspaces $\mathbf{H}_n$ is the closure of $\mathbb{C}_0^\infty(S)$ in the norm (1). We denote by $\mathrm{Y}_n(\psi)$ the projection of function $\psi \in \mathbb{L}_0^2(S)$ onto $\mathbf{H}_n$. Thus, $\psi = \sum_{n=1}^{\infty} \mathrm{Y}_n(\psi)$ for every function $\psi(x) \in \mathbb{L}_0^2(S)$.

Let $\psi(x) \in \mathbb{C}_0^\infty(S)$, and $\chi_n = n(n+1)$. We introduce the derivative $\Lambda^s = (-\Delta)^{s/2}$ of real degree s of $\psi(x)$ as

$$\Lambda^s \psi(x) = \sum_{n=1}^{\infty} \chi_n^{s/2} \mathrm{Y}_n(\psi(x)) \tag{3}$$

[11, 12].

Let s be a real. Denote by $\mathbb{H}_0^s$ the Hilbert space obtained by closing the space $\mathbb{C}_0^\infty(S)$ in the norm

$$\|\psi\|_s = \left\{\sum_{n=1}^{\infty} \chi_n^s \|\mathrm{Y}_n(\psi)\|^2\right\}^{1/2} .$$

The inner product in $\mathbb{H}_0^s$ is defined as $\langle\psi,h\rangle_s = \langle\Lambda^s\psi,\Lambda^s h\rangle$. Thus, $\|\psi\|_s = \langle\psi,\psi\rangle_s^{1/2}$. It can be shown that $\|\nabla\psi\| = \|\Lambda\psi\|$ [10].

Lemma 2. *[11] Let r, s and t be real numbers, $r < t$, $a = \sqrt{2}$, and $\psi \in \mathbb{H}_0^{s+t}$. Then*

$$\|\Lambda^r\psi\|_s \leq a^{r-t}\left\|\Lambda^t\psi\right\|_s \quad \text{and} \quad \|\psi\|_{s+t} = \left\|\Lambda^t\psi\right\|_s \ .$$

2 Existence and uniqueness of the BVE solutions

Let us consider the non-stationary nonlinear BVE problem on S [11]:

$$\Delta\psi_t + J(\psi, \Delta\psi + 2\mu) + \sigma\Delta\psi - \nu(-\Delta)^{s+1}\psi = F \tag{4}$$

$$\Delta\psi(0,x) = \Delta\psi_0(x) \ . \tag{5}$$

Here ψ is the stream function, $\Delta\psi$ is the relative vorticity, $\Delta\psi + 2\mu$ is the absolute vorticity, $F(t,x)$ is the forcing, $\sigma\Delta\psi$ is the Rayleigh friction in the planetary boundary layer,

$$J(\psi,h) = (\vec{n} \times \nabla\psi) \cdot \nabla h$$

is the Jacobian determinant, $\overrightarrow{n}$ is the outward unit normal vector to the surface of the sphere S, while the nonlinear term $J(\psi,\Delta\psi)$ and linear term $J(\psi,2\mu) = 2\psi_\lambda$ represent the advection and the rotation of sphere, respectively. The velocity vector $\overrightarrow{v} = \overrightarrow{n} \times \nabla\psi$ is nondivergent: $\nabla \cdot \overrightarrow{v} = 0$. The viscosity is modeled by the term $\nu(-\Delta)^{s+1}\psi$, where $-\Delta$ is the spherical Laplace operator and $s \geqslant 1$ is a real number.

Note that if ψ is a BVE solution then $\psi + const$ is also the solution. We ignore this constant by searching a solution in spaces of functions orthogonal to a constant on the sphere. Spaces of functions in which a solution exists are important in many applications, and, in particular, in studying the stability of solutions.

In this section, we formulate two theorems proved in [11].

