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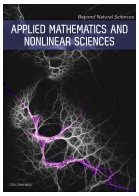

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Fractional Calculus involving (p,q)-Mathieu Type Series

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Abstract

Aim of the present paper is to establish fractional integral formulas by using fractional calculus operators involving the generalized (p,q)-Mathieu type series. Then, their composition formulas by using the integral transforms are introduced. Further, a new generalized form of the fractional kinetic equation involving the series is also developed. The solutions of fractional kinetic equations are presented in terms of the Mittag-Leffler function. The results established here are quite general in nature and capable of yielding both known and new results.

Keywords: Fractional integral operators; Fractional derivative operators; Extended generalized Mathieu series; Integral transforms.
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1 Introduction and Preliminaries

Fractional calculus is a very rapidly growing subject of mathematics which deals with the study of fractional order derivatives and integrals. Fractional calculus is an efficient tool to study many complex real world systems [1]. It is demonstrated that fractional order representation of complex processes appearing in various fields of science, engineering and finance, provides a more realistic approach with memory effects to study these problems. (see e.g. [2–14] and [15]). Among the research work developing the theory of fractional calculus and presenting some applications, we have to point out some literature (see [16–20]). Kumar *et al.* [21]

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analyzed the fractional model of modified Kawahara equation by using newly introduced Caputo-Fabrizio fractional derivative. One also *et al.* [22] studied a heat transfer problem and presented a new non-integer model for convective straight fins with temperature-dependent thermal conductivity associated with Caputo-Fabrizio fractional derivative. Recently, one *et al.* [23] presented a new fractional extension of regularized long wave equation by using Atangana-Baleano fractional operator. In *et al.* [24] one introduced a new numerical scheme for fractional Fitzhugh-Nagumo equation arising in transmission of new impulses. In *et al.* [25] one constituted a modified numerical scheme to study fractional model of Lienard's equations. Hajipour *et al.* [26] in their work formulated a new scheme for class of fractional chaotic systems. Baleanu *et al.* [27] proposed a new formulation of the fractional control problems involving Mittag-Leffler non-singular kernel. In another work, Baleanu *et al.* [28] studied the motion of a Bead sliding on a wire in fractional analysis. Jajarmi *et al.* [29] analyzed a hyperchaotic financial system and its chaos control and synchronization by using fractional calculus.

For mathematical modeling of many complex problems appearing in various fields of science and engineering such as fluid dynamics, plasma physics, astrophysics, image processing, stochastic dynamical system, controlled thermonuclear fusion, nonlinear control theory, nonlinear biological systems, quantum physics and heat transfer problems, the fractional calculus operators involving various special functions have been used successfully. There is rich literature available revealing the notable development in fractional order derivatives and integrals (see, [1, 10, 11, 18–20, 30–39]). Recently, Caputo and Fabrizio [40] introduced a new fractional derivative which is more suitable than the classical Caputo fractional derivative for many engineering and thermodynamical processes. Atangana [41] used a new fractional derivative to study the nature of Fisher's reaction diffusion equation. Riemann and Caputo fractional derivative operators both have a singular kernel which cannot exactly represent the complete memory effect of the system. To overcome these limitations of the old derivatives, very recently Atangana and Baleanu [42] presented a new non-integer order derivative having a non-local, non-singular and Mittag-Leffler type kernel.

In recent years, many researchers have extensively studied the properties, applications and extensions of various fractional integral and differential operators involving the various special functions. (for detail see McBride [43], Kalla [44, 45], Kalla and Saxena [46, 47], Saigo [48–50], Saigo and Maeda [51], Kiryakova [32, 52], [53] etc).

For our present study, we recall the following pair of Saigo hypergeometric fractional integral operators.

For $x > 0, \lambda, \sigma, \vartheta \in \mathbb{C}$ and $\Re(\lambda) > 0$, we have

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} f(t)\right)(x) = \frac{x^{-\lambda-\sigma}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} {}_2F_1\left(\lambda+\sigma, -\vartheta; \lambda; 1-\frac{t}{x}\right) f(t)\, dt \tag{1.1}$$

and

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} t^{-\lambda-\sigma} {}_2F_1\left(\lambda+\sigma, -\vartheta; \lambda; 1-\frac{x}{t}\right) f(t)\, dt \tag{1.2}$$

where the ${}_2F_1(.)$, a special case of the generalized hypergeomteric function, is the Gauss hypergeometric function.

The operator $I_{0,x}^{\lambda,\sigma,\vartheta}(.)$ contains the Riemann-Liouville $R_{0,x}^{\lambda}(.)$ fractional integral operators by means of the following relationships:

$$\left(R_{0,x}^{\lambda} f(t)\right)(x) = \left(I_{0,x}^{\lambda,-\lambda,\vartheta} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t)\, dt \tag{1.3}$$

$$\left(W_{x,\infty}^{\lambda} f(t)\right)(x) = \left(J_{x,\infty}^{\lambda,-\lambda,\vartheta} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} f(t)\, dt \tag{1.4}$$

It is noted that the operator (1.2) unifies the Erdêlyi-Kober fractional integral operators as follows:

$$\left(E_{0,x}^{\lambda,\vartheta} f(t)\right)(x) = \left(I_{0,x}^{\lambda,0,\vartheta} f(t)\right)(x) = \frac{x^{-\lambda-\vartheta}}{\Gamma(\lambda)} \int_0^{x} (x-t)^{\lambda-1} t^{\eta} f(t)\, dt \tag{1.5}$$

$$\left(K_{x,\infty}^{\lambda,\vartheta} f(t)\right)(x) = \left(J_{x,\infty}^{\lambda,0,\vartheta} f(t)\right)(x) = \frac{x^{\vartheta}}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} t^{-\lambda-\vartheta} f(t)\, dt \tag{1.6}$$

The following lemmas proved in Kilbas and Sebastin [54] are useful to prove our main results.

