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A Crank-Nicolson Approximation for the time Fractional Burgers Equation

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Abstract

In the present manuscript, Crank Nicolson finite difference method is going to be applied to get the approximate solutions for the fractional Burgers equation. The fractional derivative used in this equation is going to be taken into consideration in the Caputo sense. The L1 type discretization formula is going to be applied to this equation. For checking the efficiency of proposed methods, the error norms L_2 and L_∞ have at the same time been calculated. Those newly got solutions using the presented method illustrate the easy usage and efficiency of the approach presented in this manuscript.

Keywords: Fractional order derivatives, Crank Nicolson Finite Difference methods, Fractional Burgers equation.
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1 Introduction

Fractional order integral and derivative are the generalizations of classical integral and derivative concepts which are examined in detail by Leibniz and Newton. The concepts of fractional integral and derivative are as old as integer order integral and derivative concepts, and the fractional derivative expression is first mentioned by Leibniz's letter to L'Hospital in 1695 [1]. In the letter, Leibniz's question was 'Can the integer order derivatives be generalized to fractional order derivative'. This is known as the first emergence of the concept of fractional differential. In addition to Leibniz, many scientists such as Liouville, Riemann, Weyl, Lagrange, Laplace, Fourier, Euler and Abel have worked on the same subject [2]. Many definitions are given in the literature for fractional derivative. Some of these are Riemann-Liouville, Caputo, Grünwald-Letnikov, Wely, Riesy fractional derivatives [3]. Some studies have shown that these definitions are equivalent under certain conditions. There is more than one derivative definition in the fractional analysis, making it possible to use the most suitable one according to the problem and thus to get the best solution for this problem.

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However, if the derivative is described as how the derivative of the fractional order is defined, the expression that when the order is selected being equal to the integer is the same as the integer order of the derivative.

The description of Caputo fractional derivative was first introduced by the Italian mathematician M.Caputo in the 1960s to eliminate the problem of the calculation of Riemann-Liouville definition of initial values in Laplace transform applications. The fundamental advantage of the Caputo approach is the fact that the appropriate initial conditions defined for Caputo fractional differential equations are identical. Therefore, in recent studies in the literature, in the exact and approximate solutions of fractional differential equations, instead of Riemann-Liouville fractional derivative operator, Caputo fractional derivative operator has been more preferred. Recently, studies on the solution of fractional differential equations have increased. There are several studies about fractional problems and their computational accuracy in the literature [4–6]. Since the exact solution for many of the fractional differential equations are not found, various methods have been developed to find approximate or numerical solutions.

Definition 1. $f(t)$ function continuously and can be differentiated, and for $n-1 \leq \gamma < n$ Caputo means for the fractional order derivatives;

$$ {}_a^C D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma-n+1}} \, d\tau $$

is defined as above.

Definition 2. Finite difference methods are widely used in the solution of many linear and nonlinear partial differential equations. In general, the following way is followed in applying a finite difference method to a partial differential equation:

The given solution area of the problem is divided into meshes with geometric shapes and approximate solution for the problem is calculated on the nodes of each mesh. Proper finite difference approaches are obtained by using Taylor series instead of derivatives in differential equations. Thus, the present problem of solution of the differential equation has been converted into the problem of the solution of an algebraic system of equation consisting of difference equations. Thus the algebraic equation system obtained now may be solved easily by one of the direct or iterative methods.

2 Numerical Solution of the Model Problem

In this manuscript, we will deal with the nonlinear time fractional Burgers equation with the given initial and boundary conditions as a test problem presented as

$$ \begin{aligned} & D_t^\gamma u + u u_x - \nu u_{xx} = f(x,t) \\ & u(x,0) = g(x), \qquad 0 \leq x \leq 1. \\ & u(0,t) = h_1(t), u(1,t) = h_2(t), \ t \geq 0 \end{aligned} \tag{1} $$

in which ν is the viscosity parameter, $u(x,t)$ represents the speed of fluid media at time-space position (x,t) and

$$ D_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-\tau)^{-\gamma} \frac{\partial u(x,\tau)}{\partial \tau} d\tau, \quad 0 < \gamma \leq 1 $$

