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Fractional Calculus of the Extended Hypergeometric Function

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Abstract

Here, our aim is to demonstrate some formulae of generalization of the extended hypergeometric function by applying fractional derivative operators. Furthermore, by applying certain integral transforms on the resulting formulas and develop a new futher generalized form of the fractional kinetic equation involving the generalized Gauss hypergeometric function. Also, we obtain generating functions for generalization of extended hypergeometric function..

Keywords: Gamma function, beta function, hypergeometric functions, extended hypergeometric function, integral transforms, fractional calculus operators, generating functions.
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1 Introduction

The classical Pochhammer symbol $(\lambda)_\nu$ is given as follows: [1,4,14,23,26,34]

$$\begin{aligned}(\lambda)_\nu &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \qquad (\lambda,\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-)\\ &= \begin{cases}1 & (\nu=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N})\end{cases}\end{aligned} \tag{1}$$

and $\Gamma(\lambda)$ is the familiar Gamma function whose Euler's integral is (see, e.g., [1,4,14,23,26])

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt \quad (\Re(z)>0). \tag{2}$$

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From (1) and (2), it is easy to see the following integral formula

$$(\lambda)_\nu = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-t} t^{\lambda+\nu-1} dt \quad (\Re(\lambda+\nu) > 0). \tag{3}$$

Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ be the sets of complex numbers, non-positive integers and positive integers respectively, and assume that $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu)\} > 0$. Recently, various generalization of beta functions have been introduced and investigated (see, e.g., [7–10, 13, 16, 17, 21, 22, 27, 29, 37] and the references cited therein). In [37], Şahin et al. introduced and studied following generalization of the extended gamma function as follows:

$$\Gamma_{p,q}^{(\kappa,\mu)}(z) = \int_0^\infty t^{z-1} \exp\left(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}\right) dt, \tag{4}$$

$$(\Re(z) > 0, \Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0).$$

It is easily seen that the special cases of (4) returns to other forms of gamma functions. For example, $\Gamma_{1,0}^{(1,1)}(z) = \Gamma(z)$, $\Gamma_{1,q}^{(1,1)}(z) = \Gamma_q(z)$.

Using the above (4), Şahin et. al. [38] defined a new generalization of the extended Pochhammer symbol such as;

$$(\lambda; p,q;\kappa,\mu)_\nu := \begin{cases} \dfrac{\Gamma_{p,q}^{(\kappa,\mu)}(\lambda+\nu)}{\Gamma(\lambda)} & , \Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0 \\ (\lambda)_\nu & , p = 1\, q = 0\, \kappa = 1\, \mu = 0 \end{cases} \tag{5}$$

and, also they obtained integral representation of (5) as follows:

$$(\lambda; p,q;\kappa,\mu)_\nu := \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+\nu-1} \exp\left(-\frac{t^\kappa}{p} - \frac{q}{t^\mu}\right) dt \tag{6}$$

$$(\Re(p) > 0, \Re(q) > 0, \Re(\kappa) > 0, \Re(\mu) > 0).$$

Moreover, they gave the generalization of the extended Gauss hypergeometric function, confluent hypergeomtric function and Appell hypergeometric functions as follows [38]:

$$F_{p,q}^{\kappa,\mu}(a,b,c;z) := \sum_{n=0}^\infty \frac{(a;p,q;\kappa,\mu)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}, \tag{7}$$

$$\Phi_{p,q}^{\kappa,\mu}(a;b;z) := \sum_{n=0}^\infty \frac{(a;p,q;\kappa,\mu)_n}{(b)_n} \cdot \frac{z^n}{n!}, \tag{8}$$

$${}_{p,q}F_1^{(\kappa,\mu)}[a;b,c;d;x,y] = \sum_{m,n=0}^\infty \frac{(a;p,q;\kappa,\mu)_{m+n} (b)_m (c)_n}{(d)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \tag{9}$$

$$\max(|x|, |y|) < 1$$

and

$${}_{p,q}F_2^{(\kappa,\mu)}[a;b,c;d,e;x,y] = \sum_{m,n=0}^\infty \frac{(a;p,q;\kappa,\mu)_{m+n} (b)_m (c)_n}{(d)_m (e)_n} \frac{x^m}{m!} \frac{y^n}{n!} \tag{10}$$

$$|x| + |y| < 1,$$

respectively.

In our work, we have to recall the following fractional integral operators [13,18,19,36]. For $x > 0$, λ, ν, σ, $\in \mathbb{C}$ and $\Re(\lambda) > 0$, we have

$$(I_{0,x}^{\lambda,\sigma,\nu} f(t))(x) = \frac{x^{-\lambda-\sigma}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} {}_2F_1\left[\lambda+\sigma, -\nu; \lambda; 1-\frac{t}{x}\right] f(t)dt \tag{11}$$

and

$$(J_{x,\infty}^{\lambda,\sigma,\nu} f(t))(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} t^{-\lambda-\sigma} {}_2F_1\left[\lambda+\sigma, -\nu; \lambda; 1-\frac{t}{x}\right] f(t)dt \tag{12}$$

where the ${}_2F_1[.]$ is the Gauss hypergeometric function [1,4,14,23,26].

The Erdelyi-Kober type fractional integral operators are defined as follows [20]:

$$(E_{0,x}^{\lambda,\nu} f)(x) = \frac{x^{-\lambda-\nu}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t)dt \quad (\Re(\lambda) > 0) \tag{13}$$

and

$$(K_{x,\infty}^{\lambda,\nu} f)(x) = \frac{x^{\nu}}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} t^{-\lambda-\nu} f(t)dt \quad (\Re(\lambda) > 0). \tag{14}$$

The Riemann-Liouville fractional integral and the Weyl fractional integral operators defined as the follows [13,18,19,36]:

$$(R_{0,x}^{\lambda} f)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t)dt \tag{15}$$

and

$$(W_{x,\infty}^{\lambda} f)(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} f(t)dt. \tag{16}$$

In [36], the operator $I_{0,x}^{\lambda,\sigma,\nu}(.)$ contains both the Riemann-Liouville and Erdelyi-Kober fractional integral operators by means of the following relationships:

$$(R_{0,x}^{\lambda} f)(x) = (I_{0,x}^{\lambda,-\lambda,\nu} f)(x) \tag{17}$$

and

$$(E_{0,x}^{\lambda,\nu} f)(x) = (I_{0,x}^{\lambda,0,\nu} f)(x) \tag{18}$$

While the operator $I_{0,x}^{\lambda,\sigma,\nu}(.)$ unifies the Weyl and Erdelyi-Kober fractional integral operators as follows [36]:

$$(W_{x,\infty}^{\lambda} f)(x) = (J_{x,\infty}^{\lambda,-\lambda,\nu} f)(x) \tag{19}$$

and

$$(K_{x,\infty}^{\lambda,\nu} f)(x) = (J_{x,\infty}^{\lambda,0,\nu} f)(x). \tag{20}$$

The following equations obtained by Kilbas [18] are also required for our work.