Lemma 1. *(Kilbas and Sebastian 2008) Let $\lambda, \sigma, \vartheta \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) > max[0, \Re(\sigma - \vartheta)]$, then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)} x^{\rho-\sigma-1}. \tag{1.7}$$

Lemma 2. *(Kilbas and Sebastian 2008) Let $\lambda, \sigma, \vartheta \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) < 1 + min[\Re(\sigma), \Re(\vartheta)]$, then*

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1}\right)(x) = \frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\vartheta-\rho+1)} x^{\rho-\sigma-1}. \tag{1.8}$$

The image formulas for special functions of one or more variables are very useful in the evaluation and solution of differential and integral equations. Motivating by the above discussion, we developed new fractional calculus formulas involving extended generalized Mathieu series.

The following familiar infinite series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2}, \qquad (r \in \mathbb{R}^+), \tag{1.9}$$

is called a Mathieu series. It was introduced and studied by Émile Leonard Mathieu in his book [55] devoted to the elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two dimensional rectangular domain, see [56, Eq. (54), p. 258]. Several interesting problems and solutions dealing with integral representations and bounds for the following generalization of the Mathieu series, which is so-called generalized Mathieu series with a fractional power can be found in [57–60]:

$$S_{\mu}(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^{\mu+1}}, \qquad (\mu > 0, r > 0).$$

In [59],the authors derived the following new Laplace type integral representation series

$$S_{\mu}(r) = \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^{\infty} e^{-rt} \kappa_{\mu}(t) dt, \qquad \left(\mu > \frac{3}{2}\right) \tag{1.10}$$

$$\kappa_\mu(t) = t^{\mu+\frac{1}{2}} \sum_{k=1}^{\infty} \frac{J_{\mu+\frac{1}{2}}(kt)}{k^{\mu-\frac{1}{2}}}$$

and $J_\mu(z)$ is the Bessel function. Motivated essentially by the works of Cerone and Lenard [61], Srivastava and Tomovski in [62] defined a family of generalized Mathieu series

$$S_\mu^{(\alpha,\beta)}(r;a) = S_\mu^{(\alpha,\beta)}(r;\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu}, \quad (\alpha,\beta,\mu,r>0), \tag{1.11}$$

where it is tacitly assumed that the positive sequence

$$a = \{a_n\} = \{a_1, a_2, ...\}$$

such that

$$lim_{n\to\infty} a_n = \infty$$

is so chosen that the infinite series in definition (1.11) converges, that is, that the following auxiliary series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent.

Definition 1. *(see [63, Eq. (6.1), p. 256]) The extended Beta function $B_{p;q}(x;y)$ is defined by*

$$B_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{p,q} dt, \tag{1.12}$$

$$(x,y,p,q \in \mathbb{C}; \min\{\Re(x),\Re(y)\} > 0, \min\{\Re(p),\Re(q)) \geq 0\}$$

where $E_{p,q}(t)$ is defined by

$$E_{p,q}(t) = \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)$$

$(p,q \in \mathbb{C}$ and $\min\{\Re(p),\Re(q)\} \geq 0)$.

In particular, Chaudhry et al. [64, p. 591, Eq. (1.7)], introduced the $p-$extension of Euler's Beta function $B(x,y)$:

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}} dt$$

$(\Re(p) > 0)$

whose special case, when $p = 0$ (or $p = q = 0$ in (1.12)), is the familiar Beta integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

$(\Re(x),\Re(y) > 0)$.

Recently, Mehrez and Tomovski [65] introduces the (p,q)-Mathieu-type power series in terms of the extended Beta function (1.12), which is defined as:

$$S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{z^n}{n!} \tag{1.13}$$
$$(r,\alpha,\beta,\nu>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0;|z|<1)$$

In particular case when $p=q$; we define the p-Mathieu-type power series defined by

$$S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p;z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n B_p(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{z^n}{n!} \tag{1.14}$$

$$(\tau,\alpha,\beta,\vartheta,\xi,\tau>0,p\in C,|z|\leq 1)$$

The function $S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(\tau;a;p;z)$ has many other special cases. If we set $p=q=0$; we get

$$S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n}{(a_n^{\alpha}+r^2)^{\mu}(\xi)_n}\frac{z^n}{n!} \tag{1.15}$$

$(\tau,\alpha,\beta,\vartheta,\xi>0,|z|\leq 1)$

On the other hand, by letting $\tau=\omega$ in (1.15) we obtain [66, Eq. 5, p. 974]:

$$S^{\alpha,\beta}_{\mu,\vartheta}(r;a;z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n}{(a_n^{\alpha}+r^2)^{\mu}}\frac{z^n}{n!},\qquad(\tau,\alpha,\beta,\vartheta>0,|z|\leq 1). \tag{1.16}$$

The concept of the Hadamard product (or the convolution) of two analytic functions is very useful in our present study. It can help us to decompose a newly emerging function into two known functions. Let

$$f(z):=\sum_{n=0}^{\infty}a_n z^n,(|z|<R_f) \tag{1.17}$$

and

$$g(z):=\sum_{n=0}^{\infty}b_n z^n,(|z|<R_g) \tag{1.18}$$

be two power series whose radii of convergence are denoted byR_f and R_g, respectively.
Then their Hadamard product is the power series defined by

$$(f*g)(z):=\sum_{n=0}^{\infty}a_n b_n z^n=(g*f)(z) \tag{1.19}$$
$$(|z|<R),$$

where

$$R=\lim_{n\to\infty}\left|\frac{a_n b_n}{a_{n+1}b_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|=R_f R_g \tag{1.20}$$

Therefore, in general, we have $R\geq R_f.R_g$ [67, 68].

