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An Application of New Method to Obtain Probability Density Function of Solution of Stochastic Differential Equations

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Abstract

In this study, a new method to obtain approximate probability density function (pdf) of random variable of solution of stochastic differential equations (SDEs) by using generalized entropy optimization methods (GEOM) is developed. By starting given statistical data and Euler–Maruyama (EM) method approximating SDE are constructed several trajectories of SDEs. The constructed trajectories allow to obtain random variable according to the fixed time. An application of the newly developed method includes SDE model fitting on weekly closing prices of Honda Motor Company stock data between 02 July 2018 and 25 March 2019.

Keywords: generalized entropy optimization methods, stochastic differential equation model, Euler–Maruyama Method
AMS 2010 codes: 65C30,94A17.

1 Introduction

Stochastic differential equation (SDE) is a differential equation in which one or more of the terms are a stochastic process, thus resulting in a solution which is itself a stochastic process. In the literature, there are many interesting applications and models of SDEs [1–4]. Especially, SDEs have become standard models for financial quantities such as asset prices, interest rates, and their derivatives [5–9].

Class of SDEs for which it is possible to obtain exact solutions is narrow. Even if exact solution is known everywhere, this solution may not be available in the computational sense. For this reason numeric solution of SDEs acquires an important significance. It should be noted that the numerical solution of SDEs is a relatively new area of applied probability theory.

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A typical one-dimensional SDE has the differential form:

$$X(t,\omega) = X(0,\omega) + \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dW(s) \tag{1}$$

and differential form

$$dX(t) = f(t,X)dt + g(t,X)dW(t) \tag{2}$$

for $0 \le t \le T$ where $X(0,\cdot)\varepsilon H_{RV}$, $X(t,\omega)$ is a stochastic process but not a deterministic function.$W(t,\omega) = W(t)$ is a Wiener process or Brownian motion which satisfies the following three conditions:

1. It is nowhere differentiable.
2. $W(0) = 0$
3. $W(t), t \ge 0$ is a continuous stochastic process with stationary independent increments, hence $\int_d^c dW(s) = W(d) - W(c) \sim N(0, d-c)$ for all $0 \le c \le d$.

The function f is often called the drift coefficient of the SDE while g is referred to as the diffusion coefficient. It is assumed that the functions f and g are non-anticipating and satisfy the following conditions (c1) and (c2) for some constant $k \ge 0$ of existence and uniqueness theorem of solution of SDE model [10].

Condition (c1): $|f(t,x) - f(s,y)|^2 \le k(|t-s| + |x-y|^2)$ for $s \ge 0, T \ge t$ and $x, y\varepsilon\mathbb{R}$.

Condition (c2):$|f(t,x)|^2 \le k(1+|x|^2)$ for $0 \le t \le T$ and $x\varepsilon\mathbb{R}$.

There are many methods for determining the solution of SDE model (2). They include, Euler–Maruyama (EM) method, Milstein method, Runge–Kutta method and so on. In this study to determine the solution of SDE (2), EM method is used which is also used recently for solving SDE. EM method was used by Evans [6] for considering an autonomous system of SDE [11].

The present paper is organised as follows. In Section 1, a brief explanation on SDEs is introduced. In Section 2, Fokker–Planck-Kolmogorov (FPK) equation, an approximation of SDE using EM method and generalized entropy optimization methods (GEOM) are given. In Section 3, a new method to obtain approximate probability density function (PDF) of solution of SDEs using GEOM is described. In Section 4, an application on Honda Motor Company (HMC) stock data (02 July 2018 – 25 March 2019) is illustrated. Finally, the main results obtained in this study are summarised and some suggestions for further research are given in Section 5.

2 Auxiliary Methods

In this section, several auxiliary methods are explained for developing a new method to obtain approximate PDF of solution of SDE.

2.1 FPK Equation

The probability distribution of solutions to a discrete-valued continuous stochastic process satisfies a system of differential equations called the forward Kolmogorov or FPK equations.