is the fractional derivative given in the Caputo's form [1,7]. The Burgers' equation is the one of the most important differential equations arising in applied sciences and mathematical physics and has interesting applications in physics and astrophysics. The Burgers' equation examines the modeling of fluid mechanics, diffusive waves in fluid dynamics and also it has many application in the theory of shock waves, mathematical modeling of turbulent fluid, sound waves in a viscous medium and so on. In Mathematical modelling, fractional derivatives provide more accurate and applicable models for real life problems. For the same reasons, recent researches have concentrated on investigating and proposing models on Fractional Burgers Equations. During the years,

this attractive modelling ability of the equation have taken attention of most of scientific people, many important and useful papers have brighten science world. In this part of our manuscript, we want to remind some of them; Esen and Tasbozan [8] have investigated numerical solutions of the equation via quadratic B-spline Galerkin method which is an useful and efficient type of finite element approach, then they have obtained more accurate numerical solutions with using collocation points and raising the degree of basis in [9]. Yaseen and Abbas, in [12], have used collocation method with the help of cubic trigonometric B-splines basis. Qui *et al* [10] have proposed an implicit difference scheme with *L*1 algorithm which is a specific time discretization of the Caputo fractional derivative. A numerical method focused on a finite difference scheme in terms of time and the Chebyshev spectral collocation method in terms of space is used to get approximate solutions of the Burgers' equation in [11]. Saad and Al-Sharif [16] have applied variational iteration method for solving the equation considering several initial conditions. Asgari and Hosseini [13] have focused on generalized time fractional Burger type equation, they put forward two semi implicit Fourier pseudospectral approximations for seeking solutions of the equation. As a different view to the mentioned equation, Khan et al [14] are used the generalized version of the differential transform method and homotopy perturbation method. Lombard and Matignon [15] present an article for better understanding the competition between nonlinear effects and nonlocal relaxation.

Throughout this manuscript, in order to contribute to literature, we will consider numerical solutions of the time fractional Burgers equations using finite difference approach. For the presented numerical solutions, to get a Crank Nicolson finite difference scheme to solve the time fractional Burgers equation as utilized in explicit difference method in Ref. [17], we will also discretize the derivative respect to time using the widely-known L1 formula [18]

$$\left.\frac{\partial^\gamma f}{\partial t^\gamma}\right|_{t_n} \cong \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} b_k^\gamma \left[f(t_{n-k}) - f(t_{n-k-1})\right], \quad 0 < \gamma \le 1 \tag{2}$$

where

$$b_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}.$$

2.1 Crank Nicolson Finite Difference Scheme

Let's assume the fact that the solution domain for the present problem $0 \le x \le 1$ is discretized into regular grids with equal length Δx in the x- direction and also with equal time intervals Δt over time t such that $x_j = j\Delta x$, $j = 1(1)M - 1$ and $t_n = n\Delta t$, $n = 0(1)N$ and the numerical solution of u at the grid point $(j\Delta x, n\Delta t)$ will denote by U_j^n throughout the study. Using L1 formula in Eq.(2) instead of Caputo derivative in Eq.(1) and utilizing the following discretization in place of the terms uu_x and u_{xx} respectively:

$$\left.\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}\right|_{t_n} \cong \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} b_k^\gamma \left[U_j^{n-k} - U_j^{n-k-1}\right], \quad 0 < \gamma \le 1$$

$$uu_x \cong \frac{U_j^{n+1}}{2}\left(\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}\right) + \frac{U_j^n}{2}\left(\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x}\right)$$

and

$$u_{xx} \cong \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{2(\Delta x)^2} + \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{2(\Delta x)^2}.$$

We are able to get the following system of algebraic equations

$$U_{j-1}^{n+1}\left(-\frac{Sv}{2(\Delta x)^2} - \frac{S}{4\Delta x}U_j^n\right) + U_j^{n+1}\left(1 + \frac{Sv}{(\Delta x)^2} + \frac{S}{4\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)\right) + U_{j+1}^{n+1}\left(-\frac{Sv}{2(\Delta x)^2} + \frac{S}{4\Delta x}U_j^n\right)$$
$$= \frac{Sv}{2(\Delta x)^2}U_{j-1}^n + U_j^n\left(1 - \frac{Sv}{(\Delta x)^2}\right) + \frac{Sv}{2(\Delta x)^2}U_{j+1}^n + Sf(x,t) - \sum_{k=1}^{m-1} b_k^\gamma\left[U_j^{n+1-k} - U_j^{n-k}\right]$$

where $S = \Delta t^\gamma \Gamma(2-\gamma)$.