For various investigations involving the Hadamard product (or the convolution), the interested reader may refer to several recent papers on the subject (see, for example, [69, 70] and the references cited therein).

2 Fractional integration

In this section, we will establish some fractional integral formulas for the generalized (p,q)-Mathieu-type power series. Then their special cases also introduced here.

Theorem 1. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0$, *such that* $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, *then*

$$\begin{aligned}&\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;t)\right)(x)\\&\quad=x^{\rho-\sigma-1}\frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;x)*{}_2F_2\left[\begin{matrix}\rho,\rho+\vartheta-\sigma\\ \rho-\sigma,\rho+\lambda+\vartheta\end{matrix};x\right].\end{aligned}\tag{2.1}$$

Proof. For convenience, we denote the left-hand side of the result (2.1) by $\mathscr{I}$. Using (1.13), and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, then

$$\mathscr{I}=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_nB_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{1}{n!}\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{n+\rho-1}\right)(x),\tag{2.2}$$

applying the result (1.7), the above equation (2.2) reduced to

$$\mathscr{I}=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_nB_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{1}{n!}\frac{\Gamma(\rho+n)\Gamma(\rho+\vartheta-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\vartheta+n)}x^{\rho+n-\sigma-1},\tag{2.3}$$

after simplification, we have

$$\begin{aligned}\mathscr{I}=x^{\rho-\sigma-1}\frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_nB_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\\ \times\frac{(\rho)_n(\rho+\vartheta-\sigma)_n}{(\rho-\sigma)_n(\rho+\lambda+\vartheta)_n}\frac{x^n}{n!},\end{aligned}\tag{2.4}$$

further interpret the above equation with the view of of the function given in equation (1.13), we have

$$\mathscr{I}=x^{\rho-\sigma-1}\frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)}S_{\mu,\vartheta,\tau,\xi;\rho-\sigma,\rho+\lambda+\vartheta}^{\alpha,\beta;\rho,\rho+\vartheta-\sigma}(r;a;p,q;x),\tag{2.5}$$

employing the concept of the Hadamard product given in equation (1.19) in the above equation (2.5), required result is obtained. □

Theorem 2. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0$, *such that* $\Re(\rho)<1+\min[\Re(\sigma),\Re(\vartheta)]$, *Then*

$$\begin{aligned}\left(I_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;1/t)\right)(x)=x^{\rho-\sigma-1}\frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma-\vartheta-\rho)}\\ \times S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;1/x)*{}_2F_2\left[\begin{matrix}\sigma-\rho+1,\vartheta-\rho+1\\ 1-\rho,\lambda+\sigma-\vartheta-\rho\end{matrix};x\right].\end{aligned}\tag{2.6}$$

Proof. Proof is parallel to Theorem 1. □

2.1 Special cases of fractional integral formulae

In this section we reduces our main findings to the special cases by assigning particular values to the parameters as follows:

Case 1. *If we choose $p=q$ the findings in equations (2.1) and (2.6) reduces to the following the form:*

Corollary 1. *Let $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p\in\mathbb{C};\Re(p)\geq 0$, such that $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, then*

$$\begin{aligned}&\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p;t)\right)(x)\\&\quad=x^{\rho-\sigma-1}\frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p;x)*{}_2F_2\left[\begin{matrix}\rho,\rho+\vartheta-\sigma\\ \rho-\sigma,\rho+\lambda+\vartheta\end{matrix};x\right].\end{aligned}\tag{2.7}$$

Corollary 2. *Let $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p\in\mathbb{C};\Re(p)\geq 0$, such that $\Re(\rho)<1+\min[\Re(\sigma),\Re(\vartheta)]$, Then*

$$\begin{aligned}\left(I_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p;1/t)\right)(x)&=x^{\rho-\sigma-1}\frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma-\vartheta-\rho)}\\&\times S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p;1/x)*{}_2F_2\left[\begin{matrix}\sigma-\rho+1,\vartheta-\rho+1\\ 1-\rho,\lambda+\sigma-\vartheta-\rho\end{matrix};x\right].\end{aligned}\tag{2.8}$$

Case 2. *If we choose $p=q=0$ the findings in equations (2.1) and (2.6) reduces to the following the form:*

Corollary 3. *Let $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0$, such that $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, then*

$$\begin{aligned}&\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;t)\right)(x)\\&\quad=x^{\rho-\sigma-1}\frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;x)*{}_2F_2\left[\begin{matrix}\rho,\rho+\vartheta-\sigma\\ \rho-\sigma,\rho+\lambda+\vartheta\end{matrix};x\right].\end{aligned}\tag{2.9}$$