Theorem 1. *[11] Let $s \geq 1$, $\nu > 0$ and $\sigma \geq 0$. Suppose that $\Delta\psi_0 \in \mathbb{L}_0^2(S)$ at initial moment, and $F(t,x) \in \mathbb{L}^2(0,T;\mathbb{H}_0^s)$. Then the non-stationary BVE problem (4)-(5) has a unique weak solution $\psi(t,x) \in \mathbb{L}^\infty(0,T;\mathbb{H}_0^2)$ such that*

$$\psi(t,x) \in \mathbb{L}^\infty(0,T;\mathbb{H}^0) \cap \mathbb{L}^2(0,T;\mathbb{H}_0^s),$$
$$\Delta\psi_t \in \mathbb{L}^2(0,T;\mathbb{H}_0^{-s}), \quad \Delta\psi(0,x) = \Delta\psi_0(x)$$

and

$$\int_0^t [\langle\Delta\psi_t,h\rangle - \langle J(\psi,h),\Delta\psi+2\mu\rangle + \sigma\langle\Delta\psi,h\rangle]\,dt - \nu\int_0^t \langle\Lambda^{s+2}\psi,\Lambda^s h\rangle\,dt = \int_0^t \langle F,h\rangle\,dt$$

holds for all $t \in (0,T)$ and $h \in \mathbb{L}^2(0,T;\mathbb{H}_0^s)$.

Theorem 2. *[11] Let $s \geq 1$, $\nu > 0$ and $\sigma \geq 0$. Suppose that $F(x) \in \mathbb{H}_0^s$. Then there exists at least one weak solution $\psi(x) \in \mathbb{H}_0^{s+2}$ of the stationary equation*

$$J(\psi,\Delta\psi+2\mu) + \sigma\Delta\psi - \nu(-\Delta)^{s+1}\psi = F(x) \tag{6}$$

such that

$$\nu\langle\Lambda^{s+2}\psi,\Lambda^s h\rangle - \sigma\langle\Delta\psi,h\rangle + \langle J(\psi,h),\Delta\psi+2\mu\rangle = \langle F,h\rangle$$

holds for all $h \in \mathbb{H}_0^s$. If additionally

$$\nu^2 > 2^{1-s}M\|F(x)\|_{-s}$$

then the problem solution is unique.

Here M is the constant from the estimate $|J(\psi,h)| \leq M\|\Delta\psi\|\,\|\Delta h\|$ (see [11]). The case $s = 1$ and $\sigma = 0$ was proved in [4,5], whilst the cases $s = 1$ and $s = 2$ ($\sigma \neq 0$) were proved in [10]. Theorem 2 considers the general case when $s \geq 1$ is a real number.

3 Existence of a limited attractive set

Let us study the asymptotic behavior of the BVE solutions as $t \to \infty$.

Theorem 3. *Let $s \geq 1$, and let $F(x) \in \mathbb{H}_0^r$ be a stationary forcing of equation (4), $r \geq 1$. Then there is a limited set $\mathbf{B}$ in a space $\mathbf{X}$ that attracts all BVE solutions $\psi(t,x)$, besides,*

1. *if $r \geq 0$ then $\mathbf{X} = \mathbb{H}_0^2$ and*
$$\mathbf{B} = \{\psi \in \mathbb{H}_0^2 : \|\psi\|_2 \leq C_1(r,s)\|F\|_r \quad . \tag{7}$$
2. *if $r \in [-1,0)$ then $\mathbf{X} = \mathbb{H}_0^1$ and*
$$\mathbf{B} = \{\psi \in \mathbb{H}_0^1 : \|\psi\|_1 \leq C_2(r,s)\|F\|_r \tag{8}$$

where

$$C_1(r,s) = \frac{a^{-r}}{\sigma + 2^s\nu}\ , \quad C_2(r,s) = \frac{a^{-r-1}}{\sigma + 2^s\nu}\ , \quad a = \sqrt{2}\ .$$

Proof. Part 1. Let $r \geq 0$ and $F(x) \in \mathbb{H}_0^r$. The inner product (1) of equation (4) with $\Delta\psi$ and the use of relations

$$\langle J(g,\ \mu), \Lambda^r g\rangle = 0 \quad \text{and} \quad \langle J(g,h), g\rangle = 0 \tag{9}$$

valid for real functions due to relations

$$\langle J(\psi,\ g), h\rangle = \langle J(g,h), \psi\rangle = -\langle J(\psi,\ h),\ g\rangle$$