Lemma 1. *Let λ, σ, $\nu\,\rho \in \mathbb{C}$. Then, we have the following relations*

$$(I_{0,x}^{\lambda,\sigma,\nu} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(\sigma+\nu-\rho)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)} x^{\rho-\sigma-1} \tag{21}$$

and

$$(J_{x,\infty}^{\lambda,\sigma,\nu} t^{\rho-1})(x) = \frac{\Gamma(\sigma-\rho+1)\Gamma(\nu-\sigma+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\nu-\rho+1)} x^{\rho-\sigma-1}. \tag{22}$$

Also, taking $\sigma = -\lambda$ in equations (21) and (22), we have the following relations [18].

Lemma 2. *Let $\lambda, \rho \in \mathbb{C}$. Then, we have the following relations*

$$(R_{0,x}^{\lambda} t^{\rho-1})(x) = \frac{\Gamma(\lambda)}{\Gamma(\rho+\lambda)} x^{\rho+\lambda-1} \tag{23}$$

and

$$(W_{x,\infty}^{\lambda} t^{\rho-1})(x) = \frac{\Gamma(1-\lambda-\rho)}{\Gamma(1-\rho+)} x^{\rho+\lambda-1}. \tag{24}$$

If we choose $\sigma = 0$ in the equation (21) and (22), we have the following relations.

Lemma 3. *Let $\lambda, \rho \in \mathbb{C}$. Then, we have the following relations*

$$(E_{0,x}^{\lambda;\nu} t^{\rho-1})(x) = \frac{\Gamma(\rho+\nu)}{\Gamma(\rho+\lambda+\nu)} x^{\rho-1} \tag{25}$$

and

$$(K_{x,\infty}^{\lambda;\nu} t^{\rho-1})(x) = \frac{\Gamma(1+\nu-\rho)}{\Gamma(1+\lambda+\nu-\rho)} x^{\rho-1}. \tag{26}$$

From its birth to its today's wide use in a great number of scientific fields fractional calculus has come a long way. Despite the fact that its nearly as old as classical calculus itself, it flourished mainly over the last decades because of its good applicability on models describing complex real life problems (see. [5, 6, 28, 41]).

Here, by choosing a known generalization of the extended Gauss hypergeometric function in (7) we aim to establish certain formulas and representations for this extended Gauss hypergeometric function such as fractional derivative operators, integral transforms, fractional kinetic equations and generating functions. Also, we give some generating functions for extended Appell hypergemetric functions (9) and (10).

2 Fractional Calculus of (7)

In this section, we will present some fractional integral formulas for the generalization of the extended Gauss hypergeometric function $F_{p,q}^{\kappa,\mu}(a,b,c;z)$ (7) by using several general pair of fractional calculus operators.

We begin by recalling a known concept of Hadamard products [18, 19]

Definition 1. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ and $g(z) := \sum_{n=0}^{\infty} b_n z^n$ be two power series whose radii of convergence are given by R_f andR_g, respectively. Then their Hadamard product is power series defined by

$$(f * g) := \sum_{n=0}^{\infty} a_n b_n z^n \tag{27}$$

whose radius of convergence R satifies $R_f.R_g \leq R$.

Especially, if one of the power series defines an entire function and the radius of convergence of the grater than zero, then the Hadamard product to seperate a newly-emerged function into two known functions. For example,

$$\begin{aligned} &{}_rF_{s+m}\left[\begin{matrix}(a_1;p,q;\kappa,\mu),a_2\cdots a_r & ; \\ b_1,b_2,\cdots,b_{s+m} & ;z\end{matrix}\right] \\ &:= {}_0F_m\left[\begin{matrix}--- & ; \\ b_1,b_2,\cdots,b_m & ;z\end{matrix}\right] * {}_rF_s\left[\begin{matrix}(a_1;p,q;\kappa,\mu),a_2\cdots a_r & ; \\ b_{1+m},b_{2+m},\cdots,b_{s+m} & ;z\end{matrix}\right]. \end{aligned} \tag{28}$$

$$(|z| < \infty)$$

The main results are obtained in the following theorems.

Theorem 4. *Let* $\lambda, \sigma, \nu, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0, \Re(\rho) > \max[0, \Re(\sigma - \nu)]; \min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}; \Re(c) > \Re(b) > 0$*, then*

$$\begin{aligned}[I_{0,x}^{\lambda,\sigma,\nu} t^{\rho-1} F_{p,q}^{\kappa,\mu}(a,b,c;t)](x) = x^{\rho-\sigma-1} \frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)} \\ F_{p,q}^{\kappa,\mu}(a,b,c;x) * {}_2F_2[\rho, \rho+\nu-\sigma; \rho-\sigma, \lambda+\nu+\rho; x].\end{aligned} \tag{29}$$

Proof. Let's denote the left-hand side of the equation (29) by L. Using the definition of the generalized hypergeometric function (7) and arranging order of integration and summation, which is applicable under the conditions *Theorem 1*, we get

$$L = \sum_{n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_n (b)_n}{(c)_n} \cdot \frac{1}{n!} \left[I_{0,x}^{\lambda,\sigma,\nu} t^{\rho+n-1}\right](x), \tag{30}$$

taking advantage of the (21) in the above equality (30), we have

$$L = x^{\rho-\sigma-1} \sum_{n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_n (b)_n}{(c)_n} \frac{\Gamma(\rho+n)\Gamma(\rho+\nu-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\nu+n)} \frac{x^n}{n!}, \tag{31}$$

after simplfying the equation (31), we obtain

$$L = x^{\rho-\sigma-1} \frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)} \sum_{n=0}^{\infty} \frac{(a;p,q;\kappa,\mu)_n (b)_n}{(c)_n} \frac{(\rho)_n (\rho+\nu-\sigma)_n}{(\rho-\sigma)_n (\rho+\lambda+\nu)_n} \frac{x^n}{n!}, \tag{32}$$

further comment in the view of (7), we obtain

$$L = x^{\rho-\sigma-1} \frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)} {}_2F_{p,q,2}^{\kappa,\mu}(a,b,\rho,\rho+\nu-\sigma; c, \rho-\sigma, \rho+\lambda+\nu; x). \tag{33}$$

Finally, we have the desired result (29) in consinderation of the equation (28).

