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Periodic orbit in the frame work of restricted three bodies under the asteroids belt effect

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Abstract

In this paper, we present a comprehensive analytical study on the perturbed restricted three bodies problem. We formulate the equations of motion of this problem, in the event of the asteroids belt perturbation. We find the locations of equilibrium points (collinear and triangular points) and analysis their linear stability. Furthermore the periodic orbits around both collinear and triangular points are found.

Keywords: Restricted three bodies problem, Asteroids belt effect, Linear Stability, Periodic orbits.

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1 Introduction

In general the model of three–body problem is related to the motion of three bodies, in space under mutual gravitational forces without restrictions or specified conditions. The importance of this model in celestial mechanics will rise when the three objects move in space under the effects of their mutual gravitational attractions. One of the most familiar emerged model from the general three–body problem is the restricted model. In this model, we impose that the third body, “infinitesimal body”, is very small than the other two bodies “primaries”, and it does not affect their motion, the restricted model is called planar circular or elliptical restricted problem when the third body is moving in the plane of primaries motion [2,8,9,13,16,31], while is called spatial restricted three–body problem if the third body moves in three dimensions [35].

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In fact there are many issue of the “restricted three–body problem”, and that is regard to the existence of many disturbance forces. The studying of these issue enable us to get precise and accurate data about the dynamical features of the system. Which will have more significant particularity in space mission. The most important features of “restricted three–body problem” are the existence of libration points and their stability as well as the periodic motion around these points. There are many authors devoted their research to investigate the aforementioned properties within frame work of the “perturbed restricted three–body problem” [3, 5, 6, 10, 11, 15, 17, 33]. Furthermore, the analysis of lower or higher order of resonant periodic orbits with in frame of the photogravitational “restricted three–body problem” are studied by [28, 29].

In the frame work of studying the symmetric of periodic orbits, [27] analyzed the asymmetric solution in the restricted three–body problem. He investigated the symmetry of periodic orbits numerically. Moreover he use Levi–Civita transformation to regularize the equations of motion, in order to avoid the singularity between the third body and one of the primary bodies. [32] used theoretical and numerical approaches to investigate and study the symmetric relative periodic orbits within frame of the isosceles restricted problem three bodies. They also proved that the elastance of many families of symmetric relative periodic solution, which are emerged from heteroclinic connections between binary or triple collisions

[14] studied the real system of Saturn-Titan to explore the oblateness influence of Saturn planet on the periodic orbits and quasi-periodic motion regions around the primaries within frame restricted thee–body model. They analysed the positions, the quasi-periodic orbits and periodic size using the Poincaré surface of section technique. They proved that some quasi-periodic orbits change to periodic orbits corresponding the oblateness effect and vice-versa. [12] investigated also the periodic orbits around the libration pints, in the case of the bigger primary is radiating, while the smaller primary suffer from lack of sphericity, due to the effect of zonal harmonic coefficients, which are considered up to J_4 . In addition [7] prove that the obtained first and second kind of periodic orbits of the unperturbed restricted 3–body problem can be extended to perturbed restricted 3–body problems, under the perturbed effect of the zonal harmonic coefficients and solar sail.

In the case of the primaries in the restricted model are enclitic by a ring-type belt of material particle points, the infinitesimal body motion is not valid, if we ignore the effect of this belt. Already in stellar systems there are rings of dust particles and asteroids belts around the planetary systems. Which are regarded as the young analogues of the Kuiper belt in our Solar System, see for more details [18]. Under the effect of asteroid belt, when the massive primaries are oblate and radiating, the locations of the equilibria points and the linear stability around these points are studied by [34]. They demonstrated that there are two new equilibrium points (L_{n1} and L_{n2}) as well as the classical five points, which are found regard to the extra–gravitational asteroids belt effect.

The effect of the gravitational potential of the asteroids belt is not limited to the changes in the mathematical expressions, which represent the dynamical systems, but also its effect go to the dynamical properties of systems. This encouraged many researchers to study the dynamics of astronomical dynamical systems under the asteroids belt effect. For example, [20–22] investigated that the number and positions of equilibria, also showed that the solution curves topology will different, when the gravitational potential of asteroid belt is considered. They showed that the planetary system are affected by gravitational belt, where they proved that the probability to obtain equilibria points in the inner part of the belt is larger than to obtain near the outer part. The significant of their results is due to we can use it to investigate the observational configuration of Kuiper belt objects of the outer solar system.

[36] studied and analyzed a Chermnykh-like problem under the effect the gravitational potential of asteroid belt, and found a new equilibrium points for this problem. In addition the stability of equilibrium points when the smaller body is oblate spheroid and the bigger is a radiating body under the influence of the gravitational potential of asteroid belt, in the “restricted three–body problem” studied by [25]. The secular solution around the triangular equilibrium points when both massive bodies are oblate and radiating with the effect of asteroid belt are found and reduced to periodic one by [4] within frame restricted three–body problem.

In this paper we will study the perturbation of the gravitational potential of asteroid belt, which is constructed by [26] on the locations of the equilibrium and their stability as well as the periodic orbits around these points. This paper is organized as follow: An introduction, background on asteroids belt potential and a model descriptions are presented in Sections (1 – 3). While the locations of equilibrium points and there linear stability are studied in Sections (4 – 5). But the periodic orbits around these points are constructed in Section (6). Finally the conclusion is drawn in last Section.

2 Background on asteroids belt potential

In the solar system, the asteroid belt is similar to a ring-shaped. it can found between the Mars and Jupiter orbits. This region includes many objects (minor planets) with different sizes and shapes, which are irregular in most cases but very smaller than compered to the planets. In particularly, this belt is called the main asteroids belt, in order to characterize it from any other collection of asteroids in the solar system, such as trojan or near–earth asteroids, see Fig.1 (Source: https://en.wikipedia.org/wiki/Asteroid_belt). The asteroid belt region lies between the range of radial distances from 2.06 to 3.27 AU. It includes about 93.4% minor planets. These distances represent the inner and outer boundaries of the main belt region respectively [30]. The second law

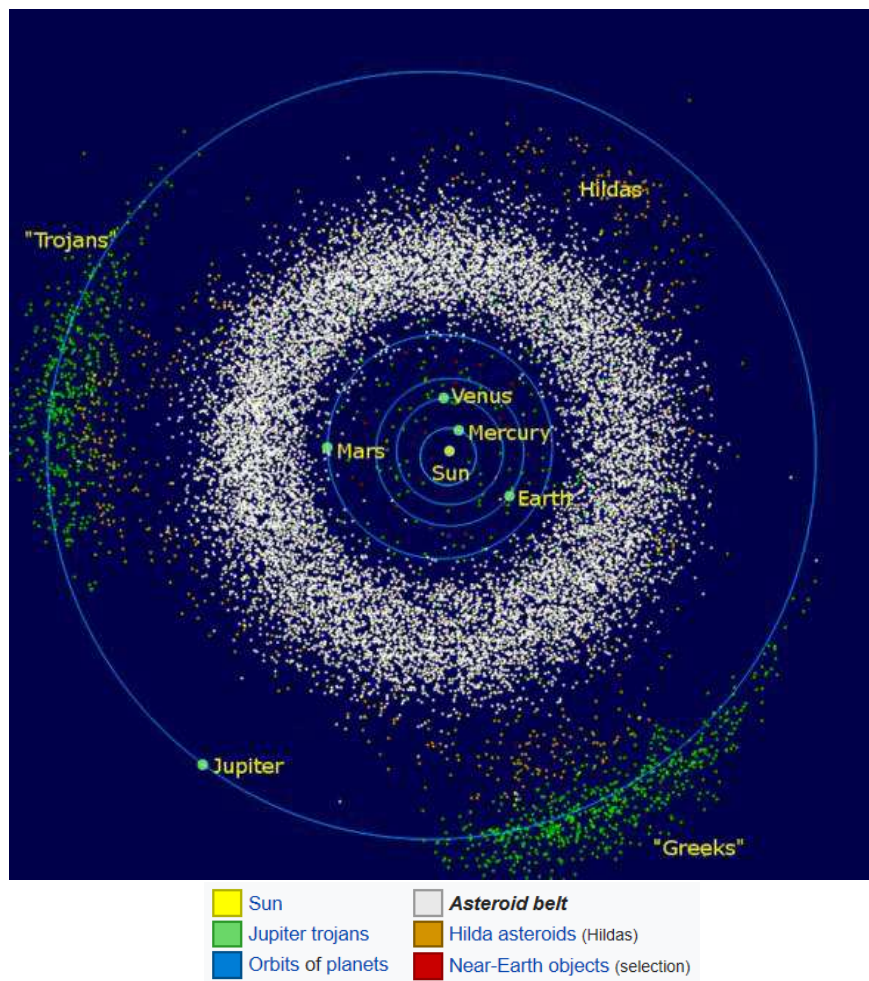


Fig. 1 The asteroids of the inner Solar System and Jupiter (Color figure online)

of motion and the universal gravitational law have been used as the most fundamental laws for the physical

sciences, since their success in investigating the celestial bodies notion in the solar system. Thus the Newtonian Law was first proved in the astronomical context. It was then applied to other fields successfully. But the obtained results of this law lacks the accuracy in cases the of stellar or planetary systems have discs of dust or asteroids belt [19].