[12], imply

$$\begin{aligned}\langle \Delta\psi_t, \Delta\psi\rangle &= -\sigma\langle \Delta\psi, \Delta\psi\rangle + \nu\left\langle (-\Delta)^{s+1}\psi, \Delta\psi\right\rangle + \langle F, \Delta\psi\rangle \\ &= -\sigma\|\Delta\psi\|^2 - \nu\left\|\Lambda^{s+2}\psi\right\|^2 + \langle F, \Delta\psi\rangle\ .\end{aligned} \tag{10}$$

Due to Lemma 2,

$$|\langle F, \Delta\psi\rangle| \leq \|F\|\,\|\Delta\psi\| \leq a^{-r}\|F\|_r\|\Delta\psi\| \quad \text{and} \quad \nu\left\|\Lambda^{s+2}\psi\right\|^2 \geq 2^s\nu\|\Delta\psi\|^2\ .$$

We denote

$$\rho = \sigma + 2^s\nu\ . \tag{11}$$

The use of the obtained inequalities in (10) leads to

$$\frac{\partial}{\partial t}\|\Delta\psi\| \leq -\rho\|\Delta\psi\| + a^{-r}\|F\|_r\ .$$

The last inequality gives

$$\|\Delta\psi(t)\| \leq \|\Delta\psi(0)\|\exp(-\rho t) + \frac{a^{-r}}{\rho}\|F\|_r[1-\exp(-\rho t)] \tag{12}$$

and hence,

$$\|\psi(t)\|_2 \to C_1(r,s)\|F\|_r \quad \text{as} \quad t \to \infty\ .$$

Part 2. Let $r \in [-1,0)$ and $F(x) \in \mathbb{H}_0^r$. The inner product (1) of equation (4) with ψ and the use of (9) give

$$\langle \Lambda\psi_t, \Lambda\psi\rangle = -\sigma\|\Lambda\psi\|^2 - \nu\left\|\Lambda^{s+1}\psi\right\|^2 - \langle F, \psi\rangle\ . \tag{13}$$

Applying Lemma 2 to the terms $\langle F, \psi\rangle$ and $\nu\left\|\Lambda^{s+1}\psi\right\|^2$ leads to

$$|\langle F, \psi\rangle| = \left|\left\langle \Lambda^r F, \Lambda^{-r}\psi\right\rangle\right| \leq \|\Lambda^r F\|\left\|\Lambda^{-r}\psi\right\| \leq a^{-r-1}\|F\|_r\|\Lambda\psi\| \tag{14}$$

and

$$\nu\left\|\Lambda^{s+1}\psi\right\|^2 \geq 2^s\nu\|\Lambda\psi\|^2\ . \tag{15}$$

Then (13) implies

$$\frac{\partial}{\partial t}\|\Lambda\psi\| \leq -\rho\|\Lambda\psi\| + a^{-r-1}\|F\|_r \tag{16}$$

where ρ is defined by (11), or

$$\|\Lambda\psi(t)\| \leq \|\Lambda\psi(0)\|\exp(-\rho t) + \frac{a^{-r-1}}{\rho}\|F\|_r[1-\exp(-\rho t)] \ . \tag{17}$$

Since $\|\Lambda\psi\| = \|\psi\|_1$, we obtain

$$\|\psi(t)\|_1 \to C_2(r,s)\|F\|_r \quad \text{as} \ \ t\to\infty \ .$$

Q.E.D.

According to (12) and (17), if some solution ψ belongs to the set $\mathbf{B}$ at time t_0 then it will belong to $\mathbf{B}$ for all $t > t_0$. Hence, all steady and periodic solutions (if they exist) belong to the set $\mathbf{B}$. Evidently, the set $\mathbf{B}$ contains the maximal BVE attractor [14]. Theorem 3 is also valid if $F(t,x)\in C(0,\omega;\mathbb{H}_0^r)$ where ω is the period. In this case, one should only replace in (7) and (8) the norms $\|F\|_r$ by the norms $\max\limits_{t\in[0,\omega]}\|F\|_r$.