Let SDE model (2) is given and $F\varepsilon C_0^\infty\mathbb{R}$. Applying Itô's formula to $F(X)$:

$$dF(X) = \left(f(t,X)\frac{\partial F(t,x)}{\partial x} + \frac{1}{2}g^2(t,x)\frac{\partial^2 F(t,x)}{\partial x^2}\right)dt + g(t,X)\frac{\partial F(t,x)}{\partial x}dW(t),$$

$$E\int_0^t g(s,X(s))\frac{\partial F(X(s))}{\partial x}dW(s) = 0,$$

then

$$\frac{dE(F)}{dt} = E\left[\frac{\partial F(X)}{\partial x}f + \frac{1}{2}g^2\frac{\partial^2 F(t,x)}{\partial x^2}\right].$$

If $p(t,x)$ is the PDF of solutions of the SDE, the earlier result implies that

$$\int_{-\infty}^{+\infty} F(x)\left[\frac{\partial p(t,x)}{\partial x} - \frac{1}{2!}\frac{\partial^2(p(t,x)a(t,x))}{\partial x^2}\right] dx = 0$$

for every $F \varepsilon C_0^\infty(\mathbb{R})$. As the above integral holds for every function $F \varepsilon C_0^\infty(\mathbb{R})$, it implies that

$$\frac{\partial p(t,x)}{\partial x} = -\frac{\partial(p(t,x)a(t,x))}{\partial x} + \frac{1}{2!}\frac{\partial^2 p(t,x)b(t,x)}{\partial x^2} \tag{3}$$

with $p(0,x) = p_0(x)$. Equation (3) is the forward Kolmogorov equation or FPK equation for the probability distribution of solutions to SDE (Eq. 2) [10].

2.2 EM Method

One of the simplest numerical approximations for the SDE is the EM method. The reason is that, EM method is an elementary method between the other applied methods, so in our investigation EM method is used. We applied the Euler method or EM method as follows:

$$X(t_{i+1}) = X(t_i) + f(X(t_i))\Delta t + g(X(t_i))\Delta W(t)$$

for $i = 0,1,2,\ldots,N-1$ with an initial value $X(t_0) = X_0$.

2.3 Generalized Entropy Optimization Methods

For the statistical data, GEOM modelling in the form of generalized entropy optimization distributions (GEOD) can be successfully applied in many scientific fields [12]. GEOD can be used in modelling, because by increasing the number and by changing the type of characterising moment vector functions MaxMaxEnt and MaxMinxEnt distributions can also be suitable for estimation [13–17].

2.3.1 MaxEnt Functional

The problem of maximising entropy function

$$H = \sum_{i=1}^{n} p_i ln p_i, \quad p_i = (p_1,\ldots,p_n) \tag{4}$$

subjects to constraints

$$\sum_{i=0}^{n} p_i g_j(x_i) = \mu_j, \quad j = 0,1,2,\ldots,m \tag{5}$$

where $\mu_0 = 1, g_0(x) = 1$; $p_i \geq 0$, $i = 0,1,2,\ldots,n$; $m+1 < n$ has solution

$$p_i = \exp\left(-\sum_{j=0}^{m} \lambda_j g_j(x_i)\right), \quad i = 1,2,\ldots,n \tag{6}$$

where λ_j, $j = 0,1,\ldots,m$ are Lagrange multipliers. In the literature, there have been numerous studies that have calculated these multipliers [12, 13]. If Equation (6) is substituted in Equation (4), the maximum entropy value is obtained in the form:

$$H_{max}(p) = U(g) = -\sum_{i=1}^{n}\left(\exp\left(-\sum_{j=0}^{m}\lambda_j g_j(x_i)\right)\right)\left(-\sum_{j=0}^{m}\lambda_j g_j(x_i)\right) = \sum_{j=0}^{m}\lambda_j \mu_j.$$