In the recent years, the researchers are studying the effect gravitational potential from a belt on the linear stability of libration points after was discovered dust ring around the star and discs around the planetary orbits [23,24]. There are perturbations in the solar system due to asteroid belt, where several of the largest asteroids are massive enough to significantly affect the orbits of other bodies for example affect the asteroids in the motion of Mars (Mars is very sensitive to perturbations from many minor planets), motion space probes affected by perturbation from asteroids and perturbations from asteroid on another asteroid when which close encounter.

In order to explore the orbital dynamics or the motion of the celestial dynamical systems, we have to build first suitable model that describing and realistically the structures and properties of the asteroid belt. One of the most important belt potential and used in the literatures introduced by Miyamoto-Nagai [26]. This model is called flattened potential and used in modelling disk galaxies. It can be controlled by

$$V_b(r, z) = \frac{M_b}{\left(r^2 + \left(a + \sqrt{z^2 + b^2}\right)^2\right)^{1/2}} \quad (1)$$

where

- M_b is the total mass of the disc.
- r is the radial distance of the infinitesimal body it is given by $r^2 = x^2 + y^2$.
- The parameter a known as the flatness parameter determine the flatness of the profile.
- The parameter b known as the core parameter determine the size of the core of density profile.

Hence the perturbed acceleration regard to the asteroid belt is $\mathbf{a}_b(r, z) = \nabla V_b(r, z)$, thereby the acceleration can be written in the following form

$$\mathbf{a}_b(r, z) = -\frac{M_b}{\left(r^2 + \left(a + \sqrt{z^2 + b^2}\right)^2\right)^{3/2}} \left(x, y, z \left[1 + \frac{a}{\sqrt{z^2 + b^2}}\right]\right) \quad (2)$$

If $a = b = 0$ the potential reduces to the one by a point mass or spherical subject whose mass is M_b . Restricting ourselves to the XY -plane ($z = 0$), and define $T \equiv a + b$ from Eq. (1) we have

$$V_b(r, 0) = \frac{M_b}{(r^2 + T^2)^{1/2}} \quad (3)$$

with a help of Eq. (3), we can get the acceleration in the XY -plane or substituting by $z = 0$ into Eq. (2), then one obtains

$$\mathbf{a}_b(r, 0) = -\frac{M_b r}{(r^2 + T^2)^{3/2}}(x, y) \quad (4)$$

3 Model description

We assume that m_1 and m_2 denote the bigger and smaller primaries masses respectively, and m is the mass of the infinitesimal body. We consider both masses m_1 and m_2 move in circular orbits around their common

center of mass. Furthermore the infinitesimal body m moves in the same plane of primaries motion under their mutual gravitational fields. We also assume that the coordinate system $OXYZ$ rotates about OZ -axis by the angular velocity n in positive direction. OX -axis is taken the joining line between the primaries, OY -axis is perpendicular to OX -axis and OZ -axis is perpendicular to the orbital plane of the primaries. Let r_1 and r_2 be the distances between m and the primaries m_1 and m_2 respectively, while R the separation distance between m_1 and m_2 . The coordinates of m_1 , m_2 and m are $(x_1, 0, 0)$, $(x_2, 0, 0)$ and $(x, y, 0)$ respectively.

Now we normalize the units as the sum of two masses m_1 and m_2 is one and the distance between them also is taken as one. In addition the gravitational constant is one. We also assume that $\mu = m_2/(m_1 + m_2)$ be the mass parameter. Consequently $m_2 = \mu$ and $m_1 = 1 - \mu$ with $m_1 > m_2$ and $0 < \mu \leq 1/2$. Then in the XY -plane, the coordinates of m_1, m_2 and m are $(x_1, 0, 0) = (\mu, 0, 0)$, $(x_2, 0, 0) = (\mu - 1, 0, 0)$ and $(x, y, 0)$ respectively. Consequently in the rotating coordinate dimensionless system, the motion equations of the infinitesimal body m under the gravitational potential of m_1 and m_2 are given by

$$\begin{aligned}\ddot{x} - 2n\dot{y} &= n^2x - \frac{(1-\mu)}{r_1^3}(x-\mu) - \frac{\mu}{r_2^3}(x-\mu+1) \\ \ddot{y} + 2n\dot{x} &= n^2y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}\end{aligned}\quad (5)$$

In Eq. (5), the angular velocity $n = 1$, see for details [35], where

$$\begin{aligned}r_1^2 &= (x-\mu)^2 + y^2 \\ r_2^2 &= (x-\mu+1)^2 + y^2\end{aligned}\quad (6)$$

In the case of the gravitational potential of asteroids belt is considered, then with a help of Eq. (4), the perturbed dynamical system of the restricted three-body problem is controlled by

$$\begin{aligned}\ddot{x} - 2n\dot{y} &= \Omega_x \\ \ddot{y} + 2n\dot{x} &= \Omega_y\end{aligned}\quad (7)$$

where

$$\Omega = \frac{n^2(x^2 + y^2)}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{M_b}{(r^2 + T^2)^{1/2}}\quad (8)$$

and r_1 and r_2 are given by Eq. (6), while the perturbed mean motion n is

$$n^2 = 1 + \frac{2M_b r_c}{(r_c^2 + T^2)^{3/2}}\quad (9)$$

here in Eq. (9), $r_c^2 = 1 - \mu + \mu^2$, see [34] for details.

Using Eqs. (7, 8) then the Jacobian integral is governed by

$$\dot{x}^2 + \dot{y}^2 - 2\Omega(x, y) + C = 0$$

In which C is the constant of integration.

4 Locations of equilibrium points

The equilibrium points are the locations of the infinitesimal body with zero velocity and zero acceleration, in the rotating reference frame. Then these locations can be found when $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$, therefore $\Omega_x = \Omega_y = 0$,

then we have to solve the following two equations

$$\begin{aligned} x \left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{M_b}{(r^2 + T^2)^{3/2}} \right] + \mu(1-\mu) \left[\frac{1}{r_1^3} - \frac{1}{r_2^3} \right] &= 0, \\ \left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{M_b}{(r^2 + T^2)^{3/2}} \right] y &= 0. \end{aligned} \quad (10)$$

Using the above two equations, we will investigate two cases for the locations of equilibrium points in the following subsection.

4.1 Location of collinear points

In the case of collinear equilibrium points (L_1, L_2 and L_3) $y = 0$, so that the equilibrium points lie on the line joining the primaries (X -axis), see Fig. 2, so we have

$$n^2 x - \frac{(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{\mu(x-\mu+1)}{|x-\mu+1|^3} - \frac{M_b x}{(r^2 + T^2)^{3/2}} = 0 \quad (11)$$

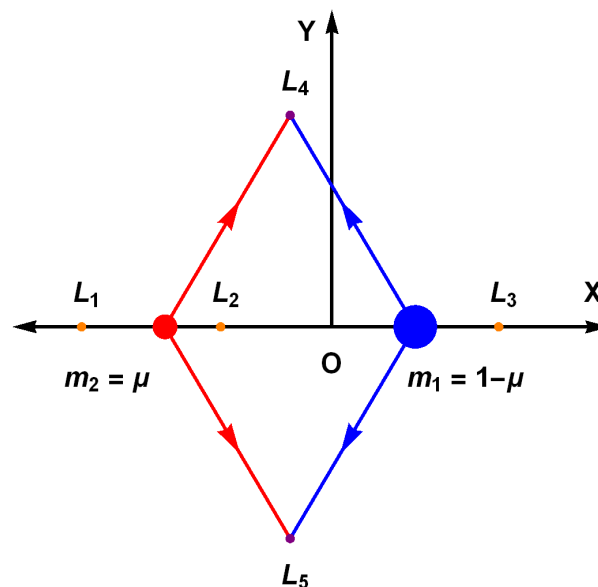


Fig. 2 Configuration of equilibrium points (Color figure online)