4 A functional for the stability study

We now introduce a positive functional for estimating arbitrary perturbations of the BVE solution. Let $\widetilde{\psi}(t,\lambda,\mu)$ be a real BVE solution. Then a perturbation $\psi'(t,\lambda,\mu)$ of the solution $\widetilde{\psi}$ satisfies the equation

$$\frac{\partial}{\partial t}\Delta\psi' + J\left(\psi',\Delta\widetilde{\psi}\right) + J(\widetilde{\psi},\Delta\psi') + 2\frac{\partial\psi'}{\partial\lambda} + J\left(\psi',\Delta\psi'\right) = -[\sigma+\nu\Lambda^{2s}]\Delta\psi' \tag{18}$$

where $s\geq 1$, $\nu>0$ and $\sigma\geq 0$. Taking the inner product (1) of equation (18) in series with ψ' and $\Delta\psi'$, then using (9), we obtain the integral equations

$$\frac{d}{dt}K + \left\langle J(\psi',\Delta\psi'),\widetilde{\psi}\right\rangle + 2\sigma K + \nu\left\|\Lambda^{s+1}\psi'\right\|^2 = 0 \tag{19}$$

and

$$\frac{d}{dt}\eta - \left\langle J(\psi',\Delta\psi'),\Delta\widetilde{\psi}\right\rangle + 2\sigma\eta + \nu\left\|\Lambda^{s+2}\psi'\right\|^2 = 0 \tag{20}$$

for the kinetic energy $K(t) = \frac{1}{2}\|\nabla\psi'(t,x)\|^2 = \frac{1}{2}\|\Lambda\psi'(t,x)\|^2$ and enstrophy $\eta(t) = \frac{1}{2}\|\Delta\psi'(t,x)\|^2$ of perturbation ψ', respectively.

One can see from (19) and (20) that the first Jacobian in (18) does not affect the behavior of the perturbation energy $K(t)$, while the second Jacobian in (18) does not affect the perturbation enstrophy $\eta(t)$. Moreover, the sphere rotation and nonlinear term (the last two terms in the LHS of (18)) do not affect the behavior of $K(t)$ and $\eta(t)$.

We will estimate the behavior of perturbations $\psi'(t,\lambda,\mu)$ of a basic flow $\widetilde{\psi}(t,\lambda,\mu)$ using the functional

$$Q(t) = pK(t) + q\eta(t) = \frac{1}{2}(p\left\|\nabla\psi'\right\|^2 + q\left\|\Delta\psi'\right\|^2)$$

where p and q are non-negative real numbers, not equal to zero simultaneously. Multiplying (19) and (20) by p and q, respectively, and combining the results, we obtain

$$\frac{d}{dt}Q(t) = -2\sigma Q(t) - R(t) - \nu p\left\|\Lambda^{s+1}\psi'\right\|^2 - \nu q\left\|\Lambda^{s+2}\psi'\right\|^2 \tag{21}$$

where

$$R(t) = \left\langle J(\psi',\Delta\psi'),\ p\widetilde{\psi} - q\Delta\widetilde{\psi}\right\rangle \ . \tag{22}$$

Applying Lemma 2 we get

$$-\left\|\Lambda^{s+1}\psi'\right\|^2 \leq -2^s\left\|\nabla\psi'\right\|^2, \qquad -\left\|\Lambda^{s+2}\psi'\right\|^2 \leq -2^s\left\|\Delta\psi'\right\|^2 \ .$$

Therefore, (21) leads to

$$\frac{d}{dt}Q(t) \leq -2\rho Q(t) - R(t) \tag{23}$$

where ρ is defined by (11).

Let us consider three examples when the basic solution is zero or represents meteorologically important flows, such as super-rotation, or a homogeneous spherical polynomial. Each basic solution is assumed to be supported by appropriate forcing.

