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A study on null cartan curve in Minkowski 3-space

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Abstract

Null cartan curves have been studied by some geometers in both Euclidean and Minkowski spaces, but some special characters of the curves are not considered. In this paper, we study weak $AW(k)-type$ and $AW(k)-type$ null cartan curve in Minkowski 3-space E_1^3. We define helix according to Bishop frame in E_1^3. Furthermore, the necessary and sufficient conditions for the helices in Minkowski 3-space are obtained.

Keywords: Null cartan curve, AW(k)-type curve, helix
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1 Introduction

Curves are one of the basic structures of differential geometry. It is safe to report that G. Monge initiated the many important results in Euclidean 3-space curve theory and G. Darboux pioneered the idea of a moving frame.

The curve theory has been one of the most studied subjects because of its many applications area from geometry to the various branch of science. Especially the characteristics of curvature and torsion play an important role in special curve types such as so-called helices. In Euclidean 3-space E^3, a general helix or a constant slope curve is defined in such a way that the tangent makes a constant angle with a fixed direction. A classical result stated by M. A Lancret in 1802 and first demonstrated by B. de Saint Venant in 1845 [5,6,11,12]. For nature's helical structures, helices arise in nano-springs, carbon nano-tubes, helices, DNA double and collagen triple helix, lipid bilayers, bacterial flagella in salmonella and escherichia coli, aerial hyphae in actinomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and shells of the sea [1,2,9]. In fractal geometry, helical structures are used.

In Minkowski 3-space, null Cartan curves are known as the curves whose and Cartan frame contains two

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null (lightlike) vector fields (see [7] for more information). Null curves of AW(k)-type are studied in the 3-dimensional Lorentzian space by M. Külahcı [8].

In [4] B. Bukcu and M. K. Karacan defined a slant helix according to Bishop frame of the timelike curve and they have given some necessary and sufficent conditions for the slant helix . Ahmad T. Ali and Rafael Lopez gave characterizations of slant helices in terms of the curvature and torsion and discussed the tangent and binormal indicatrices of slant curves in E_1^3 [3, 10, 13].

F. Gökçelik and I.Gök defined a new kind of slant helix called W-slant helix in 3-dimensional Minkowski space as a curve whose binormal lines make a constant angle with a fixed direction [14].

2 Preliminaries

Definition 1. *The Minkowski 3-space E_1^3 is the real vector space E^3 which is endowed with the standard indefinite flat metric $\langle .,. \rangle$ defined by*

$$\langle u,v\rangle = -u_1v_1 + u_2v_2 + u_3v_3, \tag{2.1}$$

for any two vectors $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ in E_1^3. Since $\langle .,. \rangle$ is an indefinite metric, an arbitrary vector $u \in E_1^3\backslash\{0\}$ can have one of three properties:

i) it can be space-like, if $\langle u,u\rangle_1 > 0$,

ii) time-like, if $\langle u,u\rangle_1 < 0$ or

iii) light-like or isotropic or null vector, if $\langle u,u\rangle_1 = 0$, but $u \neq 0$.

In particular, the norm (length) of a non lightlike vector $u \in E_1^3$ is given by

$$\|u\| = \sqrt{|\langle u,u\rangle|}.$$

Given a regular curve $\alpha : I \to E_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(t)$ satisfy $\left\langle \alpha'(t), \alpha'(t)\right\rangle_1 > 0$, $\left\langle \alpha'(t), \alpha'(t)\right\rangle_1 < 0$ or $\left\langle \alpha'(t), \alpha'(t)\right\rangle_1 = 0$, respectively, at any $t\varepsilon I$, where $\alpha'(t) = \frac{d\alpha}{dt}$.

Definition 2. *A curve $\alpha : I \to E_1^3$ is called a null curve, if its tangent vector $\alpha' = T$ is a null vector. A null curve $\alpha = \alpha(s)$ is called a **null Cartan curve**, if it is parameterized by the pseudo-arc function s defined by*

$$s(t) = \int_0^t \sqrt{\|\alpha''(u)\|}du. \tag{2.2}$$

There exists a unique Cartan frame $\{T,N,B\}$ along a non-geodesic null Cartan curve satisfying the Cartan equations

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -\tau & 0 & k \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

*where the curvature $k(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in pseudo-arc parameter s. If $\tau(s) = 0$, the null Cartan curve is called a **null Cartan cubic**. The Cartan's frame vectors satisfy the relations*

$$\begin{aligned} &\langle N,N\rangle = 1,\ \langle T,T\rangle = \langle B,B\rangle = 0, \\ &\langle T,B\rangle = -1,\ \langle T,N\rangle = \langle N,B\rangle = 0 \end{aligned} \tag{2.4}$$

and

$$T\times N = -T,\ N\times B = -B,\ B\times T = N. \tag{2.5}$$

Cartan frame $\{T,N,B\}$is positively oriented, if $det(T,N,B) = [T,N,B] = 1$.