4.1.1 Location of L_1

The equilibrium point L_1 lies beyond masse m_1 as in Fig. 2. Then in this case, we have $r_1 - r_2 = 1$ therefor $\partial r_1 / \partial x = \partial r_2 / \partial x = -1$ and $r_2 = \mu - x - 1$, $r_1 = \mu - x$ using Eq. (11), we get

$$n^2 x + \frac{(1-\mu)}{(x-\mu)^2} + \frac{\mu}{(x-\mu+1)^2} - \frac{M_b x}{(x^2 + T^2)^{3/2}} = 0 \quad (12)$$

Let $r_2 = \xi_1$, $r_1 = 1 + \xi_1$, $x_1 = \mu - 1 - \xi_1$, and $n^2 = s$, then with a help of Eq. (12), we get

$$s(\mu - 1 - \xi_1) + \frac{(1-\mu)}{(1+\xi_1)^2} + \frac{\mu}{\xi_1^2} - \frac{M_b}{(\mu - 1 - \xi_1)^2} \left(1 - \frac{3}{2} \frac{T^2}{(\mu - 1 - \xi_1)^2} \right) = 0 \quad (13)$$

after writing Eq. (13) in the series form, we get

$$\begin{aligned} & \mu + 6\mu\xi_1 + \left(\left(1 - s - M_b + \frac{3T^2M_b}{2} \right) + (10 + 5s + 2M_b)\mu \right) \xi_1^2 \\ & + \left((4 - 7s - 4M_b + 3T^2M_b) + (4 + 30s + 6M_b)\mu \right) \xi_1^3 \\ & + \left(\left(6 - 21s - 6M_b + \frac{3T^2M_b}{2} \right) + (-3 + 75s + 6M_b)\mu \right) \xi_1^4 \\ & + ((4 - 35s - 4M_b) + (-2 + 100s + 2M_b)\mu) \xi_1^5 \\ & + ((1 - 35s - M_b) + 75s\mu) \xi_1^6 + (-21s + 30s\mu) \xi_1^7 \\ & + (-7s + 5s\mu) \xi_1^8 - s\xi_1^9 = 0 \end{aligned}$$

Then the parameter μ as a function in the variation ξ_1 is given by

$$\mu = a_{11}\xi_1^2 + a_{12}\xi_1^3 + a_{13}\xi_1^4 + O[\xi_1]^5$$

where

$$\begin{aligned} a_{11} &= -1 + s + M_b - \frac{3T^2M_b}{2} \\ a_{12} &= 2 + s - 2M_b + 6T^2M_b \\ a_{13} &= -8 + 10s - 5s^2 + 10M_b - 7sM_b - \frac{45T^2M_b}{2} + \frac{15}{2}sT^2M_b \end{aligned} \quad (14)$$

Consequently, we will using the Lagrangian inversion method to inverting the above series, which represents μ , to express ξ_1 as functions of μ , then we get

$$\xi_1 = c_{11}\sqrt{\mu} + c_{12}\mu + c_{13}\mu^{3/2}$$

where

$$\begin{aligned} c_{11} &= \sqrt{\frac{1}{-1 + s + M_b - \frac{3T^2M_b}{2}}} \\ c_{12} &= \frac{(-2 - s + 2M_b - 6T^2M_b)}{(-2 + 2s + 2M_b - 3T^2M_b) \left(-1 + s + M_b - \frac{3T^2M_b}{2} \right)} \\ c_{13} &= \left(\frac{1}{2(-2 + 2s + 2M_b - 3T^2M_b)^2} \left(\frac{1}{-1 + s + M_b - \frac{3T^2M_b}{2}} \right)^{3/2} \right. \\ & \quad \times \left. \left(-12 + 92s - 55s^2 + 20s^3 + 32M_b - 128sM_b \right. \right. \\ & \quad \left. \left. + 48s^2M_b - 18T^2M_b + 240sT^2M_b - 60s^2T^2M_b \right) \right) \end{aligned} \quad (15)$$

hence

$$x_1 = \mu - 1 - c_{11}\sqrt{\mu} - c_{12}\mu - c_{13}\mu^{3/2} \quad (16)$$

4.1.2 Location of L_2

Since the point L_2 lies between the two primaries, thereby $r_1 + r_2 = 1$, $r_2 = x - \mu + 1$, $r_1 = \mu - x$ and $\partial r_2 / \partial x = -\partial r_1 / \partial x = 1$, then by using Eq. (11), we get

$$n^2 x + \frac{(1-\mu)}{(x-\mu)^2} - \frac{\mu}{(x-\mu+1)^2} - \frac{M_b x}{(x^2 + T^2)^{3/2}} = 0 \quad (17)$$

If we take $r_2 = \xi_2$, $r_1 = 1 - \xi_2$ and $x_2 = \mu - 1 + \xi_2$, then we can rewrite Eq. (17) in the following form

$$s(\mu - 1 + \xi_2) - \frac{(1-\mu)}{(1-\xi_2)^2} - \frac{\mu}{\xi_2^2} - \frac{M_b}{(\mu - 1 + \xi_2)^2} \left(1 - \frac{3}{2} \frac{T^2}{(\mu - 1 + \xi_2)^2} \right) = 0 \quad (18)$$

again Eq. (18) in the series form is

$$\begin{aligned} & -\mu + 6\mu \xi_2 + \left((1 - M_b - s + 3M_b T^2 / 2) + (-20 + 2M_b + 5s)\mu \right) \xi_2^2 \\ & + \left((-4 + 4M_b + 7s - 3M_b T^2) + (36 - 6M_b - 30s)\mu \right) \xi_2^3 \\ & + \left((6 - 6M_b - 21s + 3M_b T^2 / 2) + (-33 + 6M_b + 75s)\mu \right) \xi_2^4 \\ & + \left((-4 + 4M_b + 35s) + (14 - 2M_b - 100s)\mu \right) \xi_2^5 \\ & + \left((1 - M_b - 35s) + (-2 + 75s)\mu \right) \xi_2^6 + (21s - 30s\mu) \xi_2^7 \\ & + (-7s + 5s\mu) \xi_2^8 + s\xi_2^9 = 0 \end{aligned}$$

Then the mass ratio μ as function in ξ_2 is given by

$$\mu = -a_{11}\xi_2^2 + a_{12}\xi_2^3 + a_{23}\xi_2^4 + O\xi_2^5$$

where a_{11} and a_{12} given by Eq. (14) while a_{23} is given by

$$a_{23} = 2 + 2s - s^2 - 4M_b + sM_b + \frac{39T^2 M_b}{2} - \frac{9}{2} sT^2 M_b$$

again using the Lagrangian inversion method to inverting the above series, we get

$$\xi_2 = -c_{11}\sqrt{\mu} + c_{12}\mu + c_{23}\mu^{3/2} + O[\mu]^2$$

where c_{11} and c_{12} given as Eq. (15) while c_{23} is given by

$$\begin{aligned} c_{23} &= \left(\frac{1}{2(-2 + 2s + 2M_b - 3T^2 M_b)^2} \right) \left(\frac{1}{1 - s - M_b + \frac{3T^2 M_b}{2}} \right)^{3/2} \\ &\times \left(\frac{12 + 20s + 17s^2 - 4s^3 - 16M_b - 32sM_b}{+30T^2 M_b + 144sT^2 M_b - 12s^2 T^2 M_b} \right) \mu^{3/2} \end{aligned}$$

hence

$$x_2 = \mu - 1 + c_{21}\sqrt{\mu} + c_{22}\mu + c_{23}\mu^{3/2} + O[\mu]^2 \quad (19)$$

4.1.3 Location of L_3

The point L_3 lies beyond the large mass according to Fig. 2, $r_2 - r_1 = 1$, $r_1 = x - \mu$, $r_2 = x - \mu + 1$ and $\partial r_1 / \partial x = \partial r_2 / \partial x = 1$, then Eq. (11) can be rewritten in the form

$$n^2 x - \frac{(1-\mu)}{(x-\mu)^2} - \frac{\mu}{(x-\mu+1)^2} - \frac{M_b x}{(x^2 + T^2)^{3/2}} = 0 \quad (20)$$

after substituting $r_2 = \xi_3 + 2$ and $r_1 = \xi_3 + 1$, into Eq. (20), we get

$$s(\mu + \xi_3 + 1) - \frac{(1-\mu)}{(1+\xi_3)^2} - \frac{\mu}{(2+\xi_3)^2} - \frac{M_b}{(\mu + \xi_3 + 1)^2} \left(1 - \frac{3}{2} \frac{T^2}{(\mu + \xi_3 + 1)^2} \right) = 0 \quad (21)$$