3 Numerical Results

Numerical results obtained for the Eq. (1) are got using the Crank Nicolson finite difference methods. The efficiency of the present methods are tested using the error norm L_2

$$L_2 = \left\| u^{exact} - (U_N)_j \right\|_2 = \sqrt{\Delta x \sum_{j=0}^{M} \left(u_j^{exact} - (U_N)_j \right)^2}$$

and also the maximum error norm L_∞

$$L_\infty = \left\| u^{exact} - (U_N)_j \right\|_\infty = \max_j \left| u_j^{exact} - (U_N)_j \right|.$$

Example 1. *Firstly, we are going to take time fractional Burgers equation (1) given with the following appropriate boundary conditions*

$$U(0,t) = 0, U(1,t) = 0, t \geq 0$$

and the initial condition as

$$U(x,0) = 0, 0 \leq x \leq 1.$$

The term $f(x,t)$ is of the form [9]

$$f(x,t) = \frac{2t^{2-\gamma}\sin(2\pi x)}{\Gamma(3-\gamma)} + 2\pi t^4 \sin(2\pi x)\cos(2\pi x) + 4\nu\pi^2 t^2 \sin(2\pi x).$$

The analytic solution for the present problem is described as

$$U(x,t) = t^2 \sin(2\pi x).$$

A comparison for the exact and approximate solutions got by the Crank-Nicolson methods for fractional Burgers equation for the values of $\gamma = 0.5$, $\Delta t = 0.00025$ and $t_f = 1$ for various values of M is presented in Table 1. As one could clearly see from this table, the decreasing values of the error norms L_2 and L_∞ validate this comment. In Table 2, the approximate solutions obtained for values of M have been compared with those in Ref. [9]. Those

Table 1 The error norms L_2 and L_∞ of the time fractional Burgers equation problem using the Crank Nicolson finite difference method with $\nu = 1.0$, $\Delta t = 0.00025$ and $t_f = 1$ for various values of M

	$N = 10$	$N = 20$	$N = 40$	$M = 80$
$L_2 \times 10^3$	22.64087326	5.44483424	1.22007333	0.16846258
$L_\infty \times 10^3$	30.49445082	7.70051177	1.72552915	0.23826679

error norms for each values of M got by this method are smaller than those presented in Ref. [9].

Table 2 A comparison of the errors for Example 1 at $t_f = 1$

	$N = 40$		$N = 80$		$N = 100$	
	Present	[9]	Present	[9]	Present	[9]
$L_2 \times 10^3$	1.22007333	1.224329	0.16846258	0.177703	0.04239382	0.052299
$L_\infty \times 10^3$	1.72552915	1.730469	0.23826679	0.253053	0.05996900	0.076541

The solutions obtained at different times in Fig. 1 were given graphically with exact solutions. The results obtained at different times were found to be too close to each other on the graphics.

In Table 3, solutions for different viscosity values were given for values of $\gamma = 0.5$, $\Delta t = 0.00025$, $t_f = 1.0$, N=40. It was observed that the errors increased gradually at small values of viscosity.

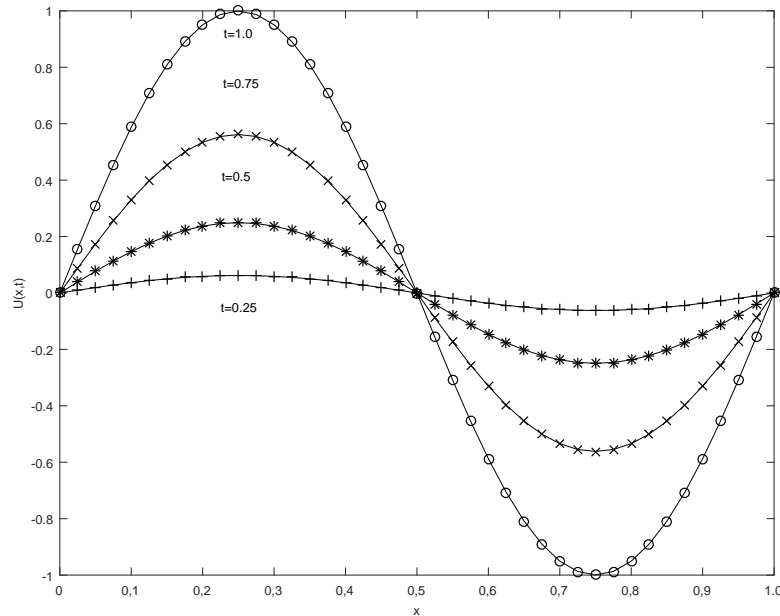*Table 4 presents the error norms obtained at various selected values of γ.*

