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A note on Bailey and WP-Bailey Pairs

Satya Prakash Singh[1], Lakshmi Narayan Mishra[2,*], Vijay Yadav[3]

[1]Department of Mathematics, TDPG College, Jaunpur-222002 (UP) India, E-mail: snsp39@gmail.com

[2]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India, E-mail: lakshminarayanmishra04@gmail.com
*Corresponding author

[3]Department of Mathematics, SPDT College, Andheri(E), Mumbai-400059, India,
E-mail: vijaychottu@yahoo.com

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Abstract

In this paper, we have established certain theorems involving Bailey pairs and WP-Bailey pairs. Further, making use of some known WP-Bailey pairs and theorems for constructing new WP-Bailey pairs, we have also established transformation formulas for q-hypergeometric series.

Keywords: Bailey pair, WP-Bailey pair, transformation formula, summation formula, basic(q-) hypergeometric series.
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1 Introduction, Notations and Definitions

We begin by recalling some standard notations and terminology. Let a,q be complex numbers with $0<|q|<1$. Then the $q-$ shifted factorial is defined by

$$(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1}) \quad \text{if} \ \ n>0,$$

$$(a;q)_0 = 1$$

and

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

For the sake of brevity, we often write

$$(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n = (a_1,a_2,a_3,\ldots,a_r;q)_n.$$

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The basic hypergeometric series is defined by

$$ {}_r\Phi_s = \begin{bmatrix} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_s \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1,a_2,...,a_r;q)_n z^n}{(q,b_1,b_2,...,b_s;q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r}. $$

In an attempt to clarify Rogers second proof [3] of the Rogers-Ramanujan identities, Bailey [1] made the following simple but very useful observation,
If

$$ \beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.1} $$

and

$$ \begin{aligned} \gamma_n &= \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} \\ &= \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \end{aligned} \tag{1.2} $$

then

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.3} $$

The proof is straightforword and merely requires an interchange of sums. Of course, in the above transform, suitable convergence conditions need to be imposed to make the definition of γ_n and interchange of sums meaningful.

In application of the transform, Bailey chose $u_r = \dfrac{1}{(q;q)_r}, v_r = \dfrac{1}{(aq;q)_r}$ and with this choice equations (1.1) and (1.2) became

$$ \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{1.4} $$

and

$$ \gamma_n = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q;q)_r(aq;q)_{r+2n}} \tag{1.5} $$

respectively.
A pair of sequence $\langle \alpha_n, \beta_n \rangle$ that satisfies (1.4) is called a Bailey pair relative to the parameter a. Similarly, the pair of sequence $\langle \gamma_n, \delta_n \rangle$ which satisfies (1.5) is called conjugate Bailey pair relative to a. For these Bailey and conjugate Bailey pairs we have

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{1.6} $$

provided series involved are convergent.

2 Theorems Involving Bailey Pairs

(i) Choosing $\delta_r = (\rho_1,\rho_2;q)_r \left(\frac{aq}{\rho_1\rho_2}\right)^r$ in (1.5) and using the summation formula [2, App. II(II.8) p. 236] we have,

$$ \gamma_n = \frac{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2};q\right)_\infty (\rho_1,\rho_2;q)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n}{\left(aq, \frac{aq}{\rho_1\rho_2};q\right)_\infty \left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2};q\right)_n}. \tag{2.1} $$

Putting these values of γ_n and δ_n in (1.6) we obtain the following theorem.

Theorem 1. *If $\langle\alpha_n,\beta_n\rangle$ is a Bailey pair satisfying (1.4) then*

$$\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\alpha_n=\frac{\left(aq,\frac{aq}{\rho_1\rho_2};q\right)_\infty}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_\infty}\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n. \tag{2.2}$$

Taking $\rho_1,\rho_2\to\infty$ in (2.2) we obtain the following theorem,

Theorem 2. *If $\langle\alpha_n,\beta_n\rangle$ is a Bailey pair then*

$$\sum_{n=0}^{\infty}q^{n^2}a^n\alpha_n=(aq;q)_\infty\sum_{n=0}^{\infty}q^{n^2}a^n\beta_n. \tag{2.3}$$

(ii) Choosing $\delta_r=(\rho_1,\rho_2;q)_r\left(\frac{a}{\rho_1\rho_2}\right)^r$ in (1.5) and using the summation formula [4, (1.4) p. 771] we have,

$$\gamma_n=\frac{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_\infty}{\left(aq,\frac{aq}{\rho_1\rho_2};q\right)_\infty}\left\{\frac{(\rho_1,\rho_2;q)_n\left(\frac{a}{\rho_1\rho_2}\right)^n}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}+\frac{(1-\rho_1)(1-\rho_2)}{(\rho_1\rho_2-a)}\frac{(\rho_1q,\rho_2q;q)_n\left(\frac{a}{\rho_1\rho_2}\right)^n}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\right\}. \tag{2.4}$$

Putting these values of γ_n and δ_n in (1.6) we have following theorem.

Theorem 3. *If $\langle\alpha_n,\beta_n\rangle$ is a Bailey pair then*

$$\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n\left(\frac{a}{\rho_1\rho_2}\right)^n}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\alpha_n+\frac{(1-\rho_1)(1-\rho_2)a}{(\rho_1\rho_2-a)}\sum_{n=0}^{\infty}\frac{(\rho_1q,\rho_2q;q)_n\left(\frac{a}{\rho_1\rho_2}\right)^n}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\alpha_n$$
$$=\frac{\left(aq,\frac{aq}{\rho_1\rho_2};q\right)_\infty}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{a}{\rho_1\rho_2}\right)^n\beta_n. \tag{2.5}$$

Taking $\rho_1,\rho_2\to\infty$ in (2.5) we have following theorem,

Theorem 4. *If $\langle\alpha_n,\beta_n\rangle$ is a Bailey pair then*

$$\sum_{n=0}^{\infty}q^{n(n-1)}a^n\alpha_n+a\sum_{n=0}^{\infty}q^{n(n+1)}a^n\alpha_n=(aq;q)_\infty\sum_{n=0}^{\infty}q^{n(n-1)}a^n\beta_n \tag{2.6}$$