Again the form series of Eq. (21) is given by

$$\begin{aligned} & a_{30} + d_{30}\mu + (a_{31} + d_{31}\mu)\xi + (a_{32} + d_{32}\mu)\xi^2 + (a_{33} + d_{33}\mu)\xi^3 \\ & + (a_{34} + d_{34}\mu)\xi^4 + (a_{35} + d_{35}\mu)\xi^5 + (a_{36} + d_{36}\mu)\xi^6 \\ & + (a_{37} + d_{37}\mu)\xi^7 + (a_{38} + d_{38}\mu)\xi^8 + (a_{39} + d_{39}\mu)\xi^9 = 0 \end{aligned}$$

where

$$\begin{aligned} a_{30} &= -1 - M_b + s + \frac{3M_b T^2}{2}, & d_{30} &= \frac{3}{4} + 2M_b + s - 6M_b T^2 \\ a_{31} &= 2 + 2M_b + s - 6M_b T^2, & d_{31} &= -\frac{7}{4} - 6M_b + 30M_b T^2 \\ a_{32} &= -3 - 3M_b + 15M_b T^2, & d_{32} &= \frac{45}{16} + 12M_b - 90M_b T^2 \\ a_{33} &= (4 + 4M_b - 30M_b T^2), & d_{33} &= -\frac{31}{8} - 20M_b + 210M_b T^2 \\ a_{34} &= -5 - 5M_b + \frac{105M_b T^2}{2}, & d_{34} &= \frac{315}{64} + 30M_b - 420M_b T^2 \\ a_{35} &= 6 + 6M_b - 84M_b T^2, & d_{35} &= -\frac{381}{64} - 42M_b + 756M_b T^2 \\ a_{36} &= -7 - 7M_b + 126M_b T^2, & d_{36} &= \frac{1785}{256} + 56M_b - 1260M_b T^2 \\ a_{37} &= 8 + 8M_b - 180M_b T^2, & d_{37} &= -\frac{511}{64} - 72M_b + 1980M_b T^2 \\ a_{38} &= -9 - 9M_b + \frac{495M_b T^2}{2}, & d_{38} &= \frac{9207}{1024} + 90M_b - 2970M_b T^2 \\ a_{39} &= 10 + 10M_b - 330M_b T^2, & d_{39} &= -\frac{10235}{1024} - 110M_b + 4290M_b T^2 \end{aligned}$$

Then μ as a function series in ξ_3 is given by

$$\mu = b_{30} + b_{31}\xi_3 + b_{32}\xi_3^2 + O[\xi_3^3]$$

where

$$b_{30} = \frac{32(-2 - 2M_b + 2s + 3MT^2)}{-48 - 128M_b - 64s + 384M_bT^2}$$

$$b_{31} = \frac{2(2 + 18M_b + 16M_b^2 - 36s - 80M_b s - 8s^2 - 129M_bT^2 + 336M_b sT^2)}{(-3 - 8M_b - 4s + 24M_bT^2)^2}$$

$$b_{32} = \frac{1}{2(-3 - 8M_b - 4s + 24M_bT^2)^3} \begin{pmatrix} -2 + 78M_b + 522s + 3608M_b s - 520s^2 - 1152M_b s^2 \\ -2097M_bT^2 - 20460M_b sT^2 + 9600M_b s^2T^2 \end{pmatrix}$$

Now using the Lagrangian inversion method to inverting the above series, we get

$$\xi_3 = c_{31}\mu + c_{32}$$

where

$$c_{31} = \frac{(-3 - 8M_b - 4s + 24M_bT^2)^2}{2(2 + 18M_b - 36s - 80M_b s - 8s^2 - 129M_bT^2)}$$

$$c_{32} = \frac{(-3 - 8M_b - 4s + 24M_bT^2)^2 \left(-\frac{32(-2 - 2M_b + 2s + 3M_bT^2)}{-48 - 128M_b - 64s + 384M_bT^2} \right)}{2(2 + 18M_b - 36s - 80M_b s - 8s^2 - 129M_bT^2 + 336M_b sT^2)}$$

hence

$$x_3 = 1 + \mu + c_{31}\mu + c_{32} \quad (22)$$

Finally we can use Eqs. (16, 19, 22) to find the locations of collinear points.

4.2 Location of triangular points

In the case of the triangular equilibrium points (L_4 and L_5) $y \neq 0$ and $\Omega_x = 0 = \Omega_y$. Using Eqs. (10), we get

$$n^2 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} - \frac{M_b}{(r^2 + T^2)^{3/2}} = 0$$

$$\frac{\mu(1-\mu)}{r_1^3} - \frac{\mu(1-\mu)}{r_2^3} = 0 \quad (23)$$

From the second equation in Eq. (23), we get

$$r_1 = r_2.$$

If we assume that the effect of the potential from the belt is neglected, i.e., $M_b = 0$, then Eqs. (23) are reduced to the classical case of Szebehely solutions $r_1 = r_2 = 1$ [35]. But, due to the perturbations from the belt, we assume that the solutions may change slightly by ε , then we can write

$$r_1 = r_2 = 1 + \varepsilon \quad (24)$$

From Eqs. (9, 23, 24), we get

$$1 - (1 + \varepsilon)^{-3} + \frac{2M_b r_c}{(r_c^2 + T^2)^{3/2}} - \frac{M_b}{(r^2 + T^2)^{3/2}} = 0 \quad (25)$$

where ε is very small quantities represents perturbation effect of asteroids belt.

Now we will keep the linear terms ε and neglecting the higher orders, then with a help of Eq. (25), we have

$$\varepsilon = -\frac{M_b(2r_c - 1)}{3(r_c^2 + T^2)^{3/2}} \quad (26)$$

From Eqs. (6, 24, 26), we get the locations of the triangular points in the following form

$$x = \mu - \frac{1}{2}, \quad y = \pm \frac{\sqrt{3}}{2} \left(1 - \frac{4M_b(2r_c - 1)}{9(r_c^2 + T^2)^{3/2}} \right) \quad (27)$$

5 Stability of motion around the libration point

After determining the locations of libration points, we will move to understand the stability motion properties around these points. In order to study the motion of the infinitesimal body in the neighborhood of an equilibrium points (x_0, y_0) , we employ small displacement (ξ, η) to the coordinate (x_0, y_0) where (x_0, y_0) represents the coordinates of one of five equilibria points. So that the vector of variation is related to the initial stat vector by $\mathbf{r} = \mathbf{r}_0 + \Delta\mathbf{r}$ where $\mathbf{r}_0 \equiv (x_0, y_0)$, $\mathbf{r} \equiv (x, y)$ and $\Delta\mathbf{r} \equiv (\xi, \eta)$, then we can write

$$\begin{aligned} x &= x_0 + \xi \\ y &= y_0 + \eta \end{aligned} \quad (28)$$

We linearize the equations of motion by using Taylor series around equilibrium. Then using Eqs. (7) and Eqs. (28), we obtain

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \Omega_x^0 + \frac{1}{1!} \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) \Omega_x^0 + \frac{1}{2!} \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right)^2 \Omega_x^0 + O(3) \\ \ddot{\eta} - 2n\dot{\xi} &= \Omega_y^0 + \frac{1}{1!} \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) \Omega_y^0 + \frac{1}{2!} \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right)^2 \Omega_y^0 + O(3) \end{aligned}$$

From the above equations the linear variational equations are

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \xi \Omega_{xx}^0 + \Omega_{xy}^0 \eta \\ \ddot{\eta} + 2n\dot{\xi} &= \xi \Omega_{xy}^0 + \Omega_{yy}^0 \eta \end{aligned} \quad (29)$$

where subscripts x and y denoted to the second partial derivatives of Ω . While the superscript 0 indicates that the partial derivatives have been evaluated at one of the equilibrium points (x_0, y_0) , where these derivatives are given by

$$\begin{aligned} \Omega_{xx} &= n^2 + \frac{3(1-\mu)(x-\mu)^2}{r_1^5} - \frac{1-\mu}{r_1^3} + \frac{3\mu(x+1-\mu)^2}{r_2^5} - \frac{\mu}{r_2^3} - \frac{M_b}{(r^2 + T^2)^{3/2}} + \frac{3M_b x^2}{(r^2 + T^2)^{5/2}} \\ \Omega_{yy} &= n^2 + \frac{3(1-\mu)y^2}{r_1^5} - \frac{1-\mu}{r_1^3} + \frac{3\mu y^2}{r_2^5} - \frac{\mu}{r_2^3} - \frac{M_b}{(r^2 + T^2)^{3/2}} + \frac{3M_b y^2}{(r^2 + T^2)^{5/2}} \\ \Omega_{xy} &= \frac{3(1-\mu)(x-\mu)y}{r_1^5} + \frac{3\mu(x+1-\mu)y}{r_2^5} + \frac{3M_b xy}{(r^2 + T^2)^{5/2}} \end{aligned} \quad (30)$$