Corollary 4. *Let $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0$, such that $\Re(\rho)<1+\min[\Re(\sigma),\Re(\vartheta)]$, Then*

$$\begin{aligned}\left(I_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;1/t)\right)(x)&=x^{\rho-\sigma-1}\frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma-\vartheta-\rho)}\\&\times S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;1/x)*{}_2F_2\left[\begin{matrix}\sigma-\rho+1,\vartheta-\rho+1\\ 1-\rho,\lambda+\sigma-\vartheta-\rho\end{matrix};x\right].\end{aligned}\tag{2.10}$$

Case 3. *If we choose $p=q=0$ and $\tau=\xi$, the findings in equations (2.1) and (2.6) reduces to the following the form:*

Corollary 5. *Let $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0$, such that $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, then*

$$\begin{aligned}&\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta}^{\alpha,\beta}(r;a;t)\right)(x)\\&\quad=x^{\rho-\sigma-1}\frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)}S_{\mu,\vartheta}^{\alpha,\beta}(r;a;x)*{}_2F_2\left[\begin{matrix}\rho,\rho+\vartheta-\sigma\\ \rho-\sigma,\rho+\lambda+\vartheta\end{matrix};x\right].\end{aligned}\tag{2.11}$$

Corollary 6. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0$, *such that* $\Re(\rho)<1+\min[\Re(\sigma),\Re(\vartheta)]$, *Then*

$$\left(I_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta}^{\alpha,\beta}(r;a;1/t)\right)(x)=x^{\rho-\sigma-1}\frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma-\vartheta-\rho)}\times S_{\mu,\vartheta}^{\alpha,\beta}(r;a;1/x)*{}_2F_2\left[\begin{matrix}\sigma-\rho+1,\vartheta-\rho+1\\1-\rho,\lambda+\sigma-\vartheta-\rho\end{matrix};x\right]. \tag{2.12}$$

3 Image Formulas Associated With Integral Transform

In this section, we establish certain theorems involving the results obtained in previous section associated with the integral transforms like, Beta transform, Laplace transform and Whittaker transform.

3.1 Beta Transform

The Beta transform of $f(z)$ is defined as [71]:

$$B\{f(z):a,b\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz \tag{3.1}$$

Theorem 3. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0$, *such that* $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, *then*

$$B\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;t)\right)(x):l,m\right\}=\Gamma(m)x^{\rho-\sigma-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;x)*{}_3\Psi_3\left[\begin{matrix}(\rho,1),(\rho+\vartheta-\sigma,1),(l,1)\\(\rho-\sigma),(\rho+\lambda+\vartheta,1),(l+m,1)\end{matrix};\frac{1}{x}\right]. \tag{3.2}$$

Proof. For convenience, we denote the left-hand side of the result (3.2) by $\mathscr{B}$. Using the definition of beta transform, the LHS of (3.1) becomes:

$$\mathscr{B}=\int_0^1 z^{l-1}(1-z)^{m-1}\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;tz)\right)(x)dz, \tag{3.3}$$

further using (1.13) and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, then

$$\mathscr{B}=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_nB_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{x^n}{n!}\left(I_{0+}^{\lambda,\sigma,\vartheta}t^{n+\rho-1}\right)(x)\int_0^1 z^{l+n-1}(1-z)^{m-1}dz \tag{3.4}$$

applying the result (1.7), after simplification the above equation (3.4) reduced to

$$\mathscr{B}=x^{\rho-\sigma-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_nB_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{x^n}{n!}\times\frac{\Gamma(\rho+n)\Gamma(\rho+\vartheta-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\vartheta+n)}\int_0^1 z^{l+n-1}(1-z)^{m-1}dz, \tag{3.5}$$

applying the definition of beta transform, the above equation (3.5) reduced to

$$\mathscr{B} = x^{\rho-\sigma-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{x^n}{n!}\frac{\Gamma(\rho+n)\Gamma(\rho+\vartheta-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\vartheta+n)} \times\frac{\Gamma(l+n)\Gamma(m)}{\Gamma(l+m+n)} \tag{3.6}$$

after simplification, we have

$$\mathscr{B} = x^{\rho-\sigma-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{x^n}{n!}\frac{(\rho)_n(\rho+\vartheta-\sigma)_n}{(\rho-\sigma)_n(\rho+\lambda+\vartheta)_n} \times\frac{(l)_n\Gamma(m)}{(l+m)_n} \tag{3.7}$$

further interpret the above equation with the view of of the function given in equation (3.7), we have

$$\mathscr{B} = x^{\rho-\sigma-1}\Gamma(m)S^{\alpha,\beta;\rho,\rho+\vartheta-\sigma,l}_{\mu,\vartheta,\tau,\xi;\rho-\sigma,\rho+\lambda+\vartheta,l+m}(r;a;p,q;x), \tag{3.8}$$

employing the concept of the Hadamard product given in equation (1.13) in the above equation (3.8), required result is obtained. □

Theorem 4. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0$, *such that* $\Re(\rho)<1+\min[\Re(\sigma),\Re(\vartheta)]$, *Then*

$$B\left\{\left(I^{\lambda,\sigma,\vartheta}_{x,\infty}t^{\rho-1}S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;z/t)\right)(x):l,m\right\} = \Gamma(m)x^{\rho-\sigma-1}S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;x) *{}_3\Psi_3\left[\begin{matrix}(\rho,1),(\vartheta-\rho+1,1),(l,1)\\(\rho-\sigma,1),(\lambda+\vartheta+\nu,1),(l+m,1)\end{matrix};1/x\right]. \tag{3.9}$$

