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Fan-Gottesman Compactification and Scattered Spaces

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Abstract

Compactification is the process or result of making a topological space into a compact space. An embedding of a topological space X as a dense subset of a compact space is called a compactification of X. There are a lot of compactification methods but we study with Fan- Gottesman compactification. A topological space X is said to be scattered if every non-empty subset S of X contains at least one point which is isolated in S. Compact scattered spaces are important for analysis and topology. In this paper, we investigate the relation between the Fan- Gottesman compactification of T_3 space and scattered spaces. We show under which conditions the Fan-Gottesman compactification X^* is a scattered.

Keywords: Fan-Gottesman compactification, Scattered spaces.
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1 introduction and some Preliminaries

The notion of compactness or compactification is an important concept in general topology as well as in a branch of mathematics and quantum physic. A compactification of a space X is a compact space containing X as a dense subspace. There are a lot of compactification methods applying different topological spaces such as Aleksandrov (one-point), Wallman, Stone-Cech. In 1929, Aleksandrov proved that all locally compact Hausdorff spaces may be completed to a compact hausdorff by the addition of one point [1]. In 1937, Cech constructed the compactification of a space by using the diagonal product of all continuous functions [3] while Stone was using Boolean algebras and rings of continuous functions [12] to do this. Henry Wallman introduced compactification of spaces having a normal base which is also called Wallman compactification [13]. Recall the Fan-Gottesman compactification is defined by Ky Fan and Noel Gottesman.

In 1952, Let β be a class of open sets in X. If it satisfies the following three conditions, it is called a *normal* base.

1. β is closed under finite intersections

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2. If $B \in \beta$, then $X - cl_X B \in \beta$, where $cl_X B$ denotes closure of B in X.

3. For every open set U in X and every $B \in \beta$ such that $cl_X B \subset U$, there exists a set $D \in \beta$ such that $cl_X B \subset D \subset cl_X D \subset U$.

We consider a regular space having a normal base for open sets i.e., which satisfies the above three properties of normal base *A chain family on* β is a non-empty family of sets of β such that

$$cl_X B_1 \cap cl_X B_2 \cap \ldots \cap cl_X B_n \neq \emptyset$$

for any finite number of sets B_i of the family. Every chain family on β is contained in at least one maximal chain family on β by Zorn's lemma. Maximal chain families on β will be denoted by letters as $a^*, b^*, \ldots$, and also the set of all maximal chain families on β will be denoted by $(X,\beta)^*$. $(X,\beta)^*$ is a compact Hausdorff spaces and is a compactification of our regular space.Whose topology is defined as follow. For each $B \in \beta$, let

$$\tau(B) = \{b^* \in (X,\beta)^* : B \in b^*\}$$

Then the topology of $(X,\beta)^*$ is defined by taking

$$\beta^* = \{\tau(B) : B \in \beta\}$$

as a base of open sets. Afterwards this compactification is called Fan-Gottesman compactification [7].

Now,we defined this compactification via ultra-open filter in [5].

Definition 1.1. Let X be a T_3 space and FX the subcollection of all open ultrafilters on X. For each open set $O \subset X$, define $O^* \subset FX$ to be the set

$$O^* = \left\{\hat{G} \in FX : V \subset ck_X V \subset O,\ O \text{ is open in } X \text{ and } V, O \in \hat{G}\right\}$$

Let Φ is the $\{O^* : O \text{ is open subset of } X\}$ set. It is clear that Φ is the base for open sets of topology on FX. FX is a compact space and the Fan-Gottesman compactifications of X.

In order to avoid the confusion between FX and $(X,\beta)^*$, we will use X^* when it regarded as Fan-Gottesman compactification of X.

On the other hand, for each closed set $D \subset X$, we define $D^* \subset X^*$ by

$$D^* = \left\{\hat{G} \in X^* : G \subseteq D \text{ for some } G \text{ in } \hat{G}\right\}.$$

The following properties of X^* are useful

i: If $U \subset X$ is open, then $X^* - U^* = (X^* - U)^*$

ii: If $D \subset X$ is closed, then $X^* - D^* = (X^* - D)^*$

iii: If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$

Properties We consider the map $f_X : X \to X^*$ be defined by $f_X(x) = \hat{G}_x$, the open ultrafilter converging to x in X. Then the following properties hold.

1. If U is open in X, then $f_X\bar{(U)} = U^*$. In particular $f_X(X)$ is dense in X^*.

2. f_X is continuous and it is an embedding of X in X^* if and only if X is a T_3-space.

3. If U_1 and U_2 are open subsets of X, then $f_X(U_1 \bar{\cap} U_2) = f_X\bar{(U_1)} \cap f_X\bar{(U_2)}$.

4. X^* is a compact T_2-space .

Recal that $X^* - X$ is called as Fan-Gottesman remainder and we get following theorems in.

Theorem 1.1. Let X be a topological space. Suppose that $X^* - X$ contains an infinite subspace having the topology of finite complements. Then there exists a sequence $H_1, H_2, \dots$ of closed noncompact subsets of X which are pairwise disjoint.

Theorem 1.2. Let X be a topological space. Then the following statements are equivalent:

1. The Fan-Gotteman compactification of X is its weak reflection in compact spaces.
2. The space X has a weak reflection in compact spaces.
3. There exists such that any pairwise disjoint family of closed sets in X contains at most k noncompact elements.
4. Every infinite sequences $F_1, F_2, \dots$ of closed sets such that $F_p \cap F_q$ is compact for $p \neq q$ has a compact member
5. The Fan-Gottesman remainder of X is finite.