Now we suppose that the solutions of the Eqs. (29) are

$$\xi = K e^{\lambda t}, \quad \eta = M e^{\lambda t} \quad (31)$$

where K , M and λ constant, thereby

$$\dot{\xi} = k\lambda e^{\lambda t}, \quad \ddot{\xi} = k\lambda^2 e^{\lambda t}, \quad \dot{\eta} = M\lambda e^{\lambda t}, \quad \ddot{\eta} = M\lambda^2 e^{\lambda t}. \quad (32)$$

substituting Eqs. (31, 32) into Eqs. (29), then the characteristic equation is determined by

$$\begin{vmatrix} \lambda^2 - \Omega_{xx} & -2n\lambda - \Omega_{xy} \\ 2n\lambda - \Omega_{xy} & \lambda^2 - \Omega_{yy} \end{vmatrix} = 0$$

hence, we get

$$\lambda^4 + (4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0)\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{xy}^0)^2 = 0 \quad (33)$$

The equilibrium point (x_0, y_0) is stable, when all the roots of the characteristic Eq. (33) are distinct pure imaginary numbers, this will be search in the next section.

5.1 Stability of collinear points

At the collinear libration points $y = 0$, and $r_1 = |x - \mu|$ and $r_2 = |x - \mu + 1|$. From Eqs. (30, 33), the characteristic equation in case of collinear libration points can be reduced to

$$\lambda^4 + (4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0)\lambda^2 + \Omega_{xx}^0\Omega_{yy}^0 = 0 \quad (34)$$

Let $\Lambda = \lambda^2$, Eq. (34) can be rewritten in the form

$$\Lambda^2 + b_c\Lambda + c_c = 0 \quad (35)$$

then Eq. (35) represents a quadratic equation, which its solution can be written in the form

$$\Lambda_{1,2} = -\frac{1}{2} \left[b_c \pm \sqrt{b_c^2 - 4c_c} \right] \quad (36)$$

where

$$\begin{aligned} b_c &= 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0 \\ c_c &= \Omega_{xx}^0\Omega_{yy}^0 \end{aligned} \quad (37)$$

Then with a help of Eqs. (36, 37), the roots of characteristic Eq. (34) can be write as $\lambda_{1,2} = \pm\sigma$, $\lambda_{3,4} = \pm i\tau$ where σ and τ are real number, which can be calculated by

$$\begin{aligned} \sigma^2 &= \frac{1}{2} \left[\sqrt{b_c^2 - 4c_c} - b_c \right] \\ \tau^2 &= \frac{1}{2} \left[\sqrt{b_c^2 - 4c_c} + b_c \right] \end{aligned}$$

From the above two equations, we can determine the eigenvalues and the properties of the equilibrium point.

5.2 Stability of triangular points

At the triangular points, we have

$$\begin{aligned} \Omega_{xx}^0 &= \frac{3}{4} \left(1 + \frac{5M_b(2r_c - 1)}{3(r_c^2 + T^2)^{3/2}} + \frac{M_b(\mu - 1/2)^2}{3(r_c^2 + T^2)^{5/2}} \right) \\ \Omega_{yy}^0 &= \frac{9}{4} \left(1 + \frac{7M_b(2r_c - 1)}{9(r_c^2 + T^2)^{3/2}} + \frac{M_b}{(r_c^2 + T^2)^{5/2}} \right) \\ \Omega_{xy}^0 &= \mp \frac{3\sqrt{3}}{4} (1 - 2\mu) \left(1 + \frac{M_b}{(r_c^2 + T^2)^{5/2}} - \frac{11M_b(2r_c - 1)}{9(r_c^2 + T^2)^{3/2}} \right) \end{aligned} \quad (38)$$

and substituting $\lambda^2 = \omega$ into the characteristic Eq. (33), then we get

$$\omega^2 + b\omega + c = 0 \quad (39)$$

where a and c can be evaluated with a help of Eqs. (38) and the following two relations

$$b = 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0$$

$$c = \Omega_{xx}^0 \Omega_{yy}^0 - (\Omega_{xy}^0)^2$$

hence we have

$$b = 1 + \frac{M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{3M_b r_c^2}{(r_c^2 + T^2)^{5/2}}$$

$$c = \left(-\frac{27}{4} - \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} - \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right) \mu^2$$

$$+ \left(\frac{27}{4} + \frac{33M_b(2r_c - 1)}{2(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{4(r_c^2 + T^2)^{5/2}} \right) \mu \quad (40)$$

Thereby the roots of Eq. (39) are given by

$$\omega_{1,2} = -\frac{1}{2} [b \mp \sqrt{D}] \quad (41)$$

where $D = b^2 - 4c$ is discriminant

$$D = \left(27 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right) \mu^2$$

$$- \left(27 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right) \mu$$

$$+ 1 + \frac{2M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{5/2}} \quad (42)$$

which can be written as of a function of the mass parameter μ in the form

$$D = \alpha\mu^2 - \beta\mu + \gamma \quad (43)$$

here α and β are the coefficients μ^2 and μ respectively. Using Eq. (41) the roots are given by

$$\lambda_{1,2} = \pm\sqrt{\omega_1}, \quad \lambda_{3,4} = \pm\sqrt{\omega_2}$$

Since $0 < \mu \leq 1/2$, then with using Eq. (43), we can study the behavior of D in the interval $(0, 1/2)$

$$D = \begin{cases} 1 + \left(\frac{2M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{6M_b r_c^2}{(r_c^2 + T^2)^{5/2}} \right) > 0, \text{ when } \mu = 0 \\ - \left(\frac{23}{4} + \frac{M_b(58r_c - 45)}{2(r_c^2 + T^2)^{3/2}} + \frac{3M_b(8r_c^2 + 9)}{4(r_c^2 + T^2)^{5/2}} \right) < 0, \text{ when } \mu = 1/2 \end{cases} \quad (44)$$

Also the derivative of D with respect to the parameter μ is given by

$$\frac{dD}{d\mu} = 2 \left(27 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right) \mu - \left(27 + \frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right)$$

thereby

$$\frac{dD}{d\mu} = \begin{cases} -27 - \left(\frac{66M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{27M_b}{(r_c^2 + T^2)^{5/2}} \right) < 0, & \text{when } \mu = 0 \\ 0, & \text{when } \mu = 1/2 \end{cases} \quad (45)$$

because $M_b \ll 1$, and using Eqs. (44, 45), we obtain

$$\frac{dD}{d\mu} \leq 0 \quad \forall \mu \in (0, 1/2)$$

Eqs. (44, 45) show that the discriminant D has two different signs at the end of interval $(0, 1/2)$, further $dD/d\mu < 0$ in the interval $(0, -\beta/2\alpha)$. Then D is strictly decreasing function in this interval, and there is only one value for μ in $(0, 1/2)$, where D vanish, which is called the critical mass parameter (μ_c). Consequently we will examine three possible cases for the value of μ .

- If $0 < \mu < \mu_c$ implies $D = b^2 - 4c > 0$, and D decreasing in the interval $(0, 1/2)$. Since $b > 0$, $b^2 - 4c < b^2$ ($\sqrt{b^2 - 4c} < b$) then $\omega < 0$, thereby the four roots of λ are distinct pure imaginary numbers. Hence the triangular points are stable in this interval.
- If $\mu = \mu_c$ ($D = 0$), then we have double equal roots of λ which lead to secular terms, thereby the triangular points are unstable.
- When $\mu_c < \mu < 1/2$, then $D < 0$ and we obtain four complex roots, with two of them whose the same real part and positive. Therefore the triangular points are also unstable.