The Frenet frame is created for the non-degenerated curves of three times continuously differentiable. But, at some points on the curve,curvature may vanish . In this case, we need an alternative frame in E^3. Bishop introduced a new frame called Bishop frame or parallel transport frame, which is well defined even if the curve has a vanishing second derivative [7] .

2.1 The Bishop Frame

The Bishop frame or relatively parallel adapted frame $\{T, N_1, N_2\}$ of a regular curve in Euclidean 3-space is introduced by R.L. Bishop. It contains the tangential vector field T and two normal vector fields N_1 and N_2, which can be obtained by rotating the Frenet vectors N and B in the normal plane $T^{\perp}$ of the curve, in such a way that they become relatively parallel. This means that their derivatives $N_1^{'}$ and $N_2^{'}$ with respect to the arc-length parameter s of the curve are **collinear** with the tangential vector field T [7].

Remark 1. *We can also define N_1 and N_2 to be relatively parallel, if the normal component $T_1^{\perp} = span\{N_1, N_2\}$ of their derivatives $N_1^{'}$ and $N_2^{'}$ is zero, which implies that the mentioned derivatives are collinear with T_1.*

2.2 The Bishop frame of a null Cartan curve in E_1^3

The Bishop frame $\{T_1, N_1, N_2\}$ of a non-geodesic null Cartan curve in E_1^3 is positively oriented pseudo-orthonormal frame consisting of the tangential vector field T_1, relatively parallel spacelike normal vector field N_1 and relatively parallel lightlike transversal vector field N_2.

Theorem 1. *Let α be a null Cartan curve in E_1^3 parameterized by pseudo-arc s with the curvature $k(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Cartan frame $\{T, N, B\}$ of α are related by*

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_2 & 1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.6}$$

and the Cartan equations of according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.7}$$

*where the first Bishop curvature $k_1(s) = 1$ and the second Bishop curvature satisfies **Riccati differential equation***

$$k_2^{'}(s) = -\frac{1}{2} k_2^2(s) - \tau(s),$$

which satisfies the conditions

$$\langle N_1, N_1 \rangle = 1, \ \langle T_1, T_1 \rangle = \langle N_2, N_2 \rangle = 0,$$
$$\langle T_1, N_2 \rangle = -1, \ \langle T_1, N_1 \rangle = \langle N_1, N_2 \rangle = 0 \ \ [7].$$

3 Curves of AW(k)- type

Proposition 1. *Let α be a Frenet curve of osculating order 3, by using the cartan equations of α according to the Bishop frame* (2.7)*, then we have*

$$\begin{aligned}
\alpha'(s) &= T_1(s),\\
\alpha''(s) &= T_1'(s) = k_2T_1 + k_1N_1,\\
\alpha'''(s) &= \left(k_2' + k_2^2\right)T_1 + \left(k_1' + k_1k_2\right)N_1 + k_1^2N_2,\\
\alpha''''(s) &= \left(k_2'' + 3k_2k_2' + k_2^3\right)T_1\\
&\quad + \left(k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2\right)N_1 + 3k_1k_1'N_2.
\end{aligned}$$

Notation 1. *Let us write*

$$M_1(s) = k_1N_1, \tag{3.1}$$

$$M_2(s) = \left(k_1' + k_1k_2\right)N_1 + k_1^2N_2, \tag{3.2}$$

$$M_3(s) = \left(k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2\right)N_1 + 3k_1k_1'N_2. \tag{3.3}$$

Corollary 1. *$\alpha'(s)$, $\alpha''(s)$, $\alpha'''(s)$ and $\alpha''''(s)$ are linearly dependent if and only if $M_1(s)$, $M_2(s)$ and $M_3(s)$ are linearly dependent.*