A topological space X is said to be scattered if every non-empty subset S of X contains at least one point which is isolated in S. Compact scattered spaces are very important tools for analysis and topology. In 1972, Mrowka and others characterized compact scattered space [11]. In1974, Kannan and Rajagopalan showed that a 0-dimensional, Lindolef, scattered first countable Hausdorff Space admits a scattered compactification [9]. In 1979, Moran characterize the class of scattered compact orders among the compact orders by means of properties of their continuous real valued functions [10]. In 2007, M.Henriksen and others established a new generalization of scattered spaces called SP-scattered [8]. In 2016, Al-Hajri and others characterized spaces such that their Aleksandrov (one-poin) compactification is a scattered space. They presented image classificaiton problem as an application of scattered space [2]. In 2018, Cioban and Budanaev present some geometrical and topological concepts to accommodate the needs of information theories. They established that Khalimsky topology on the discrete line is unique as the minimal symmetric digital topology on $\mathbb{Z}$ [4].

Definition 1.2. Let X be a topological space and $S \subseteq X$.S is called a scattered subset of X if S considered as a topological space is scattered. Also if clS is scattered, then S is scattered.

Proposition 1.1. Let X be a T_O-topological space and S be a subset of X. clS is scattered if and only if S and clS are scattered.

Proposition 1.2. Let X be a T_O-topological space and S_1, S_2 be two scattered subsets of X. The following statements hold:

1. The intersection of S_l and S_2 is a scattered subset of X.
2. The union of Sl and S_2 is a scattered subset of X.

Corollary 1.3. Let X be a topological space and $\{S_i : i \in I\}$ be a finite collection of scattered subsets of X. Then $\bigcup_{i \in I} S_i$ is a scattered subset of X.

The following remarks are very important for topologists.Because the results are scattered space definition and the construction of compactification.

Remark 1.4. Let $K(X)$ be a compactification of a topological space X. If $K(X)$ is a scattered then X is also a scattered space.

Remark 1.5. Let X be a scattered topological space and $S \subseteq X$. Then S is also a scattered space.

2 Main Results

Theorem 2.1. Let X^* be Fan-Gottesman compactification of X, T_3-space. Then X^* is a scattered if and only if X and $X^* - X$ are scattered.

Proof. ($\Rightarrow$) It is obviously that if X^* is a scattered then X and $X^* - X$ are scattered due to hereditary properties of scattered space.
($\Leftarrow$) Assume that X and $X^* - X$ are scattered and let S be a subset of X^*. We think about two cases:
Case 1: $S \subseteq X$ or $S \subseteq X^*$. So there exists an element x of S and an open set U of X such that $U \cap S = \{x\}$.
Case 2: $S \cap X \neq \emptyset$ and $S \cap X^* \neq \emptyset$. Since X is a scattered and $S \cap X \neq \emptyset$. There are an element z of $S \cap X$ and an open set U of X such that $U \cap (S \cap X) = \{x\}$. Assume that $(U \cap S) \cap (X^* - X) \neq 0$. Because $X^* - X$ is scattered there exists an element z of $(U \cap S) \cap (X^* - X)$ and an open set O of X such that $O \cap [(U \cap S) \cap (X^* - X)] = \{z\}$. Therefore either $S \cap O = \{z\}$ or $S \cap O = \{x, z\}$. It is clear that $S \cap O = \{z\}$, then z is an isolated point of S. Assume that $S \cap O = \{x, z\}$, since X^* is Hausdorff space and $z \in X^* - X$, there is an open set V of X^* such that $x \in V$ and $z \notin V$. Therefore $(V \cap U \cap O) \cap S = \{x\}$. Thus S is scattered. □

Theorem 2.2. Let X be a T_3-space such that its Fan-Gottesman compactification remainder is finite. Then the Fan-Gottesman compactifiction X^* is a scattered if and only if X is a scattered space.
Proof. We get
($\Rightarrow$) It is obviously that if the Fan-Gottesman compactification X^* is a scattered then X is a scattered space due to hereditary properties of scattered space.
($\Leftarrow$) Since $X^* - X$ is finite and X^* is Hausdorff space, $X^* - X$ is a discrete space. Then $X^* - X$ is a scattered. It is clear that X^* is a scattered space from Theorem 2.1. □

Theorem 2.3. Let X be a T_3-space such that its Fan-Gottesman compactification is a scattered space. If $\mathscr{F}$ is a collection of disjoint non-compact, closed sets of X, then there exists an open set U of X and $F \in \mathscr{F}$ such that

1. There exists a closed, non-compact set K of X such that $K \subseteq F \cap U$

2. For each non-compact, closed set $D \subseteq T$ with $T \in \mathscr{F} - \{F\}, D \nsubseteq U$.

Proof. Let $A = \{\xi \in X^* : \exists T \in \mathscr{F}, T \in \xi\}$. Since X^* is a scattered, there exists $\xi \in A$ such that ξ is an isolated point in A. Therefore there exists an open set U of X such that $U^* \cap S = \{\xi\}$ Thus $\xi \in U^*$, so that there exists $K \in \xi$ such that $K \subseteq U$. Since $\xi \in S$, let $F \in \mathscr{F}$ such that $F \in \xi$. Then $K \subseteq F \cap U$.

Let $T \in \mathscr{F} - \{F\}$ and D be a non-compact, closed set of X such that $D \subseteq T$. Then there exists $\mathscr{D} \in S$ such that $D \in \mathscr{D}$. It is clear that $\mathscr{D} \neq \xi$ for elements of T are disjoints; so that $\mathscr{D} \notin U^*$. Hence $D \nsubseteq U$. □

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