Proof. The proof of this theorem is the same as that of Theorem 3. □

3.2 Laplace Transform

The Laplace transform of $f(z)$ is defined as [71]:

$$L\{f(z)\} = \int_0^{\infty} e^{-sz}f(z)dz \tag{3.10}$$

Theorem 5. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0$, *such that* $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, *then*

$$L\left\{z^{l-1}\left(I^{\lambda,\sigma,\vartheta}_{0,x}t^{\rho-1}S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;tz)\right)(x)\right\} = \frac{x^{\rho-\sigma-1}}{s^l}S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}\left(r;a;p,q;\frac{x}{s}\right) \times{}_3\Psi_2\left[\begin{matrix}(\rho,1),(\rho+\vartheta-\sigma,1),(l,1)\\(\rho-\sigma,1),(\rho+\lambda+\vartheta,1)\end{matrix};\frac{x}{s}\right]. \tag{3.11}$$

Proof. For convenience, we denote the left-hand side of the result (3.11) by $\mathscr{L}$. Then applying the Laplace, we have:

$$\mathscr{L} = \int_0^\infty e^{-sz} z^{l-1} \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;tz) \right)(x) dz \tag{3.12}$$

further using (3.3) and then changing the order of integration and summation, which is valid under the conditions of Theorem 1, then

$$\begin{aligned} \mathscr{L} = \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{x^n}{n!} \left(I_{0+}^{\lambda,\sigma,\vartheta} t^{n+\rho-1} \right)(x) \\ \times \int_0^\infty e^{-sz} z^{n+l-1} dz \end{aligned} \tag{3.13}$$

applying the result (1.7), after simplification the above equation (3.13)reduced to

$$\begin{aligned} \mathscr{L} = \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{x^n}{n!} \frac{\Gamma(\rho+n)\Gamma(\rho+\vartheta-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\vartheta+n)} \\ \times \frac{\Gamma(n+l)}{s^{n+l}}, \end{aligned} \tag{3.14}$$

after simplification, we have

$$\mathscr{B} = \frac{x^{\rho-\sigma-1}\Gamma(m)}{s^l} \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \left(\frac{x}{s}\right)^n \frac{(\rho)_n(\rho+\vartheta-\sigma)_n}{n!(\rho-\sigma)_n(\rho+\lambda+\vartheta)_n} (l)_n \tag{3.15}$$

further interpret the above equation with the view of of the function given in equation (3.15), we have

$$\mathscr{B} = \frac{x^{\rho-\sigma-1}\Gamma(m)}{s^l} S_{\mu,\vartheta,\tau,\xi;\rho-\sigma,\rho+\lambda+\vartheta,}^{\alpha,\beta;\rho,\rho+\vartheta-\sigma,l} \left(r;a;p,q;\frac{x}{s}\right), \tag{3.16}$$

employing the concept of the Hadamard product given in equation (1.13) in the above equation (3.16), required result is obtained. □

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Theorem 6. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta > 0; \xi > \tau > 0; p,q \in \mathbb{C}; \min\{\Re(p),\Re(q)\} \geq 0$, *such that* $\Re(\rho) < 1 + \min[\Re(\sigma),\Re(\vartheta)]$, *Then*

$$\begin{aligned} L\left\{ z^{l-1} \left(I_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;z/t) \right)(x) \right\} = \frac{x^{\rho-\sigma-1}}{s^l} S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta} \left(r;a;p,q;\frac{1}{sx} \right) \\ *\, {}_3\Psi_2 \left[\begin{matrix} (\sigma-\rho+1,1),(\vartheta-\rho+1,1),(l,1) \\ (1-\rho,1),(\lambda+\sigma+\vartheta-\rho+1,1) \end{matrix} ; \frac{1}{sx} \right]. \end{aligned} \tag{3.17}$$

Proof. The proof of this theorem would run parallel as those of Theorem 5. □

3.3 Whittaker Transform

Theorem 7. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0$, *such that* $\Re(\rho)>\max[0,\Re(\sigma-\vartheta)]$, *then*

$$\begin{aligned}&\int_0^\infty z^{\xi-1}e^{-\delta z/2}W_{\tau,\omega}(\eta z)\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}(r;a;p,q;tz)\right)(x)\right\}dz\\&=\frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}}S_{\mu,\vartheta,\tau,\xi}^{\alpha,\beta}\left(r;a;p,q;\frac{x}{\eta}\right)\\&*{}_4\Psi_3\left[\begin{matrix}(\rho,1),(\rho+\vartheta-\sigma,1),(1/2+\omega+\xi,1),(1/2-\omega+\xi,1)\\(\rho-\sigma,1),(\rho+\lambda+\vartheta,1),(1/2-\tau+\xi,1)\end{matrix};\frac{x}{\eta}\right]\end{aligned}\tag{3.18}$$

Proof. For convenience, we denote the left-hand side of the result (3.25) by $\mathscr{W}$. Then using the result from (2.3), after changing the order of integration and summation, we get:

$$\begin{aligned}\mathscr{W}=x^{\rho-\sigma-1}\sum_{n=1}^{\infty}&\frac{2a_n^{\beta}(\vartheta)_nB_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{x^n}{n!}\frac{\Gamma(\rho+n)\Gamma(\rho+\vartheta-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\vartheta+n)}\\&\times\int_0^\infty z^{n+\xi-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)dz,\end{aligned}\tag{3.19}$$

by substituting $\eta z=\varsigma$, (3.19) becomes:

$$\begin{aligned}\mathscr{W}=x^{\rho-\sigma-1}\sum_{n=1}^{\infty}&\frac{2a_n^{\beta}(\vartheta)_nB_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}}\frac{x^n}{n!}\frac{\Gamma(\rho+n)\Gamma(\rho+\vartheta-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\vartheta+n)}\\&\times\int_0^\infty \varsigma^{n+\xi-1}e^{-\varsigma/2}W_{\tau,\omega}(\varsigma)d\varsigma.\end{aligned}\tag{3.20}$$