Example 1. *Let $\widetilde{\psi} = 0$ (this solution exists if $F(x) \equiv 0$). Then $\langle J(\psi', \Delta\psi'), \widetilde{\psi}\rangle = 0$ and $\langle J(\psi', \Delta\psi'), \Delta\widetilde{\psi}\rangle = 0$ in (19) and (20). Therefore, in the non-dissipative case ($\sigma = \nu = 0$), the zero solution is stable, since the perturbation energy and enstrophy are constant. In the dissipative case ($\sigma \neq 0$ and/or $\nu \neq 0$), the zero solution is globally asymptotically stable, because the energy and enstrophy of any perturbation decrease exponentially with time.*

Example 2. *The basic flow is a super-rotation: $\widetilde{\psi} \equiv \widetilde{\psi}(\mu) = C\mu$, where $C = const$. Then $R(t) = 0$ due to (9), while $Q(t)$ is the Lyapunov function. Thus, the super-rotation flow is Lyapunov stable if $\sigma = \nu = 0$, and is the global BVE attractor (asymptotically Lyapunov stable) if $\rho > 0$. It is easy to prove that the same is true for any flow from subspace $\mathbf{H}_1$, since it represents a super-rotation flow about some axis of a sphere [10].*

Example 3. *The basic flow is a homogeneous spherical polynomial: $\widetilde{\psi} \in \mathbf{H}_n$ ($n \geq 2$):*

$$\widetilde{\psi}(t, \lambda, \mu) = \sum_{m=-n}^{n} \widetilde{\psi}_n^m(t) Y_n^m(\lambda, \mu) \ . \tag{24}$$

Then $J(\psi', \Delta\psi') = 0$ for any initial perturbation ψ' from the subspace $\mathbf{H}_n$, and $R(t) \equiv 0$. Besides, such a perturbation will never leave $\mathbf{H}_n$, i.e., the subspace $\mathbf{H}_n$ is the invariant set of perturbations to the polynomial flow (24). Moreover, due to (23), $Q(t) \leq Q(0)\ \exp(-2\rho t)$, and therefore any initial perturbation $\psi'(0, \lambda, \mu)$ from $\mathbf{H}_n$ will exponentially tend to zero with time, without leaving $\mathbf{H}_n$. In other words, the invariant set $\mathbf{H}_n$ belongs to the domain of attraction of solution (24).

5 Global Asymptotic Stability of BVE solutions

Let us obtain sufficient conditions for the BVE solution to be a global attractor.

First, assume that the basic solution $\widetilde{\psi}(t, \lambda, \mu)$ of equation (4) is rather smooth, such that

$$p = \sup_{t \geq 0} \max_{(\lambda,\mu) \in S} |\nabla\Delta\widetilde{\psi}(t, \lambda, \mu)| \quad \text{and} \quad q = \sup_{t \geq 0} \max_{(\lambda,\mu) \in S} |\nabla\widetilde{\psi}(t, \lambda, \mu)| \tag{25}$$

are bounded. Then using the inequality $|J(\psi, h)| \leq |\nabla\psi| \cdot |\nabla h|$ we get

$$|R(t)| = \left|\left\langle J(p\widetilde{\psi} - q\Delta\widetilde{\psi}, \psi'), \Delta\psi'\right\rangle\right| \leq 2pq \left\|\nabla\psi'\right\| \left\|\Delta\psi'\right\| \leq 2\sqrt{pq}Q(t) \ . \tag{26}$$

The use of inequality (26) in (23) leads to

Theorem 4. *Let $s \geq 1$, $\nu > 0$ and $\sigma \geq 0$. If*

$$\sigma + 2^s\nu > \sqrt{pq} \tag{27}$$

where p and q are defined by (25), then $Q(t)$ is the Lyapunov function, and solution $\widetilde{\psi}(t, \lambda, \mu)$ is a global BVE attractor, besides, any its perturbation ψ' will exponentially decrease with time.

Note that in a limited domain on the plane, the condition for the global asymptotic stability of a smooth BVE solution was earlier obtained in [13] under the condition that rotation and linear drag are not taken into account ($\sigma = 0$) and $s = 1$. Theorem 4 extends this result to smooth flows on a rotating sphere when $s \geq 1$, and the linear drag is also taken into account ($\sigma \neq 0$).

It should be emphasized that in both assertions, the basic solution $\widetilde{\psi}(t,\lambda,\mu)$ must have continuous derivatives up to the third order. But Theorems 1 and 2 guarantee the existence of a solution that has continuous derivatives only up to the second order. Theorem 5 of this section gives new global asymptotic stability conditions for smooth solution of equation (4), in which the requirement on the smoothness is weakened. These instability conditions are in full accordance with Theorem 1.