Theorem 5. *Let* $\lambda, \sigma, \nu, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0, \Re(\rho) > \max[0, \Re(\sigma - \nu)]; \min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}; \Re(c) > \Re(b) > 0$*, then*

$$\begin{aligned}[J_{x,\infty}^{\lambda,\sigma,\nu} t^{\rho-1} F_{p,q}^{\kappa,\mu}(a,b,c;\frac{1}{t})](x) = x^{\rho-\sigma-1} \\ \times \frac{\Gamma(\sigma-\rho+1)\Gamma(\nu-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\nu-\rho+1)} F_{p,q}^{\kappa,\mu}(a,b,c;\frac{1}{x}) \\ * {}_2F_2\left[\sigma-\rho+1, \nu-\rho+1; 1-\rho, \lambda+\sigma+\nu-\rho+1; \frac{1}{x}\right]\end{aligned} \tag{34}$$

Proof. We can obtain the proof of (34) given above similar to *Theorem 1*.

Applying $\sigma = 0$ in the equations (29) and (34) yields some results asserted by the following corollaries.

Corollary 6. *Let* $\lambda, \nu, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0, \Re(\rho) > \max[0, \Re(\sigma - \nu)]; \min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}; \Re(c) > \Re(b) > 0$*, then*

$$\begin{aligned}[E_{0,x}^{\lambda,\nu} t^{\rho-1} F_{p,q}^{\kappa,\mu}(a,b,c;t)](x) = x^{\rho-1} \frac{\Gamma(\nu+\rho)}{\Gamma(\lambda+\nu+\rho)} \\ F_{p,q}^{\kappa,\mu}(a,b,c;x) * {}_1F_1[\nu+\rho; \lambda+\nu+\rho; x].\end{aligned} \tag{35}$$

Corollary 7. *Let* $\lambda, \sigma, \nu, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0, \Re(\rho) > \max[0, \Re(\sigma - \nu)]; \min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}; \Re(c) > \Re(b) > 0$*, then*

$$[K_{x,\infty}^{\lambda,\nu} t^{\rho-1} F_{p,q}^{\kappa,\mu}(a,b,c;\frac{1}{t})](x) = x^{\rho-1} \frac{\Gamma(\nu-\rho+1)}{\Gamma(\lambda+\nu-\rho+1)} F_{p,q}^{\kappa,\mu}(a,b,c;\frac{1}{x}) * {}_1F_1[\nu-\rho+1; \lambda+\nu-\rho+1; \frac{1}{x}]. \tag{36}$$

Also, replacing $\sigma = -\lambda$ in the equations (29) and (34), we get the following corollaries.

Corollary 8. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0, \Re(\rho) > \max[0, \Re(\sigma - \nu)]; \min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}; \Re(c) > \Re(b) > 0,$ *then*

$$[R^{\lambda}_{0,x} t^{\rho-1} F^{\kappa,\mu}_{p,q}(a,b,c;t)](x) = x^{\rho+\lambda-1} \frac{\Gamma(\rho)}{\Gamma(\rho+\lambda)} F^{\kappa,\mu}_{p,q}(a,b,c;x) * {}_1F_1[\rho;\rho+\lambda;x] \tag{37}$$

Corollary 9. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0, \Re(\rho) > \max[0, \Re(\sigma - \nu)]; \min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}; \Re(c) > \Re(b) > 0,$ *then*

$$[W^{\lambda}_{x,\infty} t^{\rho-1} F^{\kappa,\mu}_{p,q}(a,b,c;\frac{1}{t})](x) = x^{\rho+\lambda-1} \frac{\Gamma(1-\lambda-\rho)}{\Gamma(1-\rho)} F^{\kappa,\mu}_{p,q}(a,b,c;\frac{1}{x}) * {}_1F_1[1-\lambda-\rho;1-\rho;\frac{1}{x}] \tag{38}$$

3 Integral Transforms of (7)

In this section, we prensent some integral transforms for example, P_δ transform, Laplace transform, Sumudu transform, Hankel transform and Laguerre transform for the generalization of the extended hypergeometric function (7).

3.1 P$_\delta$ and Related Integral Transforms

The P_δ transform of $f(t)$ is defined as [14, 24]

$$P_\delta\{f(t);s\} = F_P(s) = \int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}} f(t)\,dt \qquad (\delta > 1), \tag{39}$$

on condition that the convenient existence condition given by *Lemma* 4 below are satisfied.

Lemma 10. *Let the function f(t) be integrable over any finite interval* (a,b) $(0 < a < t < b)$. *Suppose also that there exists a real number c such that each of the following assertions holds true:*
(i) For any arbitrary $b > 0$, $\int_b^\iota e^{-ct} f(t)dt$ *tends to a finite limit as* $\iota \to \infty$;
(ii) For any arbitrary $a > 0$, $\int_\varepsilon^a |f(t)|dt$ *tends to a finite limit as* $\varepsilon \to 0_+$.
Then the P_δ*-transform exists whenever*

$$\Re\left(\frac{ln[1+(\delta-1)s]}{\delta-1}\right) > c \qquad (s \in \mathbb{C}).$$

Theorem 11. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0, \Re(\rho) > \max[0, \Re(\sigma - \nu)]; \min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}; \Re(c) > \Re(b) > 0,$ *then*

$$\begin{aligned} P_\delta\{z^{\upsilon-1}[I^{\lambda,\sigma,\nu}_{0,x} t^{\rho-1} F^{\kappa,\mu}_{p,q}(a,b,c;tz)(x)];s\} &= \frac{x^{\rho-\sigma-1}}{[\Lambda(\delta;s)]^{\upsilon}} \\ &\times \frac{\Gamma(\upsilon)\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)} F^{\kappa,\mu}_{p,q}(a,b,c;\frac{x}{[\Lambda(\delta;s)]}) \\ &* {}_3F_2[\upsilon,\rho,\rho+\nu-\sigma;\rho-\sigma,\lambda+\nu+\rho;\frac{x}{[\Lambda(\delta;s)]}] \end{aligned} \tag{40}$$

where $[\Lambda(\delta;s)]^{\upsilon} = \frac{ln[1+(\delta-1)s]}{\delta-1}$ *[24].*