(iii) Taking $\delta_r=\dfrac{(\rho_1,\rho_2;q)_r\left(\frac{aq}{\rho_1\rho_2};q\right)_{N-r}}{(q;q)_{N-r}}\left(\dfrac{aq}{\rho_1\rho_2}\right)^r$ in (1.5) and using the summation formula [2, App. II (II. 12), p. 237] we obtain

$$\gamma_n=\frac{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_N(\rho_1,\rho_2;q)_n(q^{-N};q)_n\left(-\frac{aq^{1+N}}{\rho_1\rho_2}\right)^n}{(q,aq;q)_Nq^{n(n-1)/2}\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2},aq^{1+N};q\right)_n}. \tag{2.7}$$

Putting these values of γ_n and δ_n in (1.6) we have,

Theorem 5. *If $\langle\alpha_n,\beta_n\rangle$ is a Bailey pair then*

$$\sum_{n=0}^{N}\frac{(\rho_1,\rho_2,q^{-N};q)_n\left(-\frac{aq^{1+N}}{\rho_1\rho_2}\right)^n\alpha_n}{q^{n(n-1)/2}\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2},aq^{1+N};q\right)_n}=\frac{\left(aq,\frac{aq}{\rho_1\rho_2};q\right)_N}{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_N}\sum_{n=0}^{N}\frac{(\rho_1,\rho_2,q^{-N};q)_nq^n}{\left(\frac{\rho_1\rho_2}{a}q^{-N};q\right)_n}\beta_n. \tag{2.8}$$

For $N\to\infty$, (2.8) yields (2.2).

3 WP-Bailey Pairs and Related Theorems

In this section we have established certain theorems involving WP-Bailey pairs.
A WP-Bailey is a pair of sequences $\{\alpha_n(a,k),\beta_n(a,k)\}$ satisfying $\alpha_0(a,k)=\beta_0(a,k)=1$ and

$$\beta_n(a,k)=\sum_{r=0}^{n}\frac{\left(\frac{k}{a};q\right)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}}\alpha_r(a,k)$$
$$=\frac{\left(\frac{k}{a},k;q\right)_n}{(q,aq;q)_n}\sum_{r=0}^{n}\frac{(q^{-n},kq^n;q)_r}{\left(\frac{aq^{1-n}}{k},aq^{1+n};q\right)_r}\left(\frac{aq}{k}\right)^r\alpha_r(a,k). \tag{3.1}$$

The corresponding WP-conjugate Bailey pair $\langle\gamma_n(a,k),\delta_n(a,k)\rangle$ is given by,

$$\gamma_n(a,k)=\sum_{r=0}^{\infty}\frac{\left(\frac{k}{a};q\right)_r(k;q)_{r+2n}}{(q;q)_r(aq;q)_{r+2n}}\delta_{r+n}(a,k)$$
$$=\frac{(k;q)_{2n}}{(aq;q)_{2n}}\sum_{r=0}^{\infty}\frac{\left(\frac{k}{a};q\right)_r(kq^{2n};q)_r}{(q;q)_r(aq^{1+2n};q)_r}\delta_{r+n}(a,k). \tag{3.2}$$

If $\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ is a WP-Bailey pair and $\langle\gamma_n(a,k),\delta_n(a,k)\rangle$ is a WP-Conjugate Bailey pair then Bailey lemma gives,

$$\sum_{n=0}^{\infty}\alpha_n(a,k)\gamma_n(a,k)=\sum_{n=0}^{\infty}\beta_n(a,k)\delta_n(a,k), \tag{3.3}$$

provided series involved in (3.2) and (3.3) are convergent.
(i) Choosing $\delta_r(a,k)=\left(\frac{a^2q}{k^2}\right)^r$ in (3.2) and using the summation formula [2, App.II(II.8), p.236] we find,

$$\gamma_n(a,k)=\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}}\left(\frac{a^2q}{k^2}\right)^n. \tag{3.4}$$

Putting these values of $\langle\gamma_n(a,k),\delta_n(a,k)\rangle$ in (3.3) we have following theorem.

Theorem 6. *If $\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ is a WP-Bailey pair then*

$$\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}}\left(\frac{a^2q}{k^2}\right)^n\alpha_n(a,k)=\sum_{n=0}^{\infty}\left(\frac{a^2q}{k^2}\right)^n\beta_n(a,k). \tag{3.5}$$

(ii) Putting $\delta_r(a,k)=\left(\frac{a^2}{k^2}\right)^r$ in (3.2) and making use of the summation formula [4, (1.4), p.771] we get,

$$\gamma_n(a,k)=\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}}\left(\frac{k}{k+a}\right)(1+aq^{2n})\left(\frac{a^2}{k^2}\right)^n. \tag{3.6}$$

Substituting these values $\langle\gamma_n(a,k),\delta_n(a,k)\rangle$ in (3.3) we get the following theorem.

Theorem 7. *If $\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ is a WP-Bailey pair then*

$$\begin{aligned}&\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\left(\frac{k}{k+a}\right)\sum_{n=0}^{\infty}\frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}}\alpha_n(a,k)\left(\frac{a^2}{k^2}\right)^n\\&+\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\left(\frac{ka}{k+a}\right)\sum_{n=0}^{\infty}\frac{(k;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}}\left(\frac{a^2q^2}{k^2}\right)^n\alpha_n(a,k)\\&=\sum_{n=0}^{\infty}\beta_n(a,k)\left(\frac{a^2}{k^2}\right)^n.\end{aligned}\tag{3.7}$$