Definition 3. *Frenet curves of osculating order 3 are of*

i) type weak $AW(2)$ if they satisfy

$$M_3(s) = \langle M_3(s), M_2^\star(s)\rangle M_2^\star(s), \tag{3.4}$$

ii) type weak $AW(3)$ if they satisfy

$$M_3(s) = \langle M_3(s), M_1^\star(s)\rangle M_1^\star(s), \tag{3.5}$$

where

$$M_1^\star(s) = \frac{M_1(s)}{\|M_1(s)\|}, \tag{3.6}$$

$$M_2^\star(s) = \frac{M_2(s) - \langle M_2(s), M_1^\star(s)\rangle M_1^\star(s)}{\left\|M_2(s) - \left\langle M_2(s), M_1^\star(s)\right\rangle M_1^\star(s)\right\|}, \tag{3.7}$$

iii) $AW(1)$-type, if they satisfy

$$M_3(s) = 0, \tag{3.8}$$

iv) $AW(2)$-type, if they satisfy

$$\|M_2(s)\|^2 M_3(s) = \langle M_3(s), M_2(s)\rangle M_2(s), \tag{3.9}$$

v) $AW(3)$-type, if they satisfy

$$\|M_1(s)\|^2 M_3(s) = \langle M_3(s), M_1(s)\rangle M_1(s). \tag{3.10}$$

Proposition 2. *Let α be a Frenet curve of osculating order 3, then α is $AW(1)$-type if and only if*

i) k_1 is a constant function, and ii)

$$k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2 = 0. \tag{3.11}$$

Proof. Since α is a curve of type $AW(1)$, then α must satisfy (3.8)

$$M_3(s) = \left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right)N_1 + 3k_1k_1^{'}N_2,$$
$$0 = \left(k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2}\right)N_1 + 3k_1k_1^{'}N_2.$$

Since N_1 and N_2 are linearly independent, therefore

$$3k_1k_1^{'} = 0,$$

k_1 is a constant function, and

$$k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2} = 0.$$

Hence the proposition is proved. □

Proposition 3. *Let α be a Frenet curve of osculating order 3. Then, if α are of $AW(2)-$type, we have*

$$3\left(k_1^{'}\right)^2 + 3k_1^{'}k_1k_2 = k_1^{''}k_1 + 2k_1^2k_2^{'} + k_1^{'}k_1k_2 + k_1^2k_2^{2}. \tag{3.12}$$

Proof. Suppose that α is a Frenet curve of osculating order 3. From (3.2) and (3.3) we can write

$$M_2(s) = \beta(s)N_1 + \gamma(s)N_2,$$
$$M_3(s) = \delta(s)N_1 + \eta(s)N_2,$$

where $\beta(s)$, $\gamma(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_2(s)$ and $M_3(s)$ are linearly dependent , then the determinant of the coefficients of N_1 and N_2 is equal to zero, hence one can write

$$\begin{vmatrix} \beta(s) & \gamma(s) \\ \delta(s) & \eta(s) \end{vmatrix} = 0, \tag{3.13}$$

where

$$\beta(s) = k_1^{'} + k_1k_2, \ \ \gamma(s) = k_1^2,$$
$$\delta(s) = k_1^{''} + 2k_1k_2^{'} + k_1^{'}k_2 + k_1k_2^{2} \text{ and}$$
$$\eta(s) = 3k_1k_1^{'}. \tag{3.14}$$

By considering (3.14) in (3.13) we get

$$3\left(k_1^{'}\right)^2 + 3k_1^{'}k_1k_2 = k_1^{''}k_1 + 2k_1^2k_2^{'} + k_1^{'}k_1k_2 + k_1^2k_2^{2}.$$

Hence the proposition is proved. □

Proposition 4. *Let α be a Frenet curve of osculating order 3, then α is of type $AW(3)$ if and only if $k_1(s)$ is a constant function*

Proof. Suppose that α is a Frenet curve of order 3. From (3.1) and (3.3) we can write

$$M_1(s) = \beta(s)N_1 + \gamma(s)N_2,$$
$$M_3(s) = \delta(s)N_1 + \eta(s)N_2,$$

where $\beta(s)$, $\gamma(s)$, $\delta(s)$ and $\eta(s)$ are differential functions. Since $M_2(s)$ and $M_3(s)$ are linearly dependent, then the determinant of the coefficients of N_1 and N_2 is equal to zero, one can write

$$\begin{vmatrix} \beta(s) & \gamma(s) \\ \delta(s) & \eta(s) \end{vmatrix} = 0, \tag{3.15}$$

where

$$\begin{aligned} &\beta(s) = k_1, \;\; \gamma(s) = 0, \\ &\delta(s) = k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2 \;\text{ and} \\ &\eta(s) = 3k_1k_1'. \end{aligned} \tag{3.16}$$

By substituting (3.16) in (3.15) we get

$$k_1^2 k_1' = 0.$$

For $k_1^2k_1'$ to be zero, $k_1(s)$ has to be a constant function. Hence the proposition is proved. □

Proposition 5. *Let α be a Frenet curve of osculating order 3, then α is of weak $AW(2)-$type if and only if*

i) $k_1(s)$ is a constant function,

ii)

$$k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2 = 0 \;\; \textit{and} \;\; k_2(s) = \frac{2}{s-2c}, \tag{3.17}$$

where s and c are arc-length parameter and constant respectively.