Proof. Let's denote the left-hand side given in equation (40) by $\mathfrak{P}$ and using the definition of the P_δ-transform (39), we get;

$$\mathfrak{P} = \int_0^\infty z^{\upsilon-1}[1+(\delta-1)s]^{-\frac{z}{\delta-1}}\left(I_{0,x}^{\lambda,\sigma,\nu}\, t^{\rho-1}\, F_{p,q}^{\kappa,\mu}(a,b,c;tz)(x)\right)dz, \tag{41}$$

taking advantage of the equation (29) and arranging order of integration and summation, which is applicable under conditions *Theorem 3*, we have

$$\begin{aligned}\mathfrak{P} = x^{\rho-\sigma-1}\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}\\ \times\frac{\Gamma(\rho+n)\Gamma(\rho+\nu-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\nu+n)}\frac{\Gamma(\upsilon+n)}{[\Lambda(\delta;s)]^{\upsilon+n}},\end{aligned} \tag{42}$$

after simplfying the equation (42), we obtain

$$\begin{aligned}\mathfrak{P} = \frac{x^{\rho-\sigma-1}}{[\Lambda(\delta;s)]^{\upsilon}}\frac{\Gamma(\upsilon)\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)}\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}\\ \times\frac{(\rho)_n\,(\rho+\nu-\sigma)_n}{(\rho-\sigma)_n\,(\rho+\lambda+\nu)_n}\frac{\upsilon)_n}{[\Lambda(\delta;s)]^n},\end{aligned} \tag{43}$$

further comment in the view of (7), we have

$$\begin{aligned}\mathfrak{P} = \frac{x^{\rho-\sigma-1}}{[\Lambda(\delta;s)]^{\upsilon}}\frac{\Gamma(\upsilon),\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)}\\ \times\, {}_3F_{p,q,2}^{\kappa,\mu}(a,b,\upsilon,\rho,\rho+\nu-\sigma;c,\rho-\sigma,\rho+\lambda+\nu;\frac{x}{[\Lambda(\delta;s)]}).\end{aligned} \tag{44}$$

Finally, we get the required result (40) in consinderation of the equation (28).

Theorem 12. *Let* $\lambda,\rho\in\mathbb{C}$ *be such that* $\Re(\lambda)>0, \Re(\rho)>\max[0,\Re(\sigma-\nu)]$; $\min\{\Re(p),\Re(q),\Re(\kappa),\Re(\mu)>0\};\Re(c)>\Re(b)>0$, *then*

$$\begin{aligned}P_\delta\,\{z^{\upsilon-1}[J_{x,\infty}^{\lambda,\sigma,\nu}\,t^{\rho-1}\,F_{p,q}^{\kappa,\mu}(a,b,c;\frac{z}{t})](x):s\} = x^{\rho-\sigma-1}\frac{x^{\rho-\sigma-1}}{[\Lambda(\delta;s)]^{\upsilon}}\\ \times\frac{\Gamma(\upsilon)\Gamma(\sigma-\rho+1)\Gamma(\nu-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\nu-\rho+1)}F_{p,q}^{\kappa,\mu}(a,b,c;\frac{1}{x[\Lambda(\delta;s)]})\\ *\,{}_3F_2[\upsilon\sigma-\rho+1,\nu-\rho+1;\rho;1-\rho,\lambda+\sigma+\nu-\rho+1;\frac{1}{x[\Lambda(\delta;s)]}]\end{aligned} \tag{45}$$

Proof. The proof of the *Theorem 4* is parallel to the proof of *Theorem 3*

Upon letting $\delta\to 1+$ in the equation (39) is immediately reduced to the classic Laplace transform.

The Laplace transform of $f(z)$ is defined as [14, 26, 35]:

$$\mathfrak{L}\{f(z)\} = \int_0^\infty e^{-sz}f(z)dz. \tag{46}$$

The folowing theorem is a limit case of *Theorem 3* and *Theorem 4* when $\delta\to 1+$

Theorem 13. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma - \nu)]$; $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$, *then*

$$\begin{aligned}
&\mathfrak{L}\{z^{l-1}\,[I_{0,x}^{\lambda,\sigma,\nu}\,t^{\rho-1}\,F_{p,q}^{\kappa,\mu}(a,b,c;zt)](x)\}(s)\\
&= \frac{x^{\rho-\sigma-1}}{s^{l}}\,\frac{\Gamma(l)\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)}\\
&\times F_{p,q}^{\kappa,\mu}(a,b,c;\frac{x}{s}) * {}_3F_2[\rho,\rho+\nu-\sigma,l;\rho-\sigma,\lambda+\nu+\rho;\frac{x}{s}].
\end{aligned} \tag{47}$$

Proof. Let's denote the left-hand side of the equation (47) by $\mathbf{L}$. Using the definition of the Laplace transform in the above equation, we have

$$\mathbf{L} = \int_0^\infty e^{-sz}\,z^{l-1}\left(I_{0,x}^{\lambda,\sigma,\nu}\,t^{\rho-1}\,F_{p,q}^{\kappa,\mu}(a,b,c;tz)(x)\right)dz, \tag{48}$$

taking advantage of the equation (29) and arranging order of integration and summation, which is applicable under conditions *Theorem 5*, we have

$$\begin{aligned}
\mathbf{L} = x^{\rho-\sigma-1}\sum_{n=0}^{\infty}&\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}\\
&\times\frac{\Gamma(\rho+n)\Gamma(\rho+\nu-\sigma+n)}{\Gamma(\rho-\sigma+n)\Gamma(\rho+\lambda+\nu+n)}\,\frac{\Gamma(l+n)}{s^{l+n}},
\end{aligned} \tag{49}$$

after simplfying the equation (49), we obtain

$$\begin{aligned}
\mathbf{L} = x^{\rho-\sigma-1}\,\frac{\Gamma(l)}{s^{l}}\,&\frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)}\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}\\
&\times\frac{(\rho)_n\,(\rho+\nu-\sigma)_n}{(\rho-\sigma)_n\,(\rho+\lambda+\nu)_n}\,\frac{(l)_n}{s^n},
\end{aligned} \tag{50}$$

further comment in the view of (7), we have

$$\begin{aligned}
\mathbf{L} = x^{\rho-\sigma-1}\,\frac{\Gamma(l)}{s^{l}}\,&\frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)}\\
&\times {}_3F_{p,q,2}^{\kappa,\mu}(a,b\rho,\rho+\nu-\sigma,l;c,\rho-\sigma,\rho+\lambda+\nu;\frac{x}{s}).
\end{aligned} \tag{51}$$

Finally, we get the required result (47) in consinderation of the equation (28).