Now we use the following integral formula involving Whittaker function

$$\int_0^\infty t^{\nu-1}e^{-t/2}W_{\tau,\omega}(t)dt=\frac{\Gamma(1/2+\omega+\nu)\Gamma(1/2-\omega+\nu)}{\Gamma(1/2-\tau+\nu)},\qquad\left(\Re(\nu\pm\omega)>\frac{-1}{2}\right).\tag{3.21}$$

Then we have

$$\begin{aligned}\mathscr{W}=\frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}}\sum_{n=1}^{\infty}&\frac{2a_n^{\beta}(\nu)_nB_{p,q}(\tau+n,\omega-\tau)}{B(\tau,\omega-\tau)(a_n^{\alpha}+\tau^2)^{\mu}n!}\frac{\Gamma(\rho+n)\Gamma(\rho+\vartheta-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\vartheta+n)}\\&\times\frac{\Gamma(1/2+\omega+\xi+n)\Gamma(1/2-\omega+\xi+n)}{\Gamma(1/2-\tau+\xi+n)}\left(\frac{x}{\eta}\right)^n,\end{aligned}\tag{3.22}$$

after simplification, we have

$$\begin{aligned}\mathscr{W}=\frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}}\sum_{n=1}^{\infty}&\frac{2a_n^{\beta}(\nu)_nB_{p,q}(\tau+n,\omega-\tau)}{B(\tau,\omega-\tau)(a_n^{\alpha}+\tau^2)^{\mu}n!}\frac{(\rho)_n(\rho+\vartheta-\sigma)_n}{(\rho-\sigma)_n(\rho+\lambda+\vartheta)_n}\\&\times\frac{(1/2+\omega+\xi)_n(1/2-\omega+\xi)_n}{(1/2-\tau+\xi)_n}\left(\frac{x}{\eta}\right)^n,\end{aligned}\tag{3.23}$$

further interpret the above equation with the view of of the function given in equation (3.23), we have

$$\mathscr{W} = \frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}} S^{\alpha,\beta;\rho,\rho+\vartheta-\sigma,1/2+\omega+\xi,1/2-\omega+\xi}_{\mu,\vartheta,\tau,\xi;\rho-\sigma,\rho+\lambda+\vartheta,1/2-\tau+\xi}\left(r;a;p,q;\frac{x}{\eta}\right), \tag{3.24}$$

employing the concept of the Hadamard product given in equation (1.13) in the above equation (3.24), required result is obtained. □

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Theorem 8. *Let* $\lambda,\sigma,\vartheta,\rho,r,\alpha,\beta,\vartheta>0;\xi>\tau>0;p,q\in\mathbb{C};\min\{\Re(p),\Re(q)\}\geq 0$, *such that* $\Re(\rho)<1+\min[\Re(\sigma),\Re(\vartheta)]$, *Then*

$$\begin{aligned}&\int_0^\infty z^{\xi-1}e^{-\delta z/2}W_{\tau,\omega}(\eta z)\left\{\left(I^{\lambda,\sigma,\vartheta}_{x,\infty}t^{\rho-1}S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;z/t)\right)(x)\right\}dz\\&=\frac{x^{\rho-\sigma-1}}{\eta^{\xi-1}}S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}\left(r;a;p,q;\frac{x}{\eta}\right)\\&*{}_4\Psi_3\left[\begin{matrix}(\sigma-\rho+1,1),(\vartheta-\rho+1,1),(1/2+\omega+\xi,1),(1/2-\omega+\xi,1)\\(1-\rho,1),(\rho+\lambda+\vartheta,1),(1/2-\tau+\xi,1)\end{matrix};\frac{x}{\eta}\right]\end{aligned} \tag{3.25}$$

Proof. The proof of this theorem would run parallel as those of Theorem 7. □

4 Fractional Kinetic Equations

The importance of fractional differential equations in the field of applied science has gained more attention not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of many physical phenomena. Especially, the kinetic equations describe the continuity of motion of substance. The extension and generalization of fractional kinetic equations involving many fractional operators were found in [72–85].

In view of the effectiveness and a great importance of the kinetic equation in certain astrophysical problems the authors develop a further generalized form of the fractional kinetic equation involving generalized k-Mittag-Leffler function.

The fractional differential equation between rate of change of the reaction, the destruction rate and the production rate was established by Haubold and Mathai [78] given as follows:

$$\frac{dN}{dt} = -d(N_t)+p(N_t), \tag{4.1}$$

where $N=N(t)$ the rate of reaction, $d=d(N)$ the rate of destruction, $p=p(N)$ the rate of production and N_t denotes the function defined by $N_t(t^*)=N(t-t^*),t^*>0$.