Proof. Since α is of weak $AW(2)-$type , it must satisfy (3.4), by using (3.4), (3.6), (3.7) and (3.1)

$$M_1^\star(s) = \frac{M_1(s)}{\|M_1(s)\|} = \frac{k_1N_1}{\sqrt{k_1^2}} = N_1, \tag{3.18}$$

$$\begin{aligned} M_2^\star(s) &= \frac{M_2(s) - \langle M_2(s), M_1^\star(s)\rangle M_1^\star(s)}{\left\|M_2(s) - \left\langle M_2(s), M_1^\star(s)\right\rangle M_1^\star(s)\right\|}, \\ M_2^\star(s) &= \frac{\left(k_1' + k_1k_2\right)N_1 + k_1^2N_2 - \left(k_1' + k_1k_2\right)N_1}{\left\|\left(k_1' + k_1k_2\right)N_1 + k_1^2N_2 - \left(k_1' + k_1k_2\right)N_1\right\|}, \\ M_2^\star(s) &= N_2. \end{aligned} \tag{3.19}$$

Since α is of weak $AW(2)-$type, then it must satisfy

$$\begin{aligned} M_3(s) &= \langle M_3(s), M_2^\star(s)\rangle M_2^\star(s), \\ &= \left\langle \left(k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2\right)N_1 + 3k_1k_1'N_2, N_2\right\rangle N_2, \\ &= 0. \end{aligned}$$

Therefore

$$\left(k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2\right)N_1 + 3k_1k_1'N_2 = 0,$$

then

$$\begin{aligned} k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2 &= 0 \;\text{ and} \\ 3k_1k_1' &= 0. \end{aligned} \tag{3.20}$$

For the $k_1^2 k_1'$ to be zero, $k_1(s)$ has to be a constant function.
Then (3.20) turns to

$$2k_1 k_2' + k_1 k_2^2 = 0, \tag{3.21}$$

by integrating the above equation

$$\begin{aligned} 2k_2' + k_2^2 &= 0, \\ \frac{k_2'}{k_2^2} &= -\frac{1}{2}, \\ \int \frac{k_2'}{k_2^2} ds &= -\int \frac{1}{2} ds, \end{aligned} \tag{3.22}$$

let

$$k_2(s) = u, \tag{3.23}$$

$$k_2' ds = du, \tag{3.24}$$

therefore (3.22) turns to

$$\begin{aligned} \int \frac{du}{u^2} &= -\int \frac{1}{2} ds, \\ -\frac{1}{u} &= -\frac{1}{2} + c, \end{aligned}$$

by simplifying the above equation and using (3.23) we get

$$k_2(s) = \frac{2}{s - 2c}.$$

Hence the proposition is proved. □

Proposition 6. *Let α be a Frenet curve of osculating order 3, then α is of weak $AW(3)-$type if and only if $k_1(s)$ is a constant function.*

Proof. Since α is of weak $AW(3)-$type, by using (3.3) and (3.18)

$$\begin{aligned} M_3(s) &= \langle M_3(s), M_1^{\star}(s) \rangle M_1^{\star}(s), \\ &= \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2\right) N_1 \end{aligned}$$

Therefore

$$\begin{aligned} \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2\right) N_1 + 3k_1 k_1' N_2 &= \left(k_1'' + 2k_1 k_2' + k_1' k_2 + k_1 k_2^2\right) N_1 \\ k_1 k_1' &= 0. \end{aligned}$$

For the $k_1^2 k_1'$ to be zero, $k_1(s)$ has to be a constant function. Hence the proposition is proved. □

4 The helices according to Bishop frame of null cartan curve in Minkowski 3-space

Definition 4. *Helix can be defined as a curve whose tangent lines make a constant angle with a fixed direction. Helices are characterized by the fact that the ratio $\frac{k_1}{k_2}$ is constant along the curve.*