2.3.2 $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$ Distributions

GEOD indicated as $(MinMaxEnt)_m$ is closest to a given statistical data and distribution indicated as $(MaxMaxEnt)_m$ is furthest from a given statistical data in the sense of MaxEnt measure. Solving the MinMaxEnt and MaxMaxEnt problems requires to find vector functions $(g_0, g^{(1)}(x)), (g_0, g^{(2)}(x))$, where $g_0(x) \equiv 1$, $g^{(1)} \varepsilon K_{0,m}$, $g^{(2)} \varepsilon K_{0,m}$ minimising and maximising functional $U(g)$, respectively. It should be noted that $U(g)$ reaches its minimum (maximum) value subject to constraints generated by function $g_0(x)$ and all m-dimensional vector functions $g(x)$, $g\varepsilon K_{0,m}$. In other words, minimum (maximum) value of $U(g)$ is least (greatest) value of values $U(g)$ corresponding to $(g_0(x), g)$, $g\varepsilon K_{0,m}$. In other words, $(MinMaxEnt)_m((MaxMaxEnt)_m)$ is distribution giving minimum (maximum) value to functional $U(g)$ along with all distributions generated by $\binom{r}{m}$ number of moment vector functions $(g_0(x), g)$, $g\varepsilon K_{0,m})$. Therefore, we denote mentioned distributions in the forms of $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$.

Let K be the compact set of moment vector functions $(g_0(x), g)$, $g\varepsilon K_{0,m})$. $U(g)$ reaches its least and greatest values in this compact set, because of its continuity property. For this reason,

$$\min_{g\varepsilon K} U(g) = U(g^{(1)}) \quad ; \quad \min_{g\varepsilon K} U(g) = U(g^{(2)})$$

Consequently,

$$U(g^{(1)}) \leq U(g^{(2)}).$$

3 New Method to Obtain Approximate Pdf for Solution of SDE

By starting given statistical data and EM method approximating SDE are constructed several trajectories of SDEs. The constructed trajectories allow to obtain random variable according to the fixed time. The PDF of mentioned random variable is obtained by GEOM. If the distance between all neighboring fixed times is sufficiently small, then a surface passing through sections (curves) according to mentioned fixed times can be considered as some approximation to PDF of solution of SDEs. In application, the following proved theorem is used.

Theorem 1. *Assume that conditions (c1), (c2) of existence and uniqueness of solution theorem for SDE (2) satisfied and solution $X(t)$ of SDE has pdf $\varphi(t,s)$. Besides random variable $X(t_i)$ obtained by approximating EM method using given statistical data has pdf $\varphi_i(x)$. Then equality*

$$|X(t_i) - \hat{X}(t_i)| \leq E(|X(t_i) - \hat{X}(t_i)|^2)^{\frac{1}{2}} < \hat{c}\Delta t \tag{7}$$

holds. Moreover surface obtained by $Z = \varphi_i(x)$, $t\varepsilon[t_i, t_{i+1}]$, $i = 0, 1, \ldots, N-1$ represents some approximations to pdf $\varphi(t,x)$ of random variable $X(t)$ of solution SDE (2) when $\Delta t \to 0$.

4 An Application of the New Method

In this section, we illustrated the use of the developed method on weekly closing prices of HMC stock data [18] which is traded in New York Stock Exchange (NYSE) between the dates 02 July 2018 and 25 March 2019 (see Table 1). The fundamental practical steps to obtain SDE model fitting on HMC data are as follows.

1. Parameters values for the SDE model are obtained (8) according to HMC stock data in Table 1.
2. EM trajectories are constructed via estimating parameters values for this stock data and they are plotted in Figure 1.

3. Reel index values of HMC stock and the approximate EM values of $\hat{X}(t_i)$, $i = 16, 39$ are given in Tables 2, 3. HMC stock data sample path and its EM trajectories $\hat{X}(t_i)$, $i = 16, 39$ obtained by developed method are illustrated in Figures 2 and 3.

4. A lack of fit between HMC stock data and approximate values of $\hat{X}(t_i)$, $i = 16, 39$ is determined using goodness-of-fit criteria and p-values. Also, linear relationship between HMC stock data and approximate EM values of $\hat{X}(t_i)$, $i = 16, 39$ is obtained by R^2.