Fig. 1 A comparison of the analytic and approximate solutions using the Crank-Nicolson method for v=1.0, N=40, γ=0.5 and Δt = 0.0025 at various values of final time

Table 3 The error norms L_2 and L_∞ of Example 1 for γ=0.5, Δt=0.00025, t_f =1.0, N=40 and various values of v

	$v = 1$	$v = 0.5$	$v = 0.1$	$v = 0.01$	$v = 0.001$
$L_2 \times 10^3$	0.41761157	0.51259161	1.03928112	2.13880314	6.46231244
$L_\infty \times 10^3$	0.59040780	0.72342731	1.51280185	4.65707275	22.95302718

Table 4 The error norms L_2 and L_∞ of Example 1 for N=120, $t_f = 1.0$, $\Delta t = 0.00025$ for different values of γ

	$\gamma = 0.1$	$\gamma = 0.25$	$\gamma = 0.75$	$\gamma = 0.9$
$L_2 \times 10^3$	0.02411976	0.02490518	0.02669547	0.02579288
$L_\infty \times 10^3$	0.03409905	0.03521115	0.03774592	0.03646791

Example 2. *Secondly, we will consider the model problem with the appropriate boundary conditions*

$$U(0,t) = t^2, U(1,t) = -t^2, t \geq 0$$

and the initial condition as

$$U(x,0) = 0, 0 \leq x \leq 1.$$

The term

$$f(x,t) = \frac{2t^{2-\gamma}\cos(\pi x)}{\Gamma(3-\gamma)} - \pi t^4 \cos(\pi x)\sin(\pi x) + v\pi^2 t^2 \cos(\pi x)$$

and the analytic solution for the problem

$$U(x,t) = t^2 \cos(\pi x).$$

In Table 5, for various values of N, the error norms obtained for Example 2 have been illustrated. In the table, one can obviously see that the improvement in numerical solutions has been observed as expected for the increasing values of N.

Table 5 Comparison of results at $t_f = 1.0$ for $\gamma = 0.5$, $\Delta t = 0.0001$, $v = 1.0$ and various mesh sizes

	$N = 10$	$N = 20$	$N = 40$	$N = 80$
$L_2 \times 10^3$	0.81667090	0.23594859	0.09215464	0.05901177
$L_\infty \times 10^3$	1.13678094	0.32086237	0.12245871	0.09790576

Table 6 Comparison of results at $t_f = 1.0$ for v=1, $N = 40$ and various time steps

	$\Delta t = 0.0001$	$\Delta t = 0.00025$	$\Delta t = 0.0005$	$\Delta t = 0.001$	$\Delta t = 0.0025$	$\Delta t = 0.005$
$L_2 \times 10^3$	0.09215464	0.16142590	0.27906906	0.51583267	1.22775834	2.41506007
$L_\infty \times 10^3$	0.12245871	0.24577898	0.47478371	0.93280810	2.30696715	4.59744900

In Table 6, the error norms obtained for different values of Δt have been given for values of $v = 1.0$, $\Delta x = 0.025$ and $\gamma = 0.5$. As the Δt values decreased, it was observed that the errors also decreased accordingly. In Figure 2, the solutions of Problem 2 obtained at different times for values of $v = 1.0$, $\Delta x = 0.025$, $\gamma = 0.5$ and $\Delta t = 0.0025$ have been presented together with their analytical values.
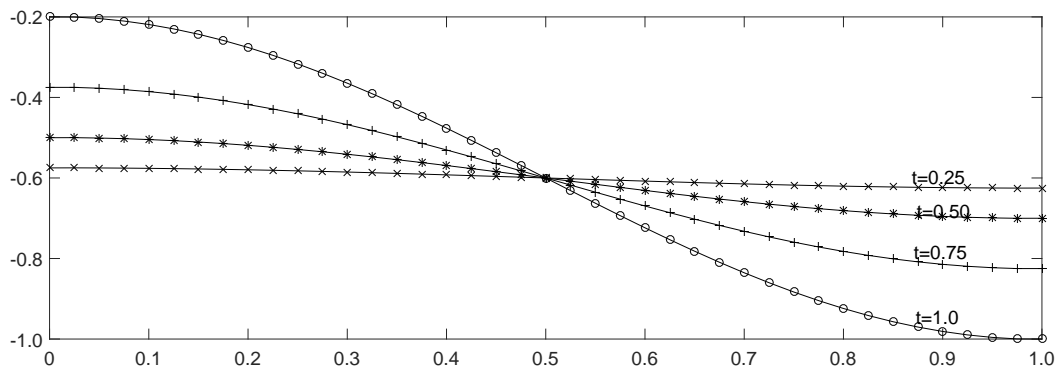
Fig. 2 A comparison of the analytic and approximate solutions using the Crank-Nicolson method for $v = 1.0$, $\Delta x = 0.025$, $\gamma = 0.5$ and $\Delta t = 0.0025$ at different values of final time