5.3 Critical mass

Under the previous discussion, when Eq. (43) is equal zero, then one can obtain the value of critical mass (μ_c), which is governed by

$$\mu_c = -\frac{1}{2\alpha} \left[\alpha + \sqrt{\alpha^2 - 4\alpha\gamma} \right] \quad (46)$$

where

$$\begin{aligned} \alpha^2 &= 729 + \frac{3564M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{1458M_b}{(r_c^2 + T^2)^{5/2}} \\ 4\alpha\gamma &= 108 + \frac{216M_b(2r_c + 3)}{(r_c^2 + T^2)^{3/2}} - \frac{648M_b r_c^2}{(r_c^2 + T^2)^{5/2}} \\ &\quad + \frac{264M_b(2r_c - 1)}{(r_c^2 + T^2)^{3/2}} + \frac{108M_b}{(r_c^2 + T^2)^{5/2}} \end{aligned} \quad (47)$$

utilizing Eqs. (46, 47) we get

$$\mu_c = \mu_0 + \mu_p \quad (48)$$

Eq. (48) represents the value of the critical mass with two parts. The first μ_0 is related to the classical restricted problem without perturbation. While the second (μ_p) is related to the effect of asteroids belt. Of course in the absence of the asteroids belt effect, the value of critical mass is given by $\mu_c = \mu_0$, where the values of μ_0 and μ_p are given by the following relations

$$\mu_0 = \frac{1}{2} \left(1 - \frac{\sqrt{69}}{9} \right), \quad \mu_p = \frac{(76 - 8r_c)M_b}{27\sqrt{69}(r_c^2 + T^2)^{3/2}} - \frac{(1 + 6r_c^2)M_b}{3\sqrt{69}(r_c^2 + T^2)^{5/2}}$$

6 Periodic orbits

6.1 Periodic orbits around collinear points

Now it is easy to obtain the periodic orbits around the collinear points. Although these points are unstable i.e. if a body in any of these points is disturbed, a body will move away. After substituting Eq. (31) into Eq. (29) with some simple computations, we will get a relation between the coefficients K_j and M_j , it is governed by

$$M_j = \alpha_j K_j \quad (49)$$

where

$$\alpha_j = \frac{(\lambda_j^2 - \Omega_{xx}^0)}{2n\lambda_j} = \frac{2n\lambda_j}{\Omega_{yy}^0 - \lambda_j^2}$$

This relation means that the coefficients K_j and M_j ($j = 1, 2, 3, 4$) are dependent. Therefore the four initial conditions $\xi_0, \eta_0, \dot{\xi}_0$ and $\dot{\eta}_0$ associated with Eqs. (29) will determine the two sets of coefficients and will completely determine the eight coefficients (K_j, M_j). Where the subscript 0 indicates to these quantities are evaluated at the initial time ($t = t_0$). Substituting Eq. (49) into Eq. (31), we get

$$\begin{aligned} \xi_0 &= \sum_{j=1}^4 K_j e^{\lambda_j t} & \dot{\xi}_0 &= \sum_{j=1}^4 K_j \lambda_j e^{\lambda_j t} \\ \eta_0 &= \sum_{j=1}^4 K_j \alpha_j e^{\lambda_j t} & \dot{\eta}_0 &= \sum_{j=1}^4 K_j \lambda_j \alpha_j e^{\lambda_j t} \end{aligned} \quad (50)$$

The coefficient can be expressed as function of the initial conditions, because the determinant (Δ) of System of Eqs. (50) is not zero.

$$\Delta = -\sqrt{\frac{\Omega_{xx}^0}{\Omega_{yy}^0}} \left[(\Omega_{xx}^0 + \Omega_{yy}^0 - 4n)^2 - \frac{1}{2} \Omega_{xx}^0 \Omega_{yy}^0 \right] \neq 0$$

The motion is bounded and consists of two harmonic motion when all roots of characteristic equation are purely imaginary [1], where the solution depends on the eigenfrequencies σ, τ , can be written in the form

$$\begin{aligned} \xi &= K_1 \cos \sigma(t - t_0) + K_2 \sin \sigma(t - t_0) + K_3 \cos \tau(t - t_0) + K_4 \sin \tau(t - t_0), \\ \eta &= M_1 \cos \sigma(t - t_0) + M_2 \sin \sigma(t - t_0) + M_3 \cos \tau(t - t_0) + M_4 \sin \tau(t - t_0). \end{aligned} \quad (51)$$

Now We can take $K_1 = K_2 = 0$, therefor the solution in Eqs. (51) can rewritten in the following form

$$\begin{aligned} \xi &= \xi_0 \cos \tau(t - t_0) + \frac{\eta_0}{\beta_3} \sin \tau(t - t_0) \\ \eta &= \eta_0 \cos \tau(t - t_0) - \xi_0 \beta_3 \sin \tau(t - t_0) \end{aligned} \quad (52)$$

where the real quantities τ and β_3 are defined by

$$\tau = \left\{ \frac{1}{2} \left[\sqrt{(4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0)^2 - 4\Omega_{xx}^0\Omega_{yy}^0} + 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0 \right] \right\}^{1/2}$$

$$\beta_3 = \frac{\tau^2 + \Omega_{xx}^0}{2n\tau} = \frac{2n\tau}{\tau^2 + \Omega_{yy}^0}$$

in which $\lambda_3 = i\tau$ and $\alpha_3 = i\beta_3$

From Eqs. (52) we get the velocity variation in the form

$$\begin{aligned}\dot{\xi} &= -\xi_0\tau \sin \tau(t-t_0) + \frac{\eta_0}{\beta_3}\tau \cos \tau(t-t_0) \\ \dot{\eta} &= -\eta_0\tau \sin \tau(t-t_0) - \xi_0\beta_3\tau \cos \tau(t-t_0)\end{aligned}\quad (53)$$

Using Eqs. (53), when $t = t_0$, one obtains

$$\dot{\xi}_0 = \eta_0\tau/\beta_3, \quad \dot{\eta}_0 = -\xi_0\beta_3\tau \quad (54)$$

With a help of Eq. (54) we can eliminate the time from Eq. (52), and the equation of periodic orbits reduced to

$$\xi^2\beta_3^2 + \eta^2 = \eta_0^2 + \xi_0^2\beta_3^2$$

which can be rewritten in the following standard form

$$\frac{\xi^2}{(\eta_0^2 + \xi_0^2\beta_3^2)/\beta_3^2} + \frac{\eta^2}{(\eta_0^2 + \xi_0^2\beta_3^2)} = 1 \quad (55)$$

Eq. (64) shows that the trajectory of the body around the collinear points is an ellipses whose center at these points. The parameters of the ellipse, the semi-major axes (a_c), the semi-minor axes (b_c) and the eccentricity (e_c) are given by

$$a_c^2 = (\eta_0^2 + \xi_0^2\beta_3^2), \quad a_c^2 = (\eta_0^2 + \xi_0^2\beta_3^2), \quad e_c^2 = \left(1 - \frac{1}{\beta_3^2}\right)$$

where the semi-major axes parallel to the η -axes while the semi-minor axes parallel to the ξ -axes. Also the periodic time (T_c) can be calculated by $T_c = 2\pi/\tau$. Since $\dot{\eta}_0 = -\xi_0\beta_3\tau$, $\dot{\xi}_0 = 0$ at $\xi_0 \neq 0$ and $\eta_0 = 0$, the motion along the orbits is retrograde.

6.2 Periodic orbits around triangular points

The triangular points are linearly stable in the range $0 < \mu < \mu_c$. And the characteristic equation has four purely imaginary roots in neighborhood of the triangular points. So we have bounded motion around the triangular points. Which composed of two harmonic motions governed by the variation ξ and η by the following relations

$$\begin{aligned}\xi &= C_1 \cos s_1 t + D_1 \sin s_1 t + C_2 \cos s_2 t + D_2 \sin s_2 t, \\ \eta &= \bar{C}_1 \cos s_1 t + \bar{D}_1 \sin s_1 t + \bar{C}_2 \cos s_2 t + \bar{D}_2 \sin s_2 t.\end{aligned}\quad (56)$$

Therefor the terms with coefficients $C_1, D_1, \bar{C}_1, \bar{D}_1$ associated with angular frequencies s_1 (mean motion) refer to the long periodic orbits and terms with coefficients $C_2, D_2, \bar{C}_2, \bar{D}_2$ associated with angular frequencies s_2 , which refer to the short periodic orbits. In addition $s_{1,2}^2 = -\omega$, thereby we get

$$s_{1,2}^2 = \frac{1}{2} [b \mp \sqrt{D}] \quad (57)$$

Substituting Eqs. (42, 40) into Eq. (57), with some simple calculations, the angular frequencies of motion s_1 and s_2 are given by

$$\begin{aligned} s_1 &= \frac{3\sqrt{3}}{2} \sqrt{\mu(1-\mu)} \left(1 + \frac{11M_b(2r_c-1)}{9(r_c^2+T^2)^{3/2}} + \frac{M_b}{2(r_c^2+T^2)^{5/2}} \right) \\ s_2 &= 1 - \frac{27}{8} \mu(1-\mu) \left(1 + \frac{22M_b(2r_c-1)}{9(r_c^2+T^2)^{3/2}} + \frac{M_b}{(r_c^2+T^2)^{5/2}} \right) \\ &\quad + \frac{M_b(2r_c+3)}{2(r_c^2+T^2)^{3/2}} - \frac{3M_b r_c^2}{2(r_c^2+T^2)^{5/2}} \end{aligned} \quad (58)$$