Theorem 14. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma - \nu)]$; $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$, *then*

$$\begin{aligned}
&\mathfrak{L}\{z^{l-1}\,[J_{x,\infty}^{\lambda,\sigma,\nu}\,t^{\rho-1}\,F_{p,q}^{\kappa,\mu}(a,b,c;\frac{z}{t})](x)\}(s)\\
&= \frac{x^{\rho-\sigma-1}}{s^{l}}\,\frac{\Gamma(l)\Gamma(\sigma-\rho+1)\Gamma(\nu-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\nu-\rho+1)}\,F_{p,q}^{\kappa,\mu}(a,b,c;\frac{1}{sx})\\
&* {}_3F_2[\sigma-\rho+1,\nu-\rho+1,l;1-\rho,\lambda+\sigma+\nu-\rho+1;\frac{1}{sx}]
\end{aligned} \tag{52}$$

Proof. The proof of the *Theorem 6* is similar to the proof of *Theorem 5*.

Now, taking $s = 1$ in the equation (46) is related to the Sumudu transform.
The Sumudu transform of $f(z)$ is given as follows [39]:

$$\mathfrak{S}\{f(z)\} = \int_0^\infty e^z f(z)\,dz. \tag{53}$$

The following corollaries is the special case of *Theorem 5* and *Theorem 6* when $s = 1$

Corollary 15. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma - \nu)]$; $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$, *then*

$$\begin{aligned}
&\mathfrak{S}\{z^{l-1}\,[I_{0,x}^{\lambda,\sigma,\nu}\,t^{\rho-1}\,F_{p,q}^{\kappa,\mu}(a,b,c;zt)](x)\}\\
&= x^{\rho-\sigma-1}\,\frac{\Gamma(l)\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)}\\
&\times F_{p,q}^{\kappa,\mu}(a,b,c;x) * {}_3F_2[\rho,\rho+\nu-\sigma,l;\rho-\sigma,\lambda+\nu+\rho;x].
\end{aligned} \tag{54}$$

Corollary 16. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma - \nu)]$; $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$, *then*

$$\begin{aligned}
&\mathfrak{S}\{z^{l-1}\,[J_{x,\infty}^{\lambda,\sigma,\nu}\,t^{\rho-1}\,F_{p,q}^{\kappa,\mu}(a,b,c;\frac{z}{t})](x)\}\\
&= x^{\rho-\sigma-1}\,\frac{\Gamma(l)\Gamma(\sigma-\rho+1)\Gamma(\nu-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\nu-\rho+1)}\,F_{p,q}^{\kappa,\mu}(a,b,c;\frac{1}{x})\\
&* {}_3F_2\left[\sigma-\rho+1,\nu-\rho+1,l;1-\rho,\lambda+\sigma+\nu-\rho+1;\frac{1}{x}\right]
\end{aligned} \tag{55}$$

3.2 Hankel transform

The Hankel transform of $f(z)$ is given as follows [14, 24]:

$$H_\alpha\{f(z)\}(u) = \int_0^\infty z J_\alpha(uz)\,f(z)\,dz, \tag{56}$$

where $J_\alpha(z)$ is the first kind of Bessel function [14, 24, 35, 40].

Theorem 17. *Let* $\lambda, \rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma - \nu)]$; $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$; $\Re(\zeta \pm \omega) > -\frac{1}{2}$, *then*

$$\begin{aligned}
&H_\alpha\{z^{\beta-1}\,[I_{0,x}^{\lambda,\sigma,\nu}\,t^{\rho-1}\,F_{p,q}^{\kappa,\mu}(a,b;c;tz^2)](x)\}(u)\\
&= \frac{1}{2}\,(\frac{2}{u})^\beta\,\frac{\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(1+\frac{\alpha-\beta}{2})}\,\frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)}\\
&\times F_{p,q}^{\kappa,\mu}(a,b;c;-\frac{4x}{u^2}) * {}_3F_3[\rho,\rho+\nu-\sigma,\frac{\alpha+\beta}{2};\\
&\qquad \rho-\sigma,\lambda+\nu+\rho,1+\frac{\alpha-\beta}{2};-\frac{4x}{u^2}]
\end{aligned} \tag{57}$$

Proof. Let's denote the left-hand side of the equation (57) by $\mathbf{H}$. Using the definition of the Whittaker transform in the above equation, we have

$$\begin{aligned}
\mathbf{H} = x^{\rho-\sigma-1}\,&\frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)}\sum_{n=0}^\infty\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\\
&\times\frac{(\rho)_n\,(\rho+\nu-\sigma)_n}{(\rho-\sigma)_n\,(\rho+\lambda+\nu)_n}\,\frac{x^n}{n!}\int_0^\infty z^{\beta+2n-1}\,J_\nu(\alpha z)dz,
\end{aligned} \tag{58}$$

Now, applying well-knowns formula for power function including Bessel function [14, 24],

$$\int_0^\infty z^{\beta-1} J_\alpha(uz)dz = 2^{\beta-1} u^{-\beta} \frac{\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(1+\frac{\alpha-\beta}{2})}, \tag{59}$$

after simplfying equation (59) and using the definition of (7), we obtain

$$\begin{aligned}\mathbf{H} = &\frac{x^{\rho-\sigma-1}}{2} (\frac{2}{u})^\beta \frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)} \frac{\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(1+\frac{\alpha-\beta}{2})} \\ &\times {}_3F^{\kappa,\mu}_{p,q,3}(a, b\rho, \rho+\nu-\sigma, \frac{\alpha+\beta}{2}; c, \rho-\sigma, \rho+\lambda+\nu, 1+\frac{\alpha-\beta}{2}; \frac{4x}{u^2}).\end{aligned} \tag{60}$$

Finally, we get the required result (57) in consinderation of the equation (28).