5. Approximate pdfs of random variables $\hat{X}(t_i)$, $i = 16, 39$ of solutions of SDE model are constructed via pdfs of random variables of $\hat{X}(t_i)$, $i = 16, 39$ using GEOM as shown in Tables 2 and 3.

Table 1 Data set of weekly closing prices of HMC stock between 2 July 2018 to 25 March 2019

Week	Price	Week	Price	Week	Price	Week	Price	Week	Price
1	28.60	9	30.00	17	25.83	25	26.31	33	27.28
2	28.60	10	29.91	18	28.82	26	25.18	34	27.47
3	29.49	11	28.12	19	28.72	27	26.26	35	28.33
4	29.49	12	30.65	20	27.34	28	29.12	36	27.15
5	30.00	13	30.53	21	27.50	29	30.06	37	26.90
6	30.05	14	30.04	22	27.39	30	30.10	38	26.80
7	29.10	15	28.24	23	27.09	31	30.58	39	26.64
8	30.37	16	27.04	24	26.00	32	27.81		

In this section, the data are fit to the SDE model,

$$dX(t) = \Theta_1 X(t)dt + \sqrt{\Theta_2 X(t)}dW(t), \quad X_0 = 28.60 \tag{8}$$

where $X(t)$ is stock price data size and $\Theta = [\Theta_1, \Theta_2]^T$ has to be determined using the nonparametric estimation method described in [3] in the following form:

$$\Theta_1 = \frac{1}{\Delta h}\frac{\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2}{\sum_{i=0}^{N-1}x_i} \quad , \quad \Theta_2 = \frac{1}{\Delta h}\frac{\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2}{\sum_{i=0}^{N-1}x_i}$$

where x_i, $i = 0, 1, \ldots, N-1$ are data values.

Applying equations to this model gives the estimates $\hat{\Theta}_1 = 0.0001$ and $\hat{\Theta}_2 = 0.051$. If these estimated parameters are taken into account in SDE model (8), then

$$dX(t) = 0.0001X(t)dt + \sqrt{(0.051X(t))}dW(t). \quad X_0 = 28.60$$

According to the theorem, by starting determined SDE model and EM method, points $(t_i, x_j^{(i)})$, $i = 0, 1, \ldots, K$, $j = 0, 1 \ldots, K$; $K = N$ can be considered as nodal points of approximate EM trajectories of SDE model. In the following figure, HMC stock data sample path and mentioned EM approximate trajectories are plotted in Fig. 1.

According to the developed method, for each of all trajectories in Figure 1 PDF can be obtained using GEOM. For this reason, we choose two EM trajectories $\hat{X}(t_i)$, $i = 16, 39$. Reel index values of HMC stock and its approximate EM values of $\hat{X}(t_i)$, $i = 16, 39$ are given in Tables 2 and 3.

HMC stock data sample path and its EM trajectories $\hat{X}(t_i)$, $i = 16, 39$ by developed method are plotted in Figure 2.

Now, a lack of fit is determined between HMC stock data and approximate EM values of $\hat{X}(t_i)$, $i = 16, 39$ using goodness-of-fit criteria and p-values. According to the values of $\hat{X}(t_i)$, $i = 16, 39$, goodness of fit criteria

Table 2 Reel index values of HMC stock and approximate values of $\hat{X}(t_{16})$ using EM method