We can determine the relation between the coefficients of short and long period terms, when we substituting Eq. (56) into Eq. (29) and equating the coefficients of sine and cosine terms, we get

$$\begin{aligned} \bar{C}_i &= \Gamma_i (2nD_i s_i - \Omega_{xy}^0 C_i) \\ \bar{D}_i &= -\Gamma_i (2nC_i s_i + \Omega_{xy}^0 D_i) \end{aligned}$$

where

$$\Gamma_i = \frac{1}{s_i^2 + \Omega_{yy}^0} = \frac{s_i^2 + \Omega_{xx}^0}{4n^2 s_i^2 + (\Omega_{xy}^0)^2} > 0$$

here ($i = 1, 2$) and $\Omega_{xx}^0, \Omega_{xy}^0$ and Ω_{yy}^0 are given by Eqs. (38).

Either the long or short period terms can be eliminated from the solving by properly selected initial conditions. The four initial conditions at $t = 0$ ($\xi_0, \eta_0, \dot{\xi}_0$ and $\dot{\eta}_0$) are linearly related to the four coefficient. Therefore we are not found difficulty in establishing the desired initial condition. Hence if we suppose that the short periodic terms are vanished, i.e. $C_2 = D_2 = \bar{C}_2 = \bar{D}_2 = 0$, then the relation between the initial conditions and the coefficients in Eqs. (56) are

$$\xi_0 = C_1, \quad \eta_0 = \bar{C}_1, \quad \dot{\xi}_0 = D_1 s_1, \quad \dot{\eta}_0 = \bar{D}_1 s_1$$

where

$$D_1 = \frac{\xi_0 \Omega_{xy}^0 + \eta_0 (\Omega_{yy}^0 + s_1^2)}{2ns_1}, \quad \bar{D}_1 = \frac{\xi_0 (\Omega_{xx}^0 + s_1^2) + \eta_0 \Omega_{xy}^0}{2ns_1}$$

6.2.1 Elliptic orbits

Now we assume that a triangular point represents the origin of the coordinates system, where the third body starts its motion at the origin of the coordinate system. So we can get the initial conditions from Eq. (27) by $(\xi_0, \eta_0) = (-x_0, -y_0)$ where

$$\xi_0 = \frac{1}{2} - \mu, \quad \eta_0 = \mp \frac{\sqrt{3}}{2} \left(1 - \frac{4M_b(2r_c-1)}{9(r_c^2+T^2)^{3/2}} \right)$$

In which the negative sign (plus) means that the infinitesimal body starts its motion from L_4 (L_5). The trajectory of infinitesimal body after the elimination the short or long periodic terms becomes an ellipse. Which can be seen when we rewritten Eq. (56) for the long periodic solution in the form

$$\begin{aligned} \xi &= C_1 \cos s_1 t + D_1 \sin s_1 t, \\ \eta &= \bar{C}_1 \cos s_1 t + \bar{D}_1 \sin s_1 t. \end{aligned} \quad (59)$$

After eliminating the time from Eqs. (59), we get

$$\alpha_1 \xi^2 + \eta^2 + 2\beta_1 \xi \eta = \gamma_1 \quad (60)$$

Since

$$\left| \frac{\Gamma_1^2 (4n^2 s_1^2 + (\Omega_{xy}^0)^2)}{\Gamma_1 \Omega_{xy}^0} \frac{\Gamma_1 \Omega_{xy}^0}{1} \right| = (2ns_1 \Gamma_1)^2 > 0$$

Eq. (60) represents of an ellipse with center at the origin coordinate system (ξ, η) , which is coinciding with L_4 or L_5 , where

$$\alpha_1 = \frac{4n^2 s_1^2 + (\Omega_{xy}^0)^2}{(s_1^2 + \Omega_{yy}^0)^2}, \quad \beta_1 = \frac{\Omega_{xy}^0}{s_1^2 + \Omega_{yy}^0}, \quad \gamma_1 = \alpha_1 \xi_0^2 + 2\beta_1 \xi_0 \eta_0 + \eta_0^2$$

6.2.2 The orientation of principal axes of the ellipse

Since Eq. (60) includes bilinear term $\xi\eta$ that appears as a result of the rotation of the principal axes of ellipses through an angle θ with respect to the coordinate system (ξ, η) . So we introduce a new coordinate reference frame $(\bar{\xi}, \bar{\eta})$ called normal coordinates such that the bilinear term dose not appear. Hence the old and new coordinates system are related by the following equation

$$\begin{aligned} \xi &= \bar{\xi} \cos \theta - \bar{\eta} \sin \theta \\ \eta &= \bar{\xi} \sin \theta + \bar{\eta} \cos \theta \end{aligned} \quad (61)$$

Substituting Eqs. (61) into Eq. (60) and equate the coefficient of $\bar{\xi} \bar{\eta}$ by zero, therefor after simplify the equation, we get

$$\frac{\bar{\xi}^2}{\bar{a}^2} + \frac{\bar{\eta}^2}{\bar{b}^2} = 1 \quad (62)$$

where the orientation of the principal axes is governed by

$$\tan 2\theta = \frac{2\Omega_{xy}^0}{\Omega_{xx}^0 - \Omega_{yy}^0} \quad (63)$$

From Eqs. (38, 63), we obtain

$$\tan 2\theta = \pm \sqrt{3} \left(1 - 2\mu + \frac{8(1-2\mu)(2r_c-1)M_b}{9(r_c^2+T^2)^{3/2}} - \frac{2\mu(1-3\mu)M_b}{(r_c^2+T^2)^{5/2}} \right)$$

where plus sign (minus sign) refers to the center of ellipse at L_4 (L_5).

Furthermore the lengths of semi-major (a), and semi-minor (b) axes are controlled by

$$\begin{aligned} \bar{a}^2 &= \frac{2\gamma_1}{((1+\alpha_1) - (1-\alpha_1)\cos 2\theta + 2\beta_1 \sin 2\theta)} \\ \bar{b}^2 &= \frac{2\gamma_1}{((1+\alpha_1) + (1-\alpha_1)\cos 2\theta - 2\beta_1 \sin 2\theta)} \end{aligned} \quad (64)$$

The eccentricity \bar{e} of the ellipse is

$$\bar{e}^2 = \frac{2((1-\alpha_1)\cos 2\theta - 2\beta_1 \sin 2\theta)}{((1+\alpha_1) + (1-\alpha_1)\cos 2\theta - 2\beta_1 \sin 2\theta)} \quad (65)$$

While the periodic of motion $T = 2\pi/s$, where s is given by the relations in Eqs. (58). Finally we demonstrate that the motion of the infinitesimal body around the triangular point will be elliptical and it is given by Eqs. (62) in normal coordinates, where the parameter of motion are given in Eqs. (64, 65).

Conclusion

We conducted a comprehensive analytical study on the effect of the gravitational force of the asteroids belt within frame of the restricted three-body problem. We have formulated the equations of motion of the restricted three-body problem, in the event of perturbation of the asteroids belt. Hence we conducted an analytical study to determine the locations of liberation points and study the linear stability of motion around these points. Furthermore we identified the elements of the periodic orbits of the infinitesimal body in the presence of the asteroids belt perturbation.