(iii) Taking $\delta_r=\dfrac{\left(\frac{a^2q}{k^2};q\right)_{N-r}\left(\frac{a^2q}{k^2}\right)^r}{(q;q)_{N-r}}$ in (3.2) we find,

$$\gamma_n(a,k)=\frac{(k;q)_{2n}}{(aq;q)_{2n}}\frac{\left(\frac{a^2q}{k^2};q\right)_{N-n}}{(q;q)_{N-n}}\left(\frac{a^2q}{k^2}\right)^n{}_3\Phi_2\left[\begin{matrix}kq^{2n},\frac{k}{a},q^{-(N-n)};q;q\\ aq^{1+2n},\frac{k^2}{a^2}q^{-(N-n)}\end{matrix}\right].\tag{3.8}$$

Now summing ${}_3\Phi_2$ series in (3.8) by making use of [2, App.II(II.12), p.237] we get,

$$\gamma_n(a,k)=\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_N}{(q,aq;q)_N}\frac{(aq;q)_{2n}}{\left(\frac{a^2q}{k};q\right)_{2n}}\frac{\left(q^{-N},\frac{a^2q^{1+N}}{k};q\right)_n\left(\frac{aq}{k}\right)^n}{\left(aq^{1+N},\frac{k}{a}q^{-N};q\right)_n}.\tag{3.9}$$

Putting these values of $\langle\gamma_n(a,k),\delta_n(a,k)\rangle$ in (3.3) we have following theorem.

Theorem 8. *If $\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ is a WP-Bailey pair then*

$$\begin{aligned}&\sum_{n=0}^{N}\frac{(k;q)_{2n}\left(q^{-N},\frac{a^2}{k}q^{1+N};q\right)_n\left(\frac{aq}{k}\right)^n}{\left(\frac{a^2q}{k};q\right)_{2n}\left(\frac{kq^{-N}}{a},aq^{1+N};q\right)_n}\alpha_n(a,k)\\&=\frac{\left(aq,\frac{a^2q}{k^2};q\right)_N}{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_N}\sum_{n=0}^{N}\frac{(q^{-N};q)_nq^n}{\left(\frac{k^2q^{-N}}{a^2};q\right)_n}\beta_n(a,k).\end{aligned}\tag{3.10}$$

(iv) Choosing $\delta_r(a,k)=\dfrac{(1-kq^{2r})}{(1-k)}\dfrac{(\rho_1,\rho_2;q)_r\left(\frac{aq}{\rho_1\rho_2}\right)^r}{\left(\frac{kq}{\rho_1},\frac{kq}{\rho_2};q\right)_r}$ in (3.2) we find,

$$\begin{aligned}\gamma_n(a,k)=&\frac{(k;q)_{2n}(\rho_1,\rho_2;q)_n(1-kq^{2n})\left(\frac{aq}{\rho_1\rho_2}\right)^n}{(aq;q)_{2n}\left(\frac{kq}{\rho_1},\frac{kq}{\rho_2};q\right)_n(1-k)}\\&\cdot{}_6\Phi_5\left[\begin{matrix}kq^{2n},q^{n+1}\sqrt{k},-q^{n+1}\sqrt{k},\rho_1q^n,\rho_2q^n,\frac{k}{a};q;\frac{aq}{\rho_1\rho_2}\\ q^n\sqrt{k},-q^n\sqrt{k},\frac{kq^{n+1}}{\rho_1},\frac{kq^{n+1}}{\rho_2},aq^{1+2n}\end{matrix}\right].\end{aligned}\tag{3.11}$$

Summing the ${}_6\Phi_5$ series in (3.11) by using [2, App. II (II 20), p. 238] we find,

$$\gamma_n(a,k)=\frac{\left(kq,\frac{kq}{\rho_1\rho_2},\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_\infty(\rho_1,\rho_2;q)_n}{\left(\frac{kq}{\rho_1},\frac{kq}{\rho_2},aq,\frac{aq}{\rho_1\rho_2};q\right)_\infty\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n.\tag{3.12}$$

Putting these values of $\langle\gamma_n(a,k),\delta_n(a,k)\rangle$ in (3.3) we get following theorem.

Theorem 9. *If $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ is a WP-Bailey pair then*

$$\frac{\left(kq, \frac{kq}{\rho_1\rho_2}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_\infty}{\left(\frac{kq}{\rho_1}, \frac{kq}{\rho_2}, aq, \frac{aq}{\rho_1\rho_2}; q\right)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n}{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n} \alpha_n(a,k)$$
$$= \sum_{n=0}^{\infty} \left(\frac{1-kq^{2n}}{1-k}\right) \frac{(\rho_1, \rho_2; q)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n}{\left(\frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_n} \beta_n(a,k). \tag{3.13}$$

As $\rho_1, \rho_2 \to \infty$, (3.13) yields following theorem.

Theorem 10. *If $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ is a WP-Bailey pair then*

$$\frac{(kq;q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} q^{n^2} a^n \alpha_n(a,k) = \sum_{n=0}^{\infty} \left(\frac{1-kq^{2n}}{1-k}\right) q^{n^2} a^n \beta_n(a,k). \tag{3.14}$$

(v) Taking $\delta_r(a,k) = \dfrac{\left(\rho_1, \rho_2, \frac{ak}{\rho_1\rho_2} q^{1+N}; q\right)_r \left(\frac{1}{k}; q\right)_{-N-r}}{\left(\frac{kq}{\rho_1}, \frac{kq}{\rho_2}, \frac{\rho_1\rho_2}{a} q^{-N}; q\right)_r (q;q)_{N-r}} \left(\dfrac{q^{-2N}}{k}\right)^r$ in (3.2) we get

$$\gamma_n(a,k) = \frac{(k;q)_{2n}(1-kq^{2n}) \left(\rho_1, \rho_2, \frac{ak}{\rho_1\rho_2} q^{1+N}; q\right)_n (q^{-N};q)_n q^n}{(aq;q)_{2n}(1-k) \left(\frac{kq}{\rho_1}, \frac{kq}{\rho_2}, \frac{\rho_1\rho_2}{a} q^{-N}; q\right)_n (kq^{1+N};q)_n} \frac{q^{N(N+1)/2} k^N}{(q,kq;q)_N}$$
$$\cdot {}_8\Phi_7 \left[\begin{matrix} kq^{2n}, q^{n+1}\sqrt{k}, -q^{n+1}\sqrt{k}, \rho_1 q^n, \rho_2 q^n, \frac{ak}{\rho_1\rho_2} q^{1+N+n}, \frac{k}{a}, q^{-(N-n)}; q; q \\ q^n\sqrt{k}, -q^n\sqrt{k}, \frac{k}{\rho_1} q^{1+n}, \frac{k}{\rho_2} q^{1+n}, \frac{\rho_1\rho_2}{a} q^{-(N-n)}, aq^{1+2n}, kq^{1+N+n} \end{matrix}\right]. \tag{3.15}$$