Theorem 18. *Let $\lambda, \rho \in \mathbb{C}$ be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma-\nu)]$; $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$; $\Re(\zeta \pm \omega) > -\frac{1}{2}$, *then*

$$\begin{aligned}&H_\alpha\{z^{\beta-1}[J^{\lambda,\sigma,\nu}_{x,0} t^{\rho-1} F^{\kappa,\mu}_{p,q}(a,b;c;\frac{z^2}{t})](x)\}(u) \\ &= \frac{1}{2}\left(\frac{2}{u}\right)^\beta \frac{\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(1+\frac{\alpha-\beta}{2})} \frac{\Gamma(\sigma-\rho+1)\Gamma(\nu-\rho+1)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\sigma)} \\ &\times F^{\kappa,\mu}_{p,q}(a,b;c;-\frac{4}{xu^2}) * {}_3F_3[\sigma-\rho+1, \nu-\rho+1, \frac{\alpha+\beta}{2}; \\ &\qquad \rho-\sigma, \lambda+\nu+\sigma+\frac{\alpha-\beta}{2}; -\frac{4}{xu^2}]\end{aligned} \tag{61}$$

Proof. The proof of the *Theorem 8* is similar to the proof of *Theorem 7*.

3.3 Laguerre Transform

The Laguerre transform of $f(z)$ is given as follows [14, 24]:

$$\mathscr{L}^{(\alpha)}\{f(z); m\} = \int_0^\infty e^{-z} z^\alpha L^\alpha_m(z) f(z)\, dz, \tag{62}$$

where $L^\alpha_m(z)$ is the Laguerre polynomial [14, 24, 35].

Theorem 19. *Let $\lambda, \rho \in \mathbb{C}$ be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma-\nu)]$; $\min\{\Re(p), \Re(q), \Re(\kappa), \Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$; $\Re(\zeta \pm \omega) > -\frac{1}{2}$, *then*

$$\begin{aligned}&\mathscr{L}^{(\alpha)}\{z^{\beta-1}[I^{\lambda,\sigma,\nu}_{0,x} t^{\rho-1} F^{\kappa,\mu}_{p,q}(a,b;c;tz)](x)\} \\ &= \frac{\Gamma(\alpha+\beta)\Gamma(2-\beta)}{\Gamma(1-\beta)} \frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)} \\ &\times F^{\kappa,\mu}_{p,q}(a,b;c;x) * {}_3F_2[\alpha+\beta, \rho, \rho+\nu-\sigma; \\ &\qquad \rho-\sigma, \lambda+\nu+\rho; x]\end{aligned} \tag{63}$$

Proof. Let's denote the left-hand side of the equation (63) by $\mathbf{M}$. Using the definition of the Laguerre transform in the above equation, we have

$$\begin{aligned}\mathbf{M} = x^{\rho-\sigma-1} &\frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\nu)} \sum_{n=0}^\infty \frac{(a;p,q;\kappa,\mu)_n (b)_n}{(c)_n} \\ &\times \frac{(\rho)_n(\rho+\nu-\sigma)_n}{(\rho-\sigma)_n(\rho+\lambda+\nu)_n} \frac{x^n}{n!} \int_0^\infty e^z z^{\alpha+\beta+n-1} L^\alpha_m(z) dz,\end{aligned} \tag{64}$$

Now, applying well-knowns integral formula for power function including Laguerre polynomial [14, 24],

$$\int_0^\infty e^z z^{\beta-1} z^{\alpha+\beta+n-1} L_m^\alpha(z)dz = \frac{\Gamma(\alpha+\beta+n)\Gamma(m-\beta+1)}{m!\Gamma(1-\beta)}, \tag{65}$$

after simplfying equation (65) and using the definition of (7), we obtain

$$\begin{aligned}\mathbf{M} = &\frac{\Gamma(\alpha+\beta)\Gamma(2-\beta)}{\Gamma(1-\beta)} \frac{\Gamma(\rho)\Gamma(\rho+\nu-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\rho)} \\ &\times {}_3F_{p,q,2}^{\kappa,\mu}(a,b,\alpha+\beta,\rho,\rho+\nu-\sigma;c,\rho-\sigma,\rho+\lambda+\nu;x).\end{aligned} \tag{66}$$

Finally, we get the required result (63) in consinderation of the equation (28).

Theorem 20. *Let* $\lambda,\rho \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\rho) > \max[0, \Re(\sigma-\nu)]$; $\min\{\Re(p),\Re(q),\Re(\kappa),\Re(\mu) > 0\}$; $\Re(c) > \Re(b) > 0$; $\Re(\zeta \pm \omega) > -\frac{1}{2}$, *then*

$$\begin{aligned}&\mathscr{L}^{(\alpha)}\{z^{\beta-1}[J_{x,0}^{\lambda,\sigma,\nu} t^{\rho-1} F_{p,q}^{\kappa,\mu}(a,b;c;\frac{z}{t})](x)\} \\ &= \frac{\Gamma(\alpha+\beta)\Gamma(2-\beta)}{\Gamma(1-\beta)} \frac{\Gamma(\sigma-\rho+1)\Gamma(\nu-\rho+1)}{\Gamma(\rho-\sigma)\Gamma(\lambda+\nu+\sigma)} \\ &\times F_{p,q}^{\kappa,\mu}(a,b;c;\frac{1}{x}) * {}_3F_2[\sigma-\rho+1,\nu-\rho+1,; \\ &\qquad \rho-\sigma,\lambda+\nu+\sigma;-\frac{1}{x}]\end{aligned} \tag{67}$$

Proof. The proof of the *Theorem 10* is similar to the proof of *Theorem 9*.

4 Generating functions of (7),(9) and (10)

In this section, we will present certain generating functions involving new generalization of extended Gauss hypergeometric function and extended Appell hypergeometric functions.