Example 3. *Lastly, we consider the model problem with the boundary conditions*

$$U(0,t) = t^2, U(1,t) = et^2, t \geq 0$$

and the initial conditions as

$$U(x,0) = 0, 0 \leq x \leq 1.$$

The function $f(x,t)$ is of the following form

$$f(x,t) = \frac{2t^{2-\gamma}e^x}{\Gamma(3-\gamma)} + t^4 e^{2x} - vt^2 e^x.$$

The analytical solution of the problem is given by

$$U(x,t) = t^2 e^x.$$

In Table 7, approximate solutions obtained for values of $t_f = 1.0$ and , $\Delta x = 0.025$ have been given at different Δt values. It is obvious that the errors get smaller and smaller as Δt values get smaller and smaller. Figure 3 shows the numerical and exact solutions obtained at different times for Problem 3. It is seen that the results are indiscriminately close to each other on the graphics.

Table 7 Comparison of results at $t_f = 1.0$ for $\nu = 1$, $\Delta x = 0.025$ and various time steps

	$\Delta t = 0.0001$	$\Delta t = 0.00025$	$\Delta t = 0.0005$	$\Delta t = 0.001$
$L_2 \times 10^3$	0.08344454	0.22252258	0.45497994	0.92040447
$L_\infty \times 10^3$	0.23648323	0.59722584	1.19848941	2.40107923
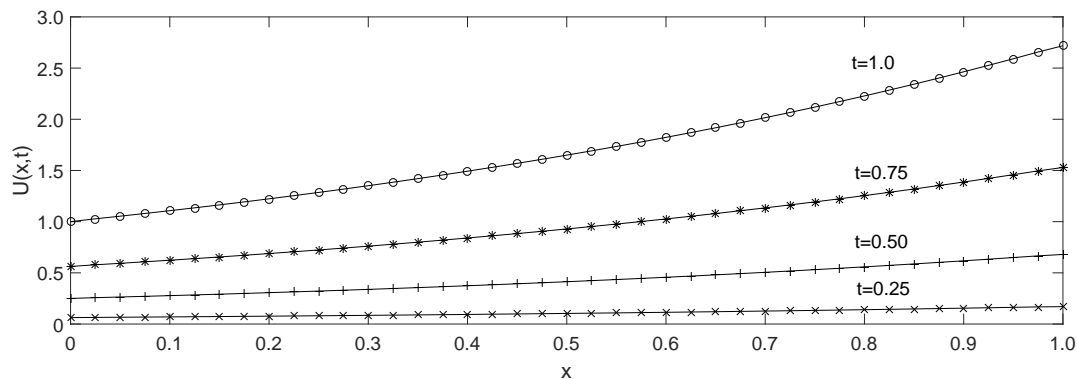
Fig. 3 A comparison of the analytic and approximate solutions using the Crank-Nicolson method for $\nu = 1.0$, $N = 40$, $\gamma = 0.5$ and $\Delta t = 0.0025$ at different values of final time

4 Conclusions

As a conclusion, in the present study the numerical solutions of the time fractional Burgers equation have been obtained by using finite difference method based on Crank-Nicolson discretization. The obtained results are compared with analytic and some of the numerical results available in the literature. This comparison has shown that the presented method is efficient and effective and can also be used for a wide range of physical and scientific applications. Moreover to illustrate the accuracy of the present method the error norms L_2 and L_∞ are computed and given in tables. Two test problems have been used to show the accuracy of the present scheme for various values of parameters in the problem. Tables and figures show the results of these various tests and also comparisons with some available results together with the error norms L_2 and L_∞. Finally the present method has been shown to be applicable for more widely used fractional differential equations.

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