Summing the ${}_8\Phi_7$ series in (3.15) by using [2, App.II (II.22), p. 238] we have

$$\gamma_n(a,k) = \frac{\left(\frac{kq}{\rho_1\rho_2}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_N q^{N(N+1)/2} k^N}{\left(q, \frac{aq}{\rho_1\rho_2}, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_N (aq;q)_N}$$
$$\cdot \frac{\left(\rho_1, \rho_2, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}, \frac{ak}{\rho_1\rho_2} q^{1+N}, q^{-N}; q\right)_n \left(\frac{aq}{k}\right)^n}{\left(aq^{1+N}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2}{k} q^{-N}, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_n}. \tag{3.16}$$

Putting these values of $\langle \gamma_n(a,k), \delta_n(a,k)\rangle$ in (3.3) we obtain the following theorem.

Theorem 11. *If $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ is a WP-Bailey pair then*

$$\frac{\left(kq, \frac{kq}{\rho_1\rho_2}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_N}{\left(aq, \frac{aq}{\rho_1\rho_2}, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}; q\right)_N} \sum_{n=0}^{N} \frac{\left(\rho_1, \rho_2, \frac{ak}{\rho_1\rho_2} q^{1+N}, q^{-N}; q\right)_n \left(\frac{aq}{k}\right)^n}{\left(aq^{1+N}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{\rho_1\rho_2}{k} q^{-N}; q\right)_n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{1-kq^{2n}}{1-k}\right) \frac{\left(\rho_1, \rho_2, \frac{ak}{\rho_1\rho_2} q^{1+N}, q^{-N}; q\right)_n q^n}{\left(kq^{1+N}, \frac{kq}{\rho_1}, \frac{kq}{\rho_2}, \frac{\rho_1\rho_2}{a} q^{-N}; q\right)_n} \beta_n(a,k). \tag{3.17}$$

As $\rho_2 \to \infty$, (3.17) yields the following theorem.

Theorem 12. *If $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ is a WP-Bailey pair then*

$$\frac{\left(kq, \frac{aq}{\rho_1}; q\right)_N}{\left(aq, \frac{kq}{\rho_1}; q\right)_N} \sum_{n=0}^{N} \frac{\left(\rho_1, q^{-N}; q\right)_n \left(\frac{aq^{1+N}}{\rho_1}\right)^n \alpha_n(a,k)}{\left(\frac{aq}{\rho_1}, aq^{1+N}; q\right)_n}$$
$$= \sum_{n=0}^{N} \left(\frac{1-kq^{2n}}{1-k}\right) \frac{(\rho_1, q^{-N}; q)_n}{\left(\frac{kq}{\rho_1}, kq^{1+N}; q\right)_n} \left(\frac{a}{\rho_1} q^{1+N}\right)^n \beta_n(a,k). \tag{3.18}$$

4 Bailey Pairs

In this section we give numerous Bailey pairs deducible from certain summation formulas.

(i) Choosing $\alpha_r = \dfrac{q^{r(r+1)/2}(a, q\sqrt{a}, -q\sqrt{a}; q)_r q^{-\frac{3}{2}r}}{(q, \sqrt{a}, -\sqrt{a}; q)_r}$ in (1.4) and using the summation formula [5, (4.1), p.76] we get,

$$\beta_n = \frac{1}{2}\left[\frac{(-q^{-\frac{1}{2}}; q)_n (1+\sqrt{a})}{(q, \sqrt{aq}, -\sqrt{a}; q)_n} + \frac{(-q^{-\frac{1}{2}}; q)_n (1-\sqrt{a})}{(q, -\sqrt{aq}, \sqrt{a}; q)_n}\right]. \tag{4.1}$$

α_n and β_n given in (4.1) form a Bailey pair.

(ii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(a, q\sqrt{a}; q)_r q^{-r}}{(q, \sqrt{a}; q)_r}$ in (1.4) and using the summation formula [5, (4.2), p.76] we get,

$$\beta_n = \frac{1+\sqrt{a}}{2} \frac{(-1;q)_n}{(q, \sqrt{aq}, -\sqrt{aq}; q)_n} + \frac{1-\sqrt{a}}{2} \frac{(-1;q)_n}{(q, \sqrt{a}, -q\sqrt{a}; q)_n}. \tag{4.2}$$

α_n and β_n given in (4.2) form a Bailey pair.

(iii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(a;q)_r q^{-\frac{1}{2}r}}{(q;q)_r}$ in (1.4) and making use of the summation formula [5, (4.3), p.76] we find,

$$\beta_n = \frac{1+\sqrt{a}}{2} \frac{(-q^{\frac{1}{2}};q)_n}{(q, -\sqrt{aq}, q\sqrt{a}; q)_n} + \frac{1-\sqrt{a}}{2} \frac{(-q^{\frac{1}{2}};q)_n}{(q, \sqrt{aq}, -q\sqrt{a}; q)_n}. \tag{4.3}$$

α_n and β_n given in (4.3) form a Bailey pair.