We now show that the restriction (25) on the smoothness of the basic solution can be weakened to be consistent with the solvability theorem (Theorem 1). To this end, let us consider a smooth solution $\widetilde{\psi}(t,\lambda,\mu)$ such that

$$p = \sup_t \max_{(\lambda,\mu)\in S} |\Delta\widetilde{\psi}(t,\lambda,\mu)| \quad \text{and} \quad q = \sup_t \max_{(\lambda,\mu)\in S} |\widetilde{\psi}(t,\lambda,\mu)| \tag{28}$$

are bounded. Using the ε-inequality one can estimate $R(t)$ as:

$$\begin{aligned} |R(t)| &\leq 2pq\,\|\nabla\psi'\|\,\|\nabla\Delta\psi'\| = 2pq\,\|\nabla\psi'\|\,\|\Lambda^3\psi'\| \leq 2pq\,\|\nabla\psi'\|\,\|\psi'\|_3 \\ &= \left(\sqrt{pq}\,\|\psi'\|_1\right)\left(2\sqrt{pq}\,\|\psi'\|_3\right) \leq 2q\varepsilon^2 Q(t) + \frac{pq}{\varepsilon^2}\,\|\psi'\|_3^2 \ . \end{aligned} \tag{29}$$

In addition, the use of Lemma 2 in (21) leads to

$$\begin{aligned} \frac{d}{dt}Q(t) \leq -2\rho Q(t) - R(t) - \nu p\,\|\psi'\|_{s+1}^2 - \nu q\,\|\psi'\|_{s+2}^2 &\leq -2\rho Q(t) - R(t) - \nu q\,\|\psi'\|_{s+2}^2 \\ &\leq -2\rho Q(t) - R(t) - \nu q a^{1-s}\,\|\psi'\|_3^2 \end{aligned} \tag{30}$$

where $a = \sqrt{2}$. Combining (29) with (30) and setting $\varepsilon^2 = pa^{s-1}/\nu$ in order to eliminate the two terms containing $\|\psi'\|_3^2$, we prove the following assertion:

Theorem 5. *Let $s \geq 1$, $\nu > 0$ and $\sigma \geq 0$. And let $\widetilde{\psi}(t,\lambda,\mu)$ be such a solution of equation (4) that the numbers p and q defined by (28) are limited. If*

$$\nu(\sigma + 2^s\nu) > 2^{(s-1)/2}pq \tag{31}$$

then $Q(t)$ is the Lyapunov function, and the solution $\widetilde{\psi}$ is a global BVE attractor, besides, any its perturbation will exponentially decrease with time.

In contrast to Theorem 4, Theorem 5 requires a non-zero viscosity coefficient ν. According to conditions (28), the basic solution must have continuous derivatives only up to the second order. Therefore, (31) can be applied to a wider class of BVE solutions. For example, the main solution to problem (4)-(5) can be one of the non-stationary modons [7, 15] supported, despite the dissipation, by the corresponding external forcing. As it is known, $\nabla\Delta\widetilde{\psi}(t,\lambda,\mu)$ is discontinuous at the boundary between the inner and outer regions of the modon. Therefore, condition (31) is applicable, while condition (27) is not.

Example 4. *Let $\sigma = 0$ and $s = 1$ in equation (4), and let $\widetilde{\psi}(\lambda,\mu)$ be a stationary solution from the subspace $\mathbf{H}_n$ of homogeneous spherical polynomials ($n \geq 2$):*

$$\widetilde{\psi}(\lambda,\mu) = \sum_{m=-n}^{n} \widetilde{\psi}_n^m\, Y_n^m(\lambda,\mu)\ . \tag{32}$$

The solution (32) is supported by a steady forcing whose Fourier coefficients F_n^m are

$$F_n^m = (-\nu\chi_n^2 + i2m)\widetilde{\psi}_n^m$$

where $i=\sqrt{-1}$ and $\chi_n = n(n+1)$. According to Example 3, subspace $\mathbf{H}_n$ is the domain of attraction of solution (32) for any $\nu \neq 0$. Moreover, it follows from (28) that $p = \chi_n q$, and due to Theorem 5, solution (32) is the global attractor of equation (4) if

$$\nu \geq q\sqrt{\frac{n(n+1)}{2}} \ .$$

Thus, in order for basic flow (32) to be a global BVE attractor, the viscosity coefficient ν must increase with increasing velocity (q) and degree n of the flow (32).