Week	Reel Price	EM App.	Week	Reel Price	EM App.	Week	Reel Price	EM App.
1	28.60	28.60	14	30.04	30.61	27	26.26	25.22
2	28.60	28.62	15	28.24	29.95	28	29.12	26.09
3	29.49	28.53	16	27.04	27.99	29	30.06	29.32
4	29.49	29.40	17	25.83	27.02	30	30.10	29.95
5	30.00	29.49	18	28.82	25.96	31	30.58	30.01
6	30.05	29.86	19	28.72	28.83	32	27.81	30.37
7	29.10	29.77	20	27.34	28.56	33	27.28	27.81
8	30.37	29.03	21	27.50	27.51	34	27.47	27.24
9	30.00	30.30	22	27.39	27.44	35	28.33	27.40
10	29.91	29.90	23	27.09	27.68	36	27.15	28.44
11	28.12	30.06	24	26.00	27.17	37	26.90	27.24
12	30.65	28.12	25	26.31	25.86	38	26.80	26.83
13	30.53	30.94	26	25.18	26.27	39	26.64	26.96

are $\chi^2 = 1.9090$, $\chi^2 = 1.9508$ and p-values are 0.1671, 0.1625, respectively. Since the p-values 0.1671 and 0.1625 are greater than the 0.05 significance level, we do not reject the null hypothesis that HMC stock data are fit to both approximate EM values of $\hat{X}(t_i)$, $i = 16, 39$. R^2 is a measure of the strength of the linear relationship between HMC stock data and approximate EM values of $\hat{X}(t_i)$, $i = 16, 39$. According to values of $\hat{X}(t_i)$, $i = 16, 39$, R^2 values are 0.4895 and 0.4923, respectively. But, in this study R^2 according to $\hat{X}(t_{39})$ is bigger than R^2 according to $\hat{X}(t_{16})$, so R^2 is acceptable for $\hat{X}(t_{39})$. So, in the sense of R^2 , $\hat{X}(t_{39})$ is more representable than $\hat{X}(t_{16})$.

Finally, approximate pdfs of random variables $\hat{X}(t_i)$, $i = 16, 39$ of solutions of SDE are constructed via pdfs of random variables of $\hat{X}(t_i)$, $i = 16, 39$ using GEOM are given in Tables 4–7.

In this study, the results show that $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$ distributions obtained by GEOM is suitable for the assessment of HMC stock data. First, MaxEnt characterising moments of given moment functions are determined according to data. Second, MaxEnt distributions subject to each of MaxEnt characterising moments are calculated. Hereafter, distributions generated by GEOM corresponding to selected MaxEnt characterising moments are obtained. Acquired results are realised by using MATLAB programme. The performances

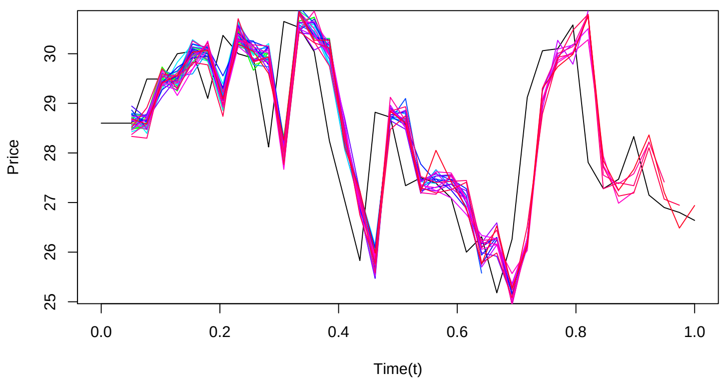

Fig. 1 HMC stock path (black line) and approximate EM trajectories $\hat{X}(t_i)$, $i = 1, \ldots, 39$ of SDE (color lines)

Table 3 Reel index values of HMC stock and approximate values of $\hat{X}(t_{39})$ using EM method

Week	Reel Price	EM App.	Week	Reel Price	EM App.	Week	Reel Price	EM App.
1	28.60	28.60	14	30.04	30.64	27	26.26	25.09
2	28.60	28.49	15	28.24	30.43	28	29.12	26.53
3	29.49	28.57	16	27.04	28.06	29	30.06	29.21
4	29.49	29.70	17	25.83	26.80	30	30.10	30.17
5	30.00	29.57	18	28.82	25.97	31	30.58	30.23
6	30.05	30.20	19	28.72	29.02	32	27.81	30.66
7	29.10	30.44	20	27.34	28.59	33	27.28	28.01
8	30.37	29.03	21	27.50	27.32	34	27.47	27.33
9	30.00	30.56	22	27.39	27.83	35	28.33	27.21
10	29.91	29.54	23	27.09	27.56	36	27.15	28.47
11	28.12	29.93	24	26.00	26.94	37	26.90	27.12
12	30.65	28.18	25	26.31	26.02	38	26.80	26.67
13	30.53	30.47	26	25.18	26.39	39	26.64	26.65