(iv) Choosing $\alpha_r = \dfrac{q^{\frac{1}{2}r^2}(a;q)_r(1-aq^{2r})}{(q;q)_r(1-a)}$ in (1.4) and using the summation formula [5, (4.5), p.77] we find,

$$\beta_n = \frac{1+\sqrt{a}}{2\sqrt{a}} \frac{(-q^{-\frac{1}{2}};q)_n}{(q, \sqrt{aq}, -q\sqrt{a}; q)_n} - \frac{1-\sqrt{a}}{2\sqrt{a}} \frac{(-q^{-\frac{1}{2}};q)_n}{(q, -\sqrt{aq}, q\sqrt{a}; q)_n}. \tag{4.4}$$

α_n and β_n given in (4.4) form a Bailey pair.

(v) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(a, q\sqrt{a}; q)_r}{(q, \sqrt{a}; q)_r}$ in (1.4) and using the summation formula [5, (4.6), p.77] we find,

$$\beta_n = \frac{1+\sqrt{a}}{2\sqrt{a}} \frac{(-1;q)_n}{(q, \sqrt{aq}, -\sqrt{aq}; q)_n} - \frac{1-\sqrt{a}}{2\sqrt{a}} \frac{(-1;q)_n}{(q, \sqrt{a}, -q\sqrt{a}; q)_n}. \tag{4.5}$$

α_n and β_n given in (4.5) form a Bailey pair.

(vi) Choosing $\alpha_r = \dfrac{q^{r(r+1)/2}(a;q)_r q^{\frac{1}{2}r}}{(q;q)_r}$ in (1.4) and making use of the summation formula [5, (4.7), p.77] we find,

$$\beta_n = \frac{1+\sqrt{a}}{2\sqrt{a}} \frac{(-q^{\frac{1}{2}};q)_n}{(q,-\sqrt{aq},q\sqrt{a};q)_n} - \frac{1-\sqrt{a}}{2\sqrt{a}} \frac{(-q^{\frac{1}{2}};q)_n}{(q,\sqrt{aq},-q\sqrt{a};q)_n}. \tag{4.6}$$

α_n and β_n given in (4.6) form a Bailey pair.

(vii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(a,-q\sqrt{a},b;q)_r \left(-\frac{\sqrt{a}}{b}\right)^r}{\left(q,-\sqrt{a},\frac{aq}{b};q\right)_r}$ in (1.4) and using the summation formula [2, App. II (II.14), p. 237] we get

$$\beta_n = \frac{\left(\frac{q\sqrt{a}}{b};q\right)_n}{\left(q,q\sqrt{a},\frac{aq}{b};q\right)_n}. \tag{4.7}$$

$\langle \alpha_n, \beta_n \rangle$ given in (4.7) form a Bailey pair.

(viii) Taking $\alpha_r = \dfrac{q^{r(r+1)/2}(a,b,c;q)_r(1-aq^{2r})\left(-\frac{a}{bc}\right)^r}{\left(q,\frac{aq}{b},\frac{aq}{c};q\right)_r(1-a)}$ in (1.4) and using the summation formula [2, App. II (II.21), p. 238] we get

$$\beta_n = \frac{\left(\frac{aq}{bc};q\right)_n}{\left(q,\frac{aq}{b},\frac{aq}{c};q\right)_n}. \tag{4.8}$$

$\langle \alpha_n, \beta_n \rangle$ given in (4.8) form a Bailey pair.

5 WP-Bailey Pairs

In this section we give certain WP-Bailey pairs out of which some are known and some are new.

(i) Taking $\alpha_r(a,k) = \dfrac{(a,b;q)_r(1-aq^{2r})}{\left(q,\frac{aq}{b};q\right)_r(1-a)}\left(\dfrac{1}{b}\right)^r$ in (3.1) and summing the series by making use of [2, App. II (II.21), p. 238] we get,

$$\beta_n(a,k) = \frac{\left(k,\frac{kb}{a};q\right)_n}{\left(q,\frac{aq}{b};q\right)_n b^n}. \tag{5.1}$$

$\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ given in (5.1) form a WP-Bailey pair.

(ii) Choosing $\alpha_r(a,k) = \dfrac{\left(a,b,c,\frac{a^2q}{bck};q\right)_r(1-aq^{2r})}{\left(q,\frac{aq}{b},\frac{aq}{c},\frac{bck}{a};q\right)_r(1-a)}\left(\dfrac{k}{a}\right)^r$ in (3.1) and using the summation formula [2, App. II (II.22), p. 238] we get,

$$\beta_n(a,k) = \frac{\left(k,\frac{aq}{bc},\frac{kb}{a},\frac{kc}{a};q\right)_n}{\left(q,\frac{aq}{b},\frac{aq}{c},\frac{kbc}{a};q\right)_n}. \tag{5.2}$$

$\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ given in (5.2) form a WP-Bailey pair.

(iii) Again, choosing $\alpha_n(a,k) = \dfrac{\left(a,\frac{a}{k};q\right)_n(1-aq^{2n})}{(q,kq;q)_n(1-a)}\left(\dfrac{k}{a}\right)^n$ in (3.1) and summing the series by making use of [2, App. II (II.21), p. 238] we get,

$$\beta_n(a,k) = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1. \end{cases} \tag{5.3}$$

So, $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ given in (5.3) form a WP-Bailey pair.

(iv) If we take $\alpha_r(a,k) = \delta_{r,0}$ in (3.1) we find,

$$\beta_n(a,k) = \frac{\left(k,\frac{k}{a};q\right)_n}{(q,aq;q)_n}. \tag{5.4}$$

$\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ given in (5.4) also form a WP-Bailey pair.
(v) In the summation formula [5, (1.3), p.71] if we take $c=\frac{a}{k}q^{1/2}$ we get,

$$
\begin{aligned}
{}_4\Phi_3\left[\begin{matrix} a,\frac{a}{k}q^{\frac12},kq^n,q^{-n};q;q\\ kq^{\frac12},\frac{a}{k}q^{1-n},aq^{1+n}\end{matrix}\right] &= \frac{1+\sqrt{a}}{2}\frac{(aq,\sqrt{q};q)_n\left(\frac{k}{\sqrt{a}},k\sqrt{\frac{q}{a}};q\right)_n}{\left(kq^{\frac12},\frac{k}{a};q\right)_n(\sqrt{aq},q\sqrt{a};q)_n}\\
&+\frac{1-\sqrt{a}}{2}\frac{(aq,\sqrt{q};q)_n\left(-\frac{k}{\sqrt{a}},-k\sqrt{\frac{q}{a}};q\right)_n}{\left(kq^{\frac12},\frac{k}{a};q\right)_n(-\sqrt{aq},-q\sqrt{a};q)_n}.
\end{aligned}\tag{5.5}
$$