6 Dimension of global spiral BVE attractor

It was shown in [3] that in the case of a stationary forcing, the Hausdorff dimension of the global attractor of the barotropic vorticity equation (4) on a sphere for $s=1$ and $s=2$ is limited by the generalized Grashof number

$$G(s) = \|F(x)\| / \chi_1^{2s-1} \nu^2(s) \tag{33}$$

where $\chi_1 = 2$ is the smallest positive eigenvalue of the spherical Laplace operator (see (2)).

This is an important hydrodynamic result. However doubts arise about its applicability to the dynamics of large-scale barotropic processes in the atmosphere. Indeed, the BVE forcing represents the influence of small-scale baroclinic processes (convection, etc.) on the large-scale dynamics of the barotropic atmosphere, and therefore is essentially nonstationary with rather complex spatio-temporal behavior. In order to show that the dimension of the global BVE attractor crucially depends on the space-time structure of the forcing, we now estimate the dimension of the global BVE attractor provided that the forcing of the BVE model is a quasiperiodic function in time and a homogeneous spherical polynomial of degree n in space:

$$F(t,x) = \sum_{m=-n}^{n} F_m(t) Y_n^m(x) \tag{34}$$

where

$$F_m(t) = f_m \exp\{im\omega_m t\}, \qquad |m| \leq n , \tag{35}$$

each f_m is a constant, and the frequencies ω_m are incommensurate. Thus, $F(t,x) \in \mathbf{H}_n$. Note that

$$\|F(t,x)\| = \left(\sum_{m=-n}^{n} |f_m|^2\right)^{1/2} \equiv Const \tag{36}$$

i.e., the norm (36) of forcing and the generalized Grashof number (33) are time-independent constants. Obviously, there are many quasiperiodic forcings of the form (34) which have the same norm (36), or the same Grashof number (33), but differ in degrees n or/and amplitudes f_m.

Since $J(\psi, \Delta\psi) = 0$ for any function $\psi \in \mathbf{H}_n$, there exists an exact BVE solution $\widetilde{\psi}(t,x) = \sum_{m=-n}^{n} \widetilde{\psi}_m(t) Y_n^m(x)$ from the subspace $\mathbf{H}_n$ defined by the Fourier coefficients

$$\widetilde{\psi}_m(t) = A_m F_m(t) \equiv -\{[\sigma + \nu\chi_n^s]\chi_n + im(\chi_n\omega_m - 2)\}^{-1} F_m(t), \qquad |m| \leq n . \tag{37}$$

Besides, it was shown in Example 3 that the subspace $\mathbf{H}_n$ is the domain of attraction of this solution.

Since the frequencies ω_m are rationally independent, the solution $\widetilde{\psi}(t,x)$ is quasiperiodic, and its path is an open endless spiral densely wound around a $2n$-dimensional torus in the $(2n+1)$-dimensional complex space $\mathbf{H}_n$. According to Theorem 3 in [9], the closure of this trajectory coincides with the torus. Hence, the Hausdorff dimension of the attractive set, that is solution (37), coincides with that of the torus and equals $2n$.

Sufficient conditions for the global asymptotic stability of smooth solution (37) are given by Theorem 4. In our case $\widetilde{\psi}(t,x) \in \mathbf{H}_n$, and (25) gives $p = \chi_n q = n(n+1)q$. Therefore, condition (27) for the global asymptotic stability of solution (37) accepts the form

$$\sigma + 2^s \nu > q\sqrt{\chi_n} \tag{38}$$

where

$$q = \sup_{t\geq 0} \max_{(\lambda,\mu)\in S} |\nabla \widetilde{\psi}(t,\lambda,\mu)| \ . \tag{39}$$

Due to Lemma 1,

$$|\nabla \widetilde{\psi}(t,x)|^2 = \left| \sum_{m=-n}^{n} \widetilde{\psi}_n^m(t) \nabla Y_n^m(x) \right|^2 \leq \|\widetilde{\psi}(t,x)\|^2 \left\{ \sum_{m=-n}^{n} |\nabla Y_n^m(x)|^2 \right\} = \chi_n K_n^2 \, \|\widetilde{\psi}(t,x)\|^2 \ . \tag{40}$$