The special case of (4.1) for spatial fluctuations and inhomogeneities in $N(t)$ the quantities are neglected , that is the equation

$$\frac{dN}{dt} = -c_iN_i(t), \tag{4.2}$$

with the initial condition that $N_i(t=0)=N_0$ is the number density of the species i at time $t=0$ and $c_i>0$. If we remove the index i and integrate the standard kinetic equation (4.2), we have

$$N(t)-N_0=-c_0D_t^{-1}N(t) \tag{4.3}$$

where ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville integral operator ${}_0D_t^{-\nu}$ defined as

$${}_0D_t^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(s)ds, \qquad (t>0, R(\nu)>0) \tag{4.4}$$

The fractional generalization of the standard kinetic equation(4.3) is given by Haubold and Mathai [78] as follows:

$$N(t)-N_0=-c^{\nu}{}_0D_t^{-1}N(t) \tag{4.5}$$

and obtained the solution of (4.5) as follows:

$$N(t)=N_0\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu k+1)}(ct)^{\nu k} \tag{4.6}$$

Further, (Saxena and Kalla [83]) considered the the following fractional kinetic equation:

$$N(t)-N_0f(t)=-c^{\nu}{}_0D_t^{-\nu}N(t), \qquad (\Re(\nu)>0), \tag{4.7}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0=N(0)$ is the number density of that species at time $t=0$, c is a constant and $f\in\mathscr{L}(0,\infty)$.

By applying the Laplace transform to (4.7) (see [79]),

$$\begin{aligned} L\{N(t);p\}&=N_0\frac{F(p)}{1+c^{\nu}p^{-\nu}}=N_0\left(\sum_{n=0}^{\infty}(-c^{\nu})^n p^{-\nu n}\right)F(p),\\ &\left(n\in N_0,\left|\frac{c}{p}\right|<1\right)\end{aligned} \tag{4.8}$$

where the Laplace transform [86] is given by

$$F(p)=L\{N(t);p\}=\int_0^{\infty}e^{-pt}f(t)dt, \qquad (\mathscr{R}(p)>0). \tag{4.9}$$

5 Solution of generalized fractional kinetic equations

In this section, we investigated the solutions of the generalized fractional kinetic equations by considering generalized (p,q)-Mathieu Type Series

Theorem 9. *If $a>0, d>0, \nu>0, r, \alpha, \beta, \vartheta>0; \xi>\tau>0; p, q \in \mathbb{C}; \min\{\Re(p), \Re(q)\} \geq 0$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.1}$$

is given by the following formula

$$\begin{aligned} N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n, \xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(a^\nu t^\nu)^n}{n!} \\ \times \Gamma(\nu n+1) E_{\nu,\nu n+1}(-a^\nu t^\nu). \end{aligned} \tag{5.2}$$

Proof. Laplace transform of Riemann-Liouville fractional integral operator is given by (Erdelyi et.al. [87], Srivastava and Saxena [88]):

$$L\left\{{}_0D_t^{-\nu} f(t); p\right\} = p^{-\nu} F(p) \tag{5.3}$$

where $F(p)$ is defined in (4.9). Now, applying Laplace transform on (5.1) gives,

$$L\{N(t); p\} = N_0 L\left\{S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;d^\nu t^\nu); p\right\} - a^\nu L\left\{{}_0D_t^{-\nu} N(t); p\right\} \tag{5.4}$$

$$i.e. \qquad N(p) = N_0 \left(\int_0^\infty e^{-pt} \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n, \xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(d^\nu t^\nu)^n}{n!} dt \right) - a^\nu p^{-\nu} N(p) \tag{5.5}$$

interchanging the order of integration and summation in (5.5), we have

$$N(p) + a^\nu p^{-\nu} N(p) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n, \xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(d^\nu)^n}{n!} \int_0^\infty e^{-pt} t^{\nu n} dt \tag{5.6}$$

$$= N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n, \xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(d^\nu)^n}{n!} \frac{\Gamma(\nu n+1)}{p^{\nu n+1}} \tag{5.7}$$

this leads to

$$\begin{aligned} N(p) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_{p,q}(\tau+n, \xi-\tau)}{B(\tau,\xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(d^\nu)^n}{n!} \\ \times \Gamma(\nu n+1) \left\{ p^{-(\nu n+1)} \sum_{l=0}^{\infty} \left[-\left(\frac{p}{a}\right)^{-\nu} \right]^l \right\} \end{aligned} \tag{5.8}$$

Taking Laplace inverse of (5.8), and by using

$$L^{-1}\left\{p^{-\nu}; t\right\} = \frac{t^{\nu-1}}{\Gamma(\nu)}, (R(\nu)>0) \tag{5.9}$$

we have,

$$L^{-1}\{N(p)\} = N_0 \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}} \frac{(a^{\nu})^n}{n!} \times \Gamma(\nu n+1) L^{-1}\left\{\sum_{l=0}^{\infty}(-1)^l a^{\nu l} p^{-[\nu(n+l)+1]}\right\} \tag{5.10}$$

$$i.e. \quad N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}} \frac{(d^{\nu})^n}{n!} \Gamma(\nu n+1) \left\{\sum_{l=0}^{\infty}(-1)^l a^{\nu l} \frac{t^{\nu(n+l)}}{\Gamma(\nu(n+l)+1)}\right\} \tag{5.11}$$

$$= N_0 \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}} \frac{(d^{\nu}t^{\nu})^n}{n!} \Gamma(\nu n+1) \left\{\sum_{l=0}^{\infty}(-1)^l \frac{(a^{\nu}t^{\nu})^l}{\Gamma(\nu(n+l)+1)}\right\} \tag{5.12}$$