Theorem 2. *Let α be a null cartan curve in E_1^3, then α is a general helix if and only if $\frac{k_1}{k_2}$ is constant.*

Proof. Let α be a general helix in E_1^3 and $\langle T,U\rangle$ is constant, then α is a general helix, from the definition we have

$$\langle T,U\rangle = c \quad \text{c is constant,} \tag{4.1}$$

by differentiating the above equation

$$\begin{aligned}
\left\langle T',U\right\rangle + \left\langle T,U'\right\rangle &= 0,\\
\left\langle T',U\right\rangle &= 0,\\
k_2\cos\theta + k_1\sin\theta &= 0,\\
\frac{k_1}{k_2} &= -\cot\theta \quad \text{(constant)},
\end{aligned} \tag{4.2}$$

as disered. □

Theorem 3. *Suppose that α is a null cartan curve in E_1^3, then α is a general helix if and only if*

$$det(T_1', T_1'', T_1''') = k_1^2\left(k_1k_2'' - k_2k_1''\right). \tag{4.3}$$

Proof. ($\Longrightarrow$) Let $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned}
T_1'(s) &= k_2T_1 + k_1N_1,\\
T_1''(s) &= \left(k_2' + k_2^2\right)T_1 + \left(k_1' + k_1k_2\right)N_1 + k_1^2N_2,\\
T_1'''(s) &= \left(k_2'' + 3k_2k_2' + k_2^3\right)T_1\\
&\quad + \left(k_1'' + 2k_1k_2' + k_1'k_2 + k_1k_2^2\right)N_1 + 3k_1k_1'N_2.
\end{aligned}$$

So we get

$$det(T_1', T_1'', T_1''') = \begin{vmatrix} k_2 & k_1 & 0\\ \left(k_2' + k_2^2\right) & \left(k_1' + k_1k_2\right) & k_1^2\\ \begin{pmatrix} k_2'' + 3k_2k_2' \\ +k_2^3\end{pmatrix} & \begin{pmatrix} k_1'' + 2k_1k_2' \\ +k_1'k_2 + k_1k_2^2\end{pmatrix} & 3k_1k_1' \end{vmatrix},$$

$$det(T_1', T_1'', T_1''') = -k_1^2k_2^3\left(\frac{k_1}{k_2}\right)' + 3k_1k_1'k_2^2\left(\frac{k_1}{k_2}\right)' + k_1^2\left(k_1k_2'' - k_2k_1''\right).$$

Since α is a general helix, and $\frac{k_1}{k_2}$ is constant. Hence, we have

$$det(T_1', T_1'', T_1''') = k_1^2\left(k_1k_2'' - k_2k_1''\right), \text{ but } k_2 \neq 0.$$

($\Longleftarrow$) Suppose that $det(T_1', T_1'', T_1''') = k_1^2\left(k_1k_2'' - k_2k_1''\right)$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the theorem is proved. □

Theorem 4. *Let α be a null cartan curve in E_1^3 , then α is a general helix if and only if*

$$det(N_1', N_1^{''}, N_1^{'''}) = 0. \tag{4.4}$$

Proof. ($\Longrightarrow$) Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned} N_1' &= k_1 N_2, \\ N_1'' &= \left(k_1' - k_1 k_2\right) N_2, \\ N_1^{'''} &= \left(k_1'' - 2k_1' k_2 - k_1 k_2' + k_1 k_2^2\right) N_2. \end{aligned}$$

So we get

$$\begin{aligned} det(N_1', N_1^{''}, N_1^{'''}) &= \begin{vmatrix} 0 & 0 & k_1 \\ 0 & 0 & -k_1^2 \\ 0 & 0 & \left(k_1'' - 2k_1' k_2 - k_1 k_2' + k_1 k_2^2\right) \end{vmatrix}, \\ det(N_1', N_1^{''}, N_1^{'''}) &= 0. \end{aligned}$$

($\Longleftarrow$) Suppose that $det(N_1', N_1^{''}, N_1^{'''}) = 0$, then it is clear that the $\frac{k_1}{k_2}$ is constant, since $\left(\frac{k_1}{k_2}\right)'$ is zero. Hence the theorem is proved. ☐

Theorem 5. *Let α be a null cartan curve in E_1^3, then α is a general helix if and only if*

$$det(N_2', N_2^{''}, N_2^{'''}) = 0. \tag{4.5}$$

From (2.7)

$$\begin{aligned} \alpha(s) &= T, \\ D_T T &= k_2 T_1 + k_1 N_1, \\ D_T N_1 &= k_1 N_2, \\ D_T N_2 &= -k_2 N_2. \end{aligned} \tag{4.6}$$