Theorem 21. *The following generating function for* (7) *holds true:*

$$\begin{aligned}\sum_{n=0}^\infty \binom{\alpha+n-1}{n} F_{p,q}^{\kappa,\mu}(a,b;1-\alpha-n,c;z)\omega^n = (1-\omega)^\alpha \\ \times F_{p,q}^{\kappa,\mu}(a,b;1-\alpha,c;z(1-\omega))\end{aligned} \tag{68}$$

Proof. Let's obtain the left hand side of the equation (68) by **S**. Then, by applting the series expression from (7) into **S**, we have that

$$\mathbf{S} = \sum_{n=0}^\infty \binom{\alpha+n-1}{n} \left[\sum_{k=0}^\infty \frac{(a;p,q;\kappa,\mu)_k(b)_k}{(1-\alpha-n)_k} \cdot \frac{z^k}{k!}\right]\omega^n, \tag{69}$$

which, upon arranging the order of summation and after some changing, gives

$$\mathbf{S} = \sum_{k=0}^\infty \frac{(a;p,q;\kappa,\mu)_k(b)_k}{(1-\alpha)_k} \left[\sum_{n=0}^\infty \binom{\alpha+n+k-1}{n}\omega^n\right]\frac{z^k}{k!}. \tag{70}$$

Finally, applying the following generalized binomial expression [34, 35]:

$$\sum_{n=0}^\infty \binom{\alpha+n-1}{n} t^n = (1-t)^n \; (|t| < 1; \alpha \in \mathbb{C}), \tag{71}$$

for calculating the inner sum in (70), we obtain the required result (68).

Theorem 22. *The following generating function for* (7) *holds true:*

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}F_{p,q}^{\kappa,\mu}(a,\alpha+n;c;z)\omega^n=(1-\omega)^{-\alpha}F_{p,q}^{\kappa,\mu}(a,\alpha;c;\frac{z}{1-w}). \tag{72}$$

Proof. The proof of the *Theorem 12* is same as the proof of *Theorem 1*.

Theorem 23. *The following generating function for* (9) *holds true:*

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\,{}_{p,q}F_1^{(\kappa,\mu)}\left[a,\alpha+k,c;d;x,y\right]\omega^k=(1-\omega)^{-\alpha}\,{}_{p,q}F_1^{(\kappa,\mu)}\left[a,\alpha,c;d;\frac{x}{1-\omega},y\right]. \tag{73}$$

Proof. Let $\mathbf{N}$ be the left side of (73), using the (9) and interchanging the order of summations, we have that

$$\mathbf{N}=\sum_{m,n=0}^{\infty}\sum_{k=0}^{\infty}(\alpha)_k(\alpha+k)_m\frac{(a;p,q;\kappa,\mu)_{m+n}(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{74}$$

Applying the equation (71) in the equation (74), we can be easily seen to lead to right-hand side of (73).

Theorem 24. *The following generating function for* (10) *holds true:*

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\,{}_{p,q}F_2^{(\kappa,\mu)}\left[a,\alpha+k,c;d,e;x,y\right]\omega^k=(1-\omega)^{-\alpha}\,{}_{p,q}F_2^{(\kappa,\mu)}\left[a,\alpha,c;d,e;\frac{x}{1-\omega},y\right]. \tag{75}$$

Proof. The proof of the *Theorem 14* is same as the proof of *Theorem 13*.

5 Fractional Differential Equations

The importance of the fractional differential equations in the field of applied sciences gained more attention not in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of physical phenomena. Specially, the kinetic equations describe the contiunity of motion of substance. The extension and generalisation of fractional kinetic equations involving many fractional operators were found in [2, 3, 9, 11, 12, 15, 31–33].

The fractional differential equation between rate of change of the reaction, the destruction rate and the production rate was established by Haubold and Mathai [15] given as follows:

$$\frac{dN}{dt}=-d(N_t)+p(N_t) \tag{76}$$

where $N=N(t)$ the rate of the reaction, $d=d(N)$ the rate of destruction, $p=p(N)$ the rate of production and N_t denotes the function defined by $N_t(t^\star)=N(t-t^\star), t^\star>0$.

The special case of equation (76) for spatial fluctuations and inhomogeneities in $N(t)$ the quantities are neglected, that is the equation

$$\frac{dN}{dt}=-c_iN_i(t) \tag{77}$$

with the initial condition that $N_i(t=0)=N_0$ is the number of density of the species i at time $t=0$ and $c_i>0$. If we shift the index i and integrate the standard kinetic equation (77), we have

$$N(t)-N_0=-c\,{}_0D_t^{-1}N(t) \tag{78}$$

where ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville integral operator ${}_0D_t^{-\nu}$ given as

$${}_0D_t^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}f(s)ds,\ (t>0,\Re(\nu)>0.) \tag{79}$$

The fractional generalisation of the standart kinetic equation (78) is given by Haubold and Mathai as follows [31, 32]:

$$N(t)-N_0=-c^{\nu}\,{}_0D_t^{-1}N(t) \tag{80}$$

and obtained the solution of (77) as follows:

$$N(t)=N_0\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu k+1)}(ct)^{\nu k}. \tag{81}$$

Furthermore, Saxena and Kalla [33] considered the following fractional kinetic equation:

$$N(t)-N_0 f(t)=-c^{\nu}\,{}_0D_t^{-1}N(t)\ \ (\Re(\nu)>0), \tag{82}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0=N(0)$ is the number of density of that species at time $t=0$, c is a constant and $f\in L(0,\infty)$.

By applying the Laplace transform (46) to the equation (82),

$$\mathfrak{L}\{N(t);p\}=N_0\frac{F(p)}{1+c^{\nu}p^{-\nu}}=N_0\left(\sum_{n=0}^{\infty}(-c^{\nu})^n p^{-\nu n}\right)F(p),\ \left(n\in N_0,\left|\frac{c}{p}\right|<1\right). \tag{83}$$

5.1 Solution of the generalised fractional kinetic equations

In this section, we will present the solution of the generalised fractional kinetic equations which by considering generalised Gauss hypergeometric function (7).