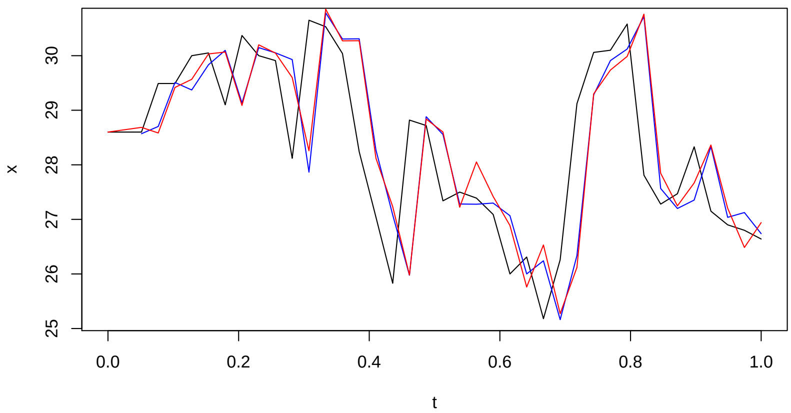

Fig. 2 HMC stock path (black line), approximate trajectories $\hat{X}(t_{16})$ (blue line) and $\hat{X}(t_{39})$ (red line) of SDE model

of distributions generated by GEOM are evaluated by statistical criteria as root mean square error (RMSE), chi-square χ^2 and MaxEnt measure (entropy).

4.1 Performance of the Method

In order to obtain $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m(m = 1,2,\dots,r)$ distributions, it is required to choose the moment vector functions giving the maximum and minimum values to the MaxEnt functional $U(g)$. For this reason, the moment functions $g_0(x) = 1$, $g_1(x) = x, g_2(x) = x^2$, $g_3(x) = \ln x$, $g_4(x) = \ln^2(x)$, $g_5(x) = \ln(1+x^2)$ are used. That is, $K_0 = \{g_0, g_1, \dots, g_5\}$. According to GEOM, all combinations of r elements of K_0 taken m elements at a time are denoted by $K_{0,m}$ $(m = 1,2,\dots,r)$. The processes of obtaining $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$ distributions are performed in this way. Out of the sets $K_{0,m}$, $(m = 1,2,3,4)$ the two moment vector functions giving least and greatest values to $U(g)$ are chosen. For example, in the set $K_{0,2}$ (triple combination), $(g_0, g^{(1)}) = (1, x, \ln x)$, $g^{(1)} \varepsilon K_{0,2}$ gives least value and $(g_0, g^{(2)}) = (1, \ln x, \ln^2(x))$, $g^{(2)} \varepsilon K_{0,2}$ gives greatest value to $U(g)$. Thus, these distributions corresponding to $g^{(1)}, g^{(2)}$ are the $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$ distributions, respectively. The evaluation of performance of these distributions for the number of values of $\hat{X}(t_i)$, $i=16,39$ in Tables 2 and 3 are given in Tables 4–7, respectively.

Table 4 Evaluation of performance of $(MinMaxEnt)_m, m=1,2,3,4$ distributions for $\hat{X}(t_{16})$

$(MinMaxEnt)_m$	entropy	R^2	RMSE	χ^2	moment functions
$(MinMaxEnt)_1$	2.5532	0.2348	0.0562	0.1169	$1, \ln(1+x^2)$
$(MinMaxEnt)_2$	2.5009	0.7036	0.0349	0.0379	$1, x, \ln x$
$(MinMaxEnt)_3$	2.5008	0.7047	0.0348	0.0378	$1, x, x^2, \