Theorem 6. *Let $\alpha : I \longrightarrow E_1^3$ be a unit speed null cartan curve with the cartan frame apparatus $\{T, N_2, N_2, k_1, k_2\}$, then α is a general helix if and only if*

$$D_T(D_T D_T N_1) + k_1 k_2 D_T N_2 = \left(k_1'' - 3k_1' k_2\right) \frac{1}{k_1} D_T N_1. \tag{4.7}$$

Proof. ($\Longrightarrow$) Suppose that α is a general helix. Then, from (4.6), we have

$$D_T N_1 = k_1 N_2,$$

$$D_T(D_T N_1) = k_1' N_2 - k_1 k_2 N_2, \tag{4.8}$$

$$D_T(D_T D_T N_1) = k_1'' N_2 - \left(k_1' k_2 + k_1 k_2'\right) N_2 - k_1' k_2 N_2 - k_1 k_2 D_T N_2. \tag{4.9}$$

Since α is a general helix

$$\frac{k_1}{k_2} = c \quad c \text{ is constant}, \tag{4.10}$$

by differentiating (4.10) we get

$$(k_1 k_2)' = 2k_1' k_2, \tag{4.11}$$

but

$$D_T N_1 = k_1 N_2,$$
$$N_2 = \frac{1}{k_1} D_T N_1. \tag{4.12}$$

By substituting (4.11) and (4.12) in (4.9) we get

$$D_T \left(D_T D_T N_1\right) = \left(k_1^{''} - 3k_1^{'} k_2\right)\left(\frac{1}{k_1} D_T N_1\right) - k_1 k_2 D_T N_2, \tag{4.13}$$
$$D_T \left(D_T D_T N_1\right) + k_1 k_2 D_T N_2 = \left(k_1^{''} - 3k_1^{'} k_2\right)\left(\frac{1}{k_1} D_T N_1\right).$$

$(\Longleftarrow)$ We will show that null cartan curve α is a general helix. By differentianting (4.12) covariently

$$N_2 = \frac{1}{k_1} D_T N_1,$$
$$D_T N_2 = -\frac{k_1^{'}}{k_1^2} D_T N_1 + \frac{1}{k_1} D_T D_T N_1, \tag{4.14}$$
$$D_T D_T N_2 = \left(-\frac{k_1^{'}}{k_1^2}\right)^{'} D_T N_1 - \frac{2k_1^{'}}{k_1^2} D_T D_T N_1 + \frac{1}{k_1} D_T D_T D_T N_1. \tag{4.15}$$

By substituting (4.8) and (4.13) in (4.15) we get

$$\begin{aligned} D_T D_T N_2 = {} & \left[\left(-\frac{k_1^{'}}{k_1^2}\right)^{'} + \left(k_1^{''} - 3k_1^{'} k_2\right)\frac{1}{k_1^2}\right] D_T N_1 \\ & - \frac{2\left(k_1^{'}\right)^2}{k_1^2} N_2 - \left(\frac{2k_1^{'}}{k_1} + k_2\right) D_T N_2. \end{aligned} \tag{4.16}$$

From (4.6)

$$D_T N_2 = -k_2 N_2,$$
$$D_T \left(D_T N_2\right) = -k_2^{'} N_2 - k_2 D_T N_2. \tag{4.17}$$

By comparing (4.16) and (4.17)

$$-\left(\frac{2k_1^{'}}{k_1} + k_2\right) = -k_2,$$

by integrating the above equation we get

$$k_1 = 1,$$

to find k_2, by comparing (4.16) and (4.17) we have

$$-\frac{2\left(k_1^{'}\right)^2}{k_1^2} = -k_2^{'}.$$

But $k_1 = 1$, therefore

$$k_2^{'} = 0,$$

which means k_2 is a constant function.

$$\frac{k_1}{k_2} \text{ is constant.}$$

Hence α is a general helix. □

Theorem 7. *Let $\alpha : I \longrightarrow E_1^3$ be a unit speed null cartan curve with the cartan frame apparatus $\{T, N_2, N_2, k_1, k_2\}$, then α is a general helix if and only if*

$$D_T(D_T D_T N_2) = \left(k_2^{''} - 3k_2 k_2^{'} + k_2^3\right)\left(\frac{1}{k_2} D_T N_2\right). \tag{4.18}$$

The above theorem can be proven analogously, so we skip its proof.

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