The equation (5.12) can be written as

$$N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}} \Gamma(\nu n+1) \frac{(a^{\nu}t^{\nu})^n}{n!} E_{\nu,\nu n+1}(-a^{\nu}t^{\nu}). \tag{5.13}$$

□

Theorem 10. *If $d>0, \nu>0, r, \alpha, \beta, \vartheta>0; \xi>\tau>0; p,q \in \mathbb{C}; \min\{\Re(p),\Re(q)\} \geq 0$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;d^{\nu}t^{\nu}) = -d^{\nu}{}_0D_t^{-\nu}N(t) \tag{5.14}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}} \frac{(d^{\nu}t^{\nu})^n}{n!} \times \Gamma(\nu n+1) E_{\nu,\nu n+1}(-d^{\nu}t^{\nu}). \tag{5.15}$$

Theorem 11. *If $d>0, \nu>0, r, \alpha, \beta, \vartheta>0; \xi>\tau>0; p,q \in \mathbb{C}; \min\{\Re(p),\Re(q)\} \geq 0$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p,q;t) = -d^{\nu}{}_0D_t^{-\nu}N(t) \tag{5.16}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\vartheta)_n B_{p,q}(\tau+n,\xi-\tau)}{B(\tau,\xi-\tau)(a_n^{\alpha}+r^2)^{\mu}} \frac{(t^{\nu})^n}{n!} \times \Gamma(\nu n+1) E_{\nu,\nu n+1}(-d^{\nu}t^{\nu}). \tag{5.17}$$

Proof. The proof of the Theorem 10 and Theorem 11 are same as that of Theorem 9, so we would like to skip here. □

5.1 Special cases

Here we introduce some special cases of our results established in this section.

Case 4. *If $p = q$, then Theorem 9, Theorem 10 and Theorem 11 reduces to*

Corollary 7. *If $a > 0, d > 0, \nu > 0, r, \alpha, \beta, \vartheta > 0; \xi > \tau > 0; p \in \mathbb{C}; \Re(p) \geq 0$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p;d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.18}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_p(\tau+n, \xi-\tau)}{B(\tau, \xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(a^\nu t^\nu)^n}{n!} \Gamma(\nu n+1) E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.19}$$

Corollary 8. *If $d > 0, \nu > 0, r, \alpha, \beta, \vartheta > 0; \xi > \tau > 0; p \in \mathbb{C}; \Re(p) \geq 0$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p;d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.20}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_p(\tau+n, \xi-\tau)}{B(\tau, \xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(d^\nu t^\nu)^n}{n!} \Gamma(\nu n+1) E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.21}$$

Corollary 9. *If $d > 0, \nu > 0, r, \alpha, \beta, \vartheta > 0; \xi > \tau > 0; p \in \mathbb{C}; \Re(p) \geq 0$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p;t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{5.22}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n B_p(\tau+n, \xi-\tau)}{B(\tau, \xi-\tau)(a_n^\alpha + r^2)^\mu} \frac{(t^\nu)^n}{n!} \Gamma(\nu n+1) E_{\nu,\nu n+1}(-d^\nu t^\nu). \tag{5.23}$$

Case 5. *If $p = q = 0$, then Theorem 9, Theorem 10 and Theorem 11 reduces to*

Corollary 10. *If $a > 0, d > 0, \nu > 0, r, \alpha, \beta, \vartheta > 0; \xi > \tau > 0$, then the solution of the fractional kinetic equation*

$$N(t) - N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{5.24}$$

is given by the following formula

$$N(t) = N_0 \sum_{n=1}^{\infty} \frac{2a_n^\beta (\vartheta)_n}{(a_n^\alpha + r^2)^\mu (\xi)_n} \frac{(a^\nu t^\nu)^n}{n!} \Gamma(\nu n+1) E_{\nu,\nu n+1}(-a^\nu t^\nu). \tag{5.25}$$

Corollary 11. *If $d>0, \nu>0, r, \alpha, \beta, \vartheta>0; \xi>\tau>0$, then the solution of the fractional kinetic equation*

$$N(t)-N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p;d^{\nu}t^{\nu}) = -d^{\nu}{}_0D_t^{-\nu}N(t) \tag{5.26}$$

is given by the following formula

$$N(t)=N_0\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n}{(a_n^{\alpha}+r^2)^{\mu}(\xi)_n}\frac{(d^{\nu}t^{\nu})^n}{n!}\Gamma(\nu n+1)E_{\nu,\nu n+1}(-d^{\nu}t^{\nu}). \tag{5.27}$$

Corollary 12. *If $d>0, \nu>0, r, \alpha, \beta, \vartheta>0; \xi>\tau>0$, then the solution of the fractional kinetic equation*

$$N(t)-N_0 S^{\alpha,\beta}_{\mu,\vartheta,\tau,\xi}(r;a;p;t) = -d^{\nu}{}_0D_t^{-\nu}N(t) \tag{5.28}$$

is given by the following formula

$$N(t)=N_0\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\vartheta)_n}{(a_n^{\alpha}+r^2)^{\mu}(\xi)_n}\frac{(t^{\nu})^n}{n!}\Gamma(\nu n+1)E_{\nu,\nu n+1}(-d^{\nu}t^{\nu}). \tag{5.29}$$

6 Conclusion

In the present work, fractional integral formulae involving (p,q)-Mathieu Type series has established. The image formulae of our findings by employing integral transform has been also introduced. Further in this work we gave the solution of fractional kinetic equation in terms of Mittag-Leffler function. All the results are general in nature and give numerous results as their special cases.

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