Now, choosing $\alpha_r(a,k)=\frac{\left(a,\frac{aq^{\frac12}}{k};q\right)_n}{\left(q,kq^{\frac12};q\right)_n}\left(\frac{k}{a}\right)^n$ in (3.1) and summing the series by using (5.5) we get

$$
\begin{aligned}
\beta_n(a,k) &= \frac{1+\sqrt{a}}{2}\frac{\left(k,q^{\frac12},\frac{k}{\sqrt{a}},k\sqrt{\frac{q}{a}};q\right)_n}{\left(q,kq^{\frac12},q\sqrt{a},\sqrt{aq};q\right)_n}\\
&+\frac{1-\sqrt{a}}{2}\frac{\left(k,q^{\frac12},-\frac{k}{\sqrt{a}},-k\sqrt{\frac{q}{a}};q\right)_n}{\left(q,kq^{\frac12},-q\sqrt{a},-\sqrt{aq};q\right)_n}.
\end{aligned}\tag{5.6}
$$

$\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ given in (5.6) form a WP-Bailey pair.
(vi) Again, taking $c=\frac{a}{k}q^{\frac12}$ in the summation formula [5, (4.4), p.77] we get,

$$
\begin{aligned}
{}_4\Phi_3\left[\begin{matrix} a,\frac{a}{k}q^{\frac12},kq^n,q^{-n};q;q^2\\ kq^{\frac12},\frac{a}{k}q^{1-n},aq^{1+n}\end{matrix}\right] &= \frac{1+\sqrt{a}}{2\sqrt{a}}\frac{(aq,\sqrt{q};q)_n\left(\frac{k}{\sqrt{a}},k\sqrt{\frac{q}{a}};q\right)_n}{\left(kq^{\frac12},\frac{kq^{\frac12}}{a};q\right)_n(\sqrt{aq},q\sqrt{a};q)_n}\\
&+\frac{1-\sqrt{a}}{2\sqrt{a}}\frac{(aq,\sqrt{a};q)_n\left(-\frac{k}{\sqrt{a}},-k\sqrt{\frac{q}{a}};q\right)_n}{\left(kq^{\frac12},\frac{kq^{\frac12}}{a};q\right)_n(-\sqrt{aq},-q\sqrt{a};q)_n}.
\end{aligned}\tag{5.7}
$$

Now, choosing $\alpha_n(a,k)=\frac{\left(a,\frac{aq^{\frac12}}{k};q\right)_n}{\left(q,kq^{\frac12};q\right)_n}\left(\frac{kq}{a}\right)^n$ in (3.1) and using (5.7) we get

$$
\begin{aligned}
\beta_n(a,k) &= \frac{1+\sqrt{a}}{2\sqrt{a}}\frac{\left(k,q^{\frac12},\frac{k}{\sqrt{a}},k\sqrt{\frac{q}{a}};q\right)_n}{\left(q,kq^{\frac12},q\sqrt{a},\sqrt{aq};q\right)_n}\\
&-\frac{1-\sqrt{a}}{2\sqrt{a}}\frac{\left(k,q^{\frac12},-\frac{k}{\sqrt{a}},-k\sqrt{\frac{q}{a}};q\right)_n}{\left(q,kq^{\frac12},-q\sqrt{a},-\sqrt{aq};q\right)_n}.
\end{aligned}\tag{5.8}
$$

$\langle\alpha_n(a,k),\beta_n(a,k)\rangle$ given in (5.8) form a WP-Bailey pair. Bailey pairs given in (5.6) and (5.8) are believed to be new.

6 Bailey Chain

If $\langle \alpha_n, \beta_n \rangle$ is a Bailey pair i.e. $\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_r (aq;q)_r}$, then so $\langle \alpha'_n, \beta'_n \rangle$ is also a Bailey pair, where

$$\alpha'_n = \frac{(\rho_1, \rho_2; q)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n}{\left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n} \alpha_n. \tag{6.1}$$

and

$$\begin{aligned}\beta'_n &= \sum_{r=0}^{\infty} \frac{(\rho_1, \rho_2; q)_r \left(\frac{aq}{\rho_1\rho_2}; q\right)_{n-r} \left(\frac{aq}{\rho_1\rho_2}\right)^r \beta_r}{(q;q)_{n-r} \left(\frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n} \\ &= \frac{\left(\frac{aq}{\rho_1\rho_2}; q\right)_n}{\left(q, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}; q\right)_n} \sum_{r=0}^{\infty} \frac{(\rho_1, \rho_2; q)_r (q^{-n}; q)_r q^r}{\left(\frac{\rho_1\rho_2}{a} q^{-n}; q\right)_r} \beta_r.\end{aligned} \tag{6.2}$$

Thus, we find that if one Bailey pair $\langle \alpha_n, \beta_n \rangle$ is known then a new Bailey pair $\langle \alpha'_n, \beta'_n \rangle$ can be constructed as shown above in (6.2). Repeating this process we can have infinite number of Bailey pairs if one initial pair is known. These Bailey pairs so constructed from an initial Bailey pair form a chain called Bailey chain.

7 WP-Bailey Tree

Andrews proved following two theorems for constructing WP-Bailey pairs from a initial known Bailey pair.