Theorem 25. *If* $d>0,\nu>0;p,q,\kappa,\mu,a,b,c,\delta\in\mathbb{C}$ *be such that* $\Re(c)>\Re(b)>0;\min\{\Re(p),\Re(q),\Re(\kappa),\Re(\mu)>0\}$*, then the solution of the following fractional equation*

$$N(t)-N_0F_{p,q}^{\kappa,\mu}(a,b,c;d^{\nu}t^{\nu})=-\delta^{\nu}\,{}_0D_t^{-\nu}N(t) \tag{84}$$

is given by

$$N(t)=N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n(b)_n}{(c)_n}\frac{\Gamma(\nu n+1)(d^{\nu}t^{\nu})^n}{n!}E_{\nu,\nu n+1}(-\delta^{\nu}t^{\nu}). \tag{85}$$

where $E_{\nu,\nu n+1}(-\delta^{\nu}t^{\nu})$ *is the Mittag-Leffler function [25].*

Proof. The Laplace transform of the Riemann-Liouville fractional integral operator is defined by [14, 36]:

$$\mathfrak{L}\{{}_0D_t^{-\nu}f(t);s\}=s^{-\nu}F(s) \tag{86}$$

where $F(p)$ is given in (46). Now, applying the Laplace transform to the both sides of (84), we obtain

$$\begin{aligned}
&\mathfrak{L}\{N(t);s\}=N_0\,\mathfrak{L}\{F_{p,q}^{\kappa,\mu}(a,b,c;d^{\nu}t^{\nu});s\}-\delta^{\nu}\,\mathfrak{L}\{{}_0D_t^{-\nu}N(t);s\}\\
&N(s)=N_0\left(\int_0^{\infty}e^{-st}\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n(b)_n}{(c)_n}\frac{(d^{\nu}t^{\nu})^n}{n!}\right)-\delta^{\nu}s^{-\nu}N(s)\\
&N(s)+\delta^{\nu}s^{-\nu}N(s)\\
&=N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n(b)_n}{(c)_n}\frac{(d^{\nu})^n}{n!}\int_0^{\infty}e^{-st}t^{\nu n}dt\\
&=N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n(b)_n}{(c)_n}\frac{(d^{\nu})^n}{n!}\frac{\Gamma(\nu n+1)}{s^{\nu n+1}}\\
&N(s)=N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n(b)_n}{(c)_n}\frac{\Gamma(\nu n+1)(d^{\nu})^n}{n!}s^{-(\nu n+1)}\sum_{r=0}^{\infty}\left[-\left(\frac{s}{\delta}\right)^{-\nu}\right]^r
\end{aligned} \tag{87}$$

The inverse Laplace transform of (87) is given by [14, 24, 36]

$$\mathfrak{L}^{-1}\{s^{-\nu};t\}=\frac{t^{\nu-1}}{\Gamma(\nu)},\ (\Re(\nu)>0), \tag{88}$$

we get

$$\begin{aligned} L^{-1}\{N(s)\} &= N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{\Gamma(\nu n+1)\,(d^{\nu})^n}{n!}L^{-1}\{s^{-(\nu n+1)}\sum_{r=0}^{\infty}\left[-\left(\frac{s}{\delta}\right)^{-\nu}\right]^r\} \\ N(t) &= N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{\Gamma(\nu n+1)\,(d^{\nu}t^{\nu})^n}{n!}\left[\sum_{r=0}^{\infty}(-1)^r\delta^{\nu r}\frac{t^{\nu r}}{\Gamma(\nu n+\nu r+1)}\right] \\ N(t) &= N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{\Gamma(\nu n+1)\,(d^{\nu}t^{\nu})^n}{n!}E_{\nu,\nu n+1}(-\delta^{\nu}t^{\nu}). \end{aligned} \tag{89}$$

So, we can be yield the required result (84).

Theorem 26. *If $d>0,\nu>0;p,q,\kappa,\mu,a,b,c,\delta\in\mathbb{C}$ be such that $\Re(c)>\Re(b)>0;\min\{\Re(p),\Re(q),\Re(\kappa),\Re(\mu)>0\}$, then the solution of the following fractional equation:*

$$N(t)-N_0F_{p,q}^{\kappa,\mu}(a,b,c;d^{\nu}t^{\nu})=-d^{\nu}\,{}_0D_t^{-\nu}N(t) \tag{90}$$

is given by

$$N(t)=N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{\Gamma(\nu n+1)(d^{\nu}t^{\nu})^n}{n!}E_{\nu,\nu n+1}(-d^{\nu}t^{\nu}). \tag{91}$$

Proof. Choosing $\delta=d$ in equation (84), we can be easily yield the desired result (90).

Theorem 27. *If $d>0;p,q,\kappa,\mu,a,b,c,\delta\in\mathbb{C}$ be such that $\Re(c)>\Re(b)>0;\min\{\Re(p),\Re(q),\Re(\kappa),\Re(\mu)>0\}$, then the solution of the following fractional equation*

$$N(t)-N_0F_{p,q}^{\kappa,\mu}(a,b,c;t)=-\delta\,{}_0D_t^{-1}N(t) \tag{92}$$

is given by

$$N(t)=N_0\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{\Gamma(n+2)(t)^n}{n!}E_{1,n+2}(-\delta t). \tag{93}$$

Proof. Choosing $\nu=d=1$ in equation (84), we can be easily yield the desired result (90).

Corollary 28. *If $d>0;p,q,\kappa,\mu,a,b,c,\delta\in\mathbb{C}$ be such that $\Re(c)>\Re(b)>0;\min\{\Re(p),\Re(q),\Re(\kappa),\Re(\mu)>0\}$, then the solution of the following fractional equation*

$$N(t)-N_0F_{p,q}^{\kappa,\mu}(a,b,c;t)=-\delta\,{}_0D_t^{-1}N(t) \tag{94}$$

is given by

$$\begin{aligned} N(t) &= N_0-\sum_{n=0}^{\infty}\frac{(a;p,q;\kappa,\mu)_n\,(b)_n}{(c)_n}\frac{(n+1)(t)^{n-1}}{\delta}E_{1,n+1}(-\delta t) \\ &+\frac{1}{\delta t}F_{p,q}^{\kappa,\mu}(a,b,c;t)+\frac{a.b}{c.\delta}F_{p,q}^{\kappa,\mu}(a+1,b+1,c+1;t). \end{aligned} \tag{95}$$

Proof. Applying the above Mittag-Leffler function properties [14, 24]

$$E_{\alpha,\beta}(z)=\frac{1}{z}E_{\alpha,\beta-\alpha}(z)-\frac{1}{z\Gamma(\beta-\alpha)}, \tag{96}$$

in the equation (93). Then, making some simple arrangment, we can be easily yield the desired result (95)

6 Conclusions

We may also give point to that results obtained in this work are of general character and can appropriate to give farther interesting and potentially practical formulas involving integral transform, fractional calculus and generating functions. Also we give a new fractional generalization of the standard kinetic equation and obtained solution for the same. From the close relationship of family of extended generalized Gauss hypergeometric functions with many special functions, we can easily construct various known and new fractional kinetic equations.

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