Also, according to (37),

$$\|\widetilde{\psi}(t,x)\| \leq \frac{1}{(\sigma + \nu \chi_n^s)\chi_n} \|F\| \tag{41}$$

where the norm $\|F\|$ of forcing (34), (35) is time-independent (see (36)). Due to (39)-(41),

$$q\sqrt{\chi_n} \leq \frac{K_n}{\sigma + \nu \chi_n^s} \|F\|$$

and hence condition (38) is satisfied if

$$\sigma + 2^s \nu > \frac{K_n}{[\sigma + \nu \chi_n^s]} \|F\|$$

and also if

$$2^s \nu > \frac{K_n}{\nu \chi_n^s} \|F\| \ .$$

Using the definition (33) where $\chi_1 = 2$, the last inequality can be written in terms of the generalized Grashof number:

$$2^{2-s} \chi_n^s \sqrt{\frac{\pi}{2n+1}} > G(s) \ . \tag{42}$$

Since

$$\chi_n^s \frac{1}{\sqrt{2n+1}} = n^s (n+1)^s \frac{1}{\sqrt{2}\sqrt{n+1}} \geq \frac{1}{\sqrt{2}} n^{2s-1/2}$$

condition (42) holds if

$$2^{3/2-s} \sqrt{\pi} n^{2s-1/2} > G(s) \ . \tag{43}$$

Thus, we prove the following assertion:

Theorem 6. *Let $s \geq 1$, $\nu > 0$, $\sigma \geq 0$, and let $F(t,x) \in \mathbf{H}_n$ be a quasiperiodic forcing (34)-(35) of the BVE equation (4). Then solution (37) from the subspace $\mathbf{H}_n$ is a global attractor provided that condition (43) is satisfied.*

In particular, for $s = 2$ and $s = 1$, solution (37) is the global BVE attractor if

$$2^{-1/2} \sqrt{\pi} n^{7/2} > G(2) \quad \text{and} \quad 2^{1/2} \sqrt{\pi} n^{3/2} > G(1) \ , \tag{44}$$

respectively. Note that $s = 1$ corresponds to the Navier-Stokes equations.

It follows from (44) that for a fixed finite value of the generalized Grashof number G, it is always possible to determine such an integer $n(G)$ that the spiral solution generated by any quasiperiodic forcing (34)-(35) from subspace $\mathbf{H}_n$ with $n \geq n(G)$ is a global BVE attractor. For example, if we take $G(2) = \|F(x)\| / \chi_1^3 \nu^2(2) = 1500$ for the barotropic atmosphere (this value was used in [3]), then condition (44) is satisfied for any forcing (34)-(35) whose degree n is equal to or greater than 8.

The result obtained is not unexpected. Indeed, for a fixed coefficient $\nu(s)$, the number $G(s)$ is fixed if the L_2-norm (36) of the forcing is a constant independent of n. Let the amplitudes $|f_m|$ of oscillations of forcing be nonzero for all m. Then they must decrease as n grows, and for a sufficiently large number n (or for sufficiently small amplitudes $|f_m|$), the viscosity $\nu(s)$ can become sufficient to satisfy condition (27) for the global asymptotic stability of the quasiperiodic solution (37).

Thus, unlike the case of stationary forcing when the Hausdorff dimension of the global BVE attractor is limited above by the generalized Grashof number G [3], in the case of the quasiperiodic forcing (34), the Hausdorff dimension $2n$ of the global spiral attractor (37) is not limited by the generalized Grashof number G and can become arbitrarily large as the degree n of the BVE forcing increases.

This result is of particular meteorological interest, since it shows that the dimension of the global attractor in the barotropic atmosphere can be unlimited, even if the generalized Grashof number (33) is bounded. Thus, the dimension of the global attractor crucially depends not only on the generalized Grashof number, but also on the time-space structure of the BVE forcing. This also shows that the search for a global attractor of small dimension in the barotropic atmosphere [1] is theoretically unjustified due to the fact that forcing usually has a very complex structure with a huge number of degrees of freedom.

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