Theorem 13. *If $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ is a WP-Bailey pair, then so is the pair $\langle \alpha'_n(a,k),\ \beta'_n(a,k) \rangle$ given by,*

$$\begin{aligned}\alpha'_n(a,k) &= \frac{(b,c;q)_n}{\left(\frac{aq}{b}, \frac{aq}{c}; q\right)_n} \left(\frac{k}{m}\right)^n \alpha_n(a,m), \\ \beta'_n(a,k) &= \frac{\left(\frac{mq}{b}, \frac{mq}{c}; q\right)_n}{\left(\frac{aq}{b}, \frac{aq}{c}; q\right)_n} \sum_{r=0}^{n} \frac{1-mq^{2r}}{1-m} \frac{(b,c;q)_r \left(\frac{k}{m}; q\right)_{n-r} (k;q)_{n+r}}{\left(\frac{mq}{b}, \frac{mq}{c}; q\right)_r (q;q)_{n-r} (mq;q)_{m+r}} \left(\frac{k}{m}\right)^r \beta_r(a,m),\end{aligned} \tag{7.1}$$

[6, Theorem (2.1)]

where $m = \frac{bck}{aq}$.

Theorem 14. *If $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ is a WP-Bailey pair, then so is the pair $\langle \alpha'_n(a,k),\ \beta'_n(a,k) \rangle$ given by,*

$$\begin{aligned}\alpha'_n(a,k) &= \frac{(m;q)_{2n}}{(k;q)_{2m}} \left(\frac{k}{m}\right)^n \alpha_n(a,m), \\ \beta'_n(a,k) &= \sum_{r=0}^{n} \frac{\left(\frac{k}{m}; q\right)_{n-r}}{(q;q)_{n-r}} \left(\frac{k}{m}\right)^r \beta_r(a,m),\end{aligned} \tag{7.2}$$

[6, Theorem (2.2)]

where $m = \dfrac{a^2q}{k}$.
From these two theorems, each WP-Bailey pair gives rise to a binary tree of WP-Bailey pairs. Andrews coined this the WP-Bailey tree. The following four theorems due to Warnaar give additional branches to the Bailey tree.

Theorem 15. *If $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ is a WP-Bailey pair, then so is the pair $\langle \alpha_n'(a,k),\ \beta_n'(a,k)\rangle$ given by,*

$$\alpha_n'(a,k) = \frac{1-\sigma k^{\frac{1}{2}}}{1-\sigma k^{\frac{1}{2}}q^n}\,\frac{1+\sigma m^{\frac{1}{2}}q^n}{1+\sigma m^{\frac{1}{2}}}\,\frac{(m;q)_{2n}}{(k;q)_{2m}}\left(\frac{k}{m}\right)^n \alpha_n(a,m),$$
$$\beta_n'(a,k) = \frac{1-\sigma k^{\frac{1}{2}}}{1-\sigma k^{\frac{1}{2}}q^n}\sum_{r=0}^{n}\frac{1+\sigma m^{\frac{1}{2}}q^n}{1+\sigma m^{\frac{1}{2}}}\,\frac{\left(\frac{k}{m};q\right)_{n-r}}{(q;q)_{n-r}}\left(\frac{k}{m}\right)^r \beta_r(a,m), \tag{7.3}$$

[6, Theorem (2.3)]

where $m = \dfrac{a^2}{k}$ and $\sigma = \{-1,1\}$.
The freedom in the choice of σ simply reflects that the above expressions are invariant under the simultaneous negation of $k^{\frac{1}{2}}$, $m^{\frac{1}{2}}$ and σ.

Theorem 16. *If $\langle \alpha_n(a,k), \beta_n(a,k)\rangle$ is a WP-Bailey pair, then so is the pair $\langle \alpha_n'(a,k),\ \beta_n'(a,k)\rangle$ given by,*

$$\alpha_n'(a^2,k;q^2) = \alpha_n(a,m;q),$$
$$\beta_n'(a^2,k;q^2) = \frac{(-mq;q)_{2n}}{(-aq;q)_{2n}}\sum_{r=0}^{n}\frac{(1-mq^{2r})}{(1-m)}\,\frac{\left(\frac{k}{m^2};q^2\right)_{n-r}(k;q^2)_{n+r}\left(\frac{m}{a}\right)^{n-r}}{(q^2;q^2)_{n-r}(m^2q^2;q^2)_{n+r}}\beta_r(a,m;q), \tag{7.4}$$

[6, Theorem (2.4)]

where $m = \dfrac{k}{aq}$.

Theorem 17. *If $\langle \alpha_n(a,k;q), \beta_n(a,k;q)\rangle$ is a WP-Bailey pair, then so is the pair $\langle \alpha_n'(a,k;q),\ \beta_n'(a,k;q)\rangle$ given by,*

$$\alpha_n'(a^2,k;q^2) = q^{-n}\frac{1+aq^{2n}}{1+a}\alpha_n(a,m;q),$$
$$\beta_n'(a^2,k;q^2) = q^{-n}\frac{(-mq;q)_{2n}}{(-a;q)_{2n}}\sum_{r=0}^{n}\frac{(1-mq^{2r})}{(1-m)}\,\frac{\left(\frac{k}{m^2};q^2\right)_{n-r}(k;q^2)_{n+r}\left(\frac{m}{a}\right)^{n-r}}{(q^2;q^2)_{n-r}(m^2q^2;q^2)_{n+r}}\beta_r(a,m;q), \tag{7.5}$$

[6, Theorem (2.5)]

where $m = \dfrac{k}{a}$.

Theorem 18. *If $\langle \alpha_n(a,k;q), \beta_n(a,k;q)\rangle$ is a WP-Bailey pair, then so is the pair $\langle \alpha_n'(a,k;q),\ \beta_n'(a,k;q)\rangle$ given by,*

$$\alpha_{2n}'(a,k;q) = \alpha_n(a,m;q^2), \quad \alpha_{2n+1}'(a,k;q) = 0,$$
$$\beta_n'(a,k;q) = \frac{(mq;q^2)_n}{(aq;q)_n}\sum_{r=0}^{[\frac{n}{2}]}\frac{(1-mq^{2r})}{(1-m)}\,\frac{\left(\frac{k}{m};q\right)_{n-r}(k;q)_{n+2r}\left(-\frac{k}{a}\right)^{n-2r}}{(q;q)_{n-r}(mq;q)_{n+2r}}\beta_r(a,m;q^2), \tag{7.6}$$

[6, Theorem (2.6)]

where $m = \dfrac{k}{a}$.