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References

- [1] Abouelmagd, E. I., Alhothuali, M. S., Guirao, L. G. J., Malaikah, H. M. (2015), *The effect of zonal harmonic coefficients in the framework of the restricted three-body problem*, *Astrophys. Space Sci.* 55: 1660 – 1672. Doi:10.1016/j.asr.2014.12.030
- [2] Abouelmagd E. I., Alzahrani F., Guirao J. L. G., Hobiny A. (2016), *Periodic orbits around the collinear libration points*, *J. Nonlinear Sci. Appl. (JNSA)*. 9 (4): 1716 – 1727. Doi:10.22436/jnsa.009.04.27
- [3] Abouelmagd E. I., Asiri H. M., Sharaf M. A. (2013), *The effect of oblateness in the perturbed restricted three-body problem*, *Meccanica* 48: 2479 – 2490. Doi 10.1007/S11012-013-9762-3
- [4] Abouelmagd, E.I., Awad, M.E., Elzayat, E.M.A., Abbas, I.A. (2014), *Reduction the secular solution to periodic solution in the generalized restricted three-body problem*, *Astrophys. Space Sci.* 341: 495 – 505. Doi:10.1007/s10509-013-1756-z
- [5] Abouelmagd E. I., El-Shaboury S. M. (2012), *Periodic orbits under combined effects of oblateness and radiation in the restricted problem of three bodies*, *Astrophys. Space Sci.* 341: 331 – 341. Doi : 10.1007/s10509-012-1093-7
- [6] Abouelmagd E. I. (2012), *Existence and stability of triangular points in the restricted three-body problem with numerical applications*, *Astrophys. Space Sci.* 342: 45 – 53. Doi 10.1007/s10509-012-1162-y
- [7] Abouelmagd E. I., Guirao J. L.G., Llibre J. (2019), *Periodic orbits for the perturbed planar circular restricted 3-body problem*, *Discrete and Continuous Dynamical Systems - Series B (DCDS-B)* 24 (3): 1007 – 1020. Doi:10.3934/dcdsb.2019003
- [8] Abouelmagd E. I., Mostafa A., Guirao J. L. G. (2015), *A first order automated Lie transform*. *International Journal of Bifurcation and Chaos*. 25 (14): 1540026. Doi:10.1142/S021812741540026X
- [9] Abouelmagd E. I., Guirao Juan L. G., Mostafa A. (2014), *Numerical integration of the restricted three-body problem with Lie series*, *Astrophys. Space Sci.* 354 (2)P: 369 – 378. Doi 10.1007/s10509-014-2107-4
- [10] Abouelmagd E. I. (2013), *The effect of photogravitational force and oblateness in the perturbed restricted three-body problem*, *Astrophys. Space Sci.* 346: 51 – 69. Doi: 10.1007/s10509-013-1439-9
- [11] Abouelmagd E. I. (2013), *Stability of the triangular points under combined effects of radiation and oblateness in the restricted three-body problem*, *Earth Moon and Planets* 110: 143 – 155. Doi: 10.1007/s11038-013-9415-5
- [12] Abouelmagd E. I., Sharaf M. A. (2013), *The motion around the libration points in the restricted three-body problem with the effect of radiation and oblateness*, *Astrophys. Space Sci.* 344: 321 – 332. Doi: 10.1007/s10509-012-1335-8
- [13] Alzahrani F., Abouelmagd E. I., Guirao J. L. G., Hobiny A. (2017), *On the libration collinear points in the restricted three-body problem*, *Open Physics* 15 (1): 58 – 67. Doi:10.1515/phys-2017-0007
- [14] Beevi A. S., Sharma, R. K. (2012), *Oblateness effect of Saturn on periodic orbits in the Saturn-Titan restricted three-body problem*, *Astrophys. Space Sci.* 340: 245 – 261. Doi: 10.1007/s10509-012-1052-3
- [15] Elipse A., Lara M. (1997), *Periodic orbits in the restricted three body problem with radiation pressure*, *Celest. Mech. Dyn. Astron.* 68: 1 – 11.
- [16] Elshaboury S. M., Abouelmagd E. I., Kalantonis V. S., Perdios E. A. (2016), *The planar restricted three-body problem when both primaries are triaxial rigid bodies: Equilibrium points and periodic orbits*, *Astrophys. Space Sci.* 361 (9): 315. Doi:10.1007/s10509-016-2894-x
- [17] Ishwar B., Elipse A. (2001), *Secular solutions at triangular equilibrium point in the generalized photogravitational restricted three body problem*, *Astrophys. Space Sci.* 277: 437 – 446. Doi: 10.1023/A:1012528929233

- [18] Greaves J. S., Holland W. S., Moriarty-Schieven G., Jenness T., Dent W. R. F., Zuckerman B., McCarthy C., Webb R. A., Butner H. M., Gear W. K., Walker H. J. (1998), *A dust ring around Eridani: analog to the young Solar System*, *Astrophys. J.* 506: 133 – 137. Doi:10.1086/311652
- [19] Jayawardhana R., Holland W. S., Greaves J. S., Dent W. R. F., Marcy G. W., Hartmann L. W., Fazio G. G. (2000), *Dust in the 55 Cancri planetary system*, *The Astrophysical Journal* 536, 425 – 428. Doi:10.1086/308942
- [20] Jiang I. G., Yeh L. C. (2003), *Bifurcation for dynamical systems of planet-belt interaction*, *Int. J. Bifurc. Chaos* 13(3): 617 – 630. Doi:10.1142/s0218127403006807.
- [21] Jiang, I.G., Yeh, L.C. (2004b), *On the chaotic orbits of disk-star-planet systems*, *Astron. J.* 128: 923 – 932. Doi:10.1086/422018.
- [22] Jiang I. G., Yeh L. C. (2004a), *The drag-induced resonant capture for Kuiper belt objects*, *Mon. Not. R. Astron. Soc.* 355: 29 – 32. Doi:10.1111/j.1365-2966.2004.08504.x.
- [23] Jiang Y., Baoyin H. (2019), *Periodic orbits related to the equilibrium points in the potential of Irregular-shaped minor celestial bodies*, *Results in Physics* 12: 368 – 374. Doi.org/10.1016/j.rinp.2018.11.049.
- [24] Jung C., Zotos E. E. (2015), *Order and chaos in a three dimensional galaxy model*, *Mechanics Research Communications* 69: 45 – 53 Doi.org/10.1016/j.mechrescom.2015.06.005
- [25] Kushvah B.S. (2008), *The effect of radiation pressure on the equilibrium points in the generalised photogravitational restricted three body problem*, *Astrophys. Space Sci.* 315: 231– 241. Doi:10.1007/s10509-008-9823-6
- [26] Miyamoto M., Nagai R. (1975), *Three-dimensional models for the distribution of mass in galaxies*, *Publ. Astron. Soc. Jpn.* 27: 533 – 543.
- [27] Papadakis K.E. (2008), *Families of asymmetric periodic orbits in the restricted three-body problem*, *Earth, Moon and Planets*, 103: 25 – 42. Doi: 10.1007/s11038-008-9232-4
- [28] Pathak N., Abouelmagd E. I., Thomas V. O. (2019), *On higher order of resonant periodic orbits in the photogravitational restricted three body problem*, *J of Astronaut Sci* 66(4): 475 – 505. DOI: 10.1007/s40295-019-00178-z
- [29] Pathak N., Thomas V. O., Abouelmagd E. I. (2019), *The photogravitational restricted three-body problem: Analysis of resonant periodic orbits*, *Discrete and Continuous Dynamical Systems - Series S (DCDS-S)* 12 (4&5): 849 – 875. Doi: 10.3934/dcdss.2019057
- [30] Pitjeva E. V., Pitjev N. P. (2016), *Masses of asteroids and total mass of the main asteroid belt*, *International Astronomical Union*, 318: 212 – 217, Doi:10.1017/S1743921315008388
- [31] Selim H. H., Guirao J. L. G., Abouelmagd E. I. (2019), *Libration points in the restricted three-body problem: Euler angles, existence and stability*, *Discrete and Continuous Dynamical Systems - Series S (DCDS-S)* 12 (4&5): 703 – 710. Doi: 10.3934/dcdss.2019044
- [32] Shibayama M., Yagasaki K. (2011) *Families of symmetric relative periodic orbits Originating from the circular Euler solution in the isosceles three-body Problem*, *Celest. Mech. Dyn. Astron.* 110: 53 – 70. Doi:10.1007/s10569-011-9338-2
- [33] Singh J., Begha J. M. (2011), *Periodic orbits in the generalized perturbed restricted three-body problem*, *Astrophys. Space Sci.* 332: 319 – 324. Doi: 10.1007/s10509-010-0545-1
- [34] Singh J., Taura J.J. (2013), *Motion in the generalized restricted three-body problem*, *Astrophys. Space Sci.* 343(1): 95 – 106. Doi:10.1007/s10509-012-1225-0.
- [35] Szebehely V. (1967), *Theory of Orbits: The Restricted Problem of Three Bodies* Academic Press, New York
- [36] Yeh L. C., Jiang I. G. (2006), *On the Chermnykh-like problems: II. The equilibrium points*, *Astrophys. Space Sci.* 306: 189 – 200. Doi:10.1007/s10509-006-9170-4.