8 Applications

In this section we shall establish certain transformation formulas by making use of the results established in previous sections.

(a) Substituting the Bailey pairs given in (4.1) in (2.2) we get,

$$\frac{\left(\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_\infty}{\left(aq,\frac{aq}{\rho_1\rho_2};q\right)_\infty}\,{}_5\Phi_5\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},\rho_1,\rho_2;q;-\frac{aq^{\frac{1}{2}}}{\rho_1\rho_2}\\ \sqrt{a},-\sqrt{a},\frac{aq}{\rho_1},\frac{aq}{\rho_2},0\end{matrix}\right]$$
$$=\frac{1+\sqrt{a}}{2}\,{}_3\Phi_2\left[\begin{matrix}\rho_1,\rho_2,-q^{-\frac{1}{2}};q;\frac{aq}{\rho_1\rho_2}\\ \sqrt{aq},-\sqrt{a}\end{matrix}\right]$$
$$+\frac{1-\sqrt{a}}{2}\,{}_3\Phi_2\left[\begin{matrix}\rho_1,\rho_2,-q^{-\frac{1}{2}};q;\frac{aq}{\rho_1\rho_2}\\ -\sqrt{aq},\sqrt{a}\end{matrix}\right],\quad\left|\frac{aq^{\frac{1}{2}}}{\rho_1\rho_2}\right|<1. \tag{8.1}$$

Similar transformations can be established by putting Bailey pairs given in (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8) in any one of the results given in (2.2), (2.3), (2.5), (2.6) and (2.8).

(b) If we put WP-Bailey pair given in (5.1) and (3.5) we get,

$$\frac{\left(\frac{a^2q}{k},\frac{aq}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\,{}_8\Phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,\sqrt{k},-\sqrt{k},\sqrt{kq},-\sqrt{kq};q;-\frac{a^2q}{bk^2}\\ \sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{\sqrt{k}},-\frac{aq}{\sqrt{k}},a\sqrt{\frac{q}{k}},-a\sqrt{\frac{q}{k}}\end{matrix}\right]$$
$$={}_2\Phi_1\left[\begin{matrix}k,\frac{kb}{a};q;\frac{a^2q}{bk^2}\\ \frac{aq}{b}\end{matrix}\right]. \tag{8.2}$$

Similar transformations can be established by substituting WP-Bailey pairs given in (5.2), (5.3), (5.4), (5.6) and (5.8) in any one of the results given in (3.5), (3.7), (3.10), (3.13), (3.14), (3.17) and (3.18).

(c) Replacing b,c by ρ_1 and ρ_2 in (7.1) respectively and then putting the values of $\alpha_n(a,m)$, $\beta_n(a,m)$ from (5.1) we get new Bailey pairs.

$$\alpha_n'(a,k)=\frac{(a,q\sqrt{a},-q\sqrt{a},b,\rho_1,\rho_2;q)_n}{\left(q,\sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\left(\frac{k}{mb}\right)^n,$$
$$\beta_n'(a,k)=\frac{\left(k,\frac{k}{m},\frac{mq}{\rho_1},\frac{mq}{\rho_2};q\right)_n}{\left(q,mq,\frac{aq}{\rho_1},\frac{aq}{\rho_2};q\right)_n}\sum_{r=0}^{n}\frac{1-mq^{2r}}{1-m}\frac{\left(m,\frac{mb}{a},\rho_1,\rho_2,kq^n,q^{-n};q\right)_r\left(\frac{q}{b}\right)^r}{\left(q,\frac{aq}{b},\frac{mq}{\rho_1},\frac{mq}{\rho_2},\frac{m}{k}q^{1-n},mq^{1+n};q\right)_r}, \tag{8.3}$$

where $m=\dfrac{k\rho_1\rho_2}{aq}$.

Now, putting these values of $\alpha_n'(a,k)$ and $\beta_n'(a,k)$ given in (8.3) in (3.1) we get,

$$\frac{\left(\frac{k}{m},\frac{mq}{\rho_1},\frac{mq}{\rho_2},aq;q\right)_n}{\left(mq,\frac{aq}{\rho_1},\frac{aq}{\rho_2},\frac{k}{a};q\right)_n}\,{}_8\Phi_7\left[\begin{matrix}m,q\sqrt{m},-q\sqrt{m},\frac{mb}{a},\rho_1,\rho_2,kq^n,q^{-n};q;\frac{q}{b}\\ \sqrt{m},-\sqrt{m},\frac{aq}{b},\frac{mq}{\rho_1},\frac{mq}{\rho_2},\frac{m}{k}q^{1-n},mq^{1+n}\end{matrix}\right]$$
$$={}_8\Phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,\rho_1,\rho_2,kq^n,q^{-n};q;\frac{aq}{mb}\\ \sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{\rho_1},\frac{aq}{\rho_2},\frac{a}{k}q^{1-n},aq^{1+n}\end{matrix}\right]. \tag{8.4}$$

where $m=\frac{k\rho_1\rho_2}{aq}$.

Putting the WP-Bailey pairs given in (5.1), (5.2), (5.3), (5.4), (5.5), (5.6), (5.7) and (5.8) in any one of the results given in (7.1), (7.2), (7.3), (7.4), (7.5) and (7.6) one finds new WP-Bailey pairs, on substituting these new Bailey pairs in (3.1) we get transformations similar to (8.4).

9 Conclusions

In this paper, certain transformation formulas involving q-hypergeometric series have been obtained by making use of theorems, Bailey Pairs and WP-Bailey Pairs established herein. From these transformation formulas q-series identities can be deduced which may have partition theoretic interpretations. Results of this paper are quite useful and we hope that these results will form the base of further research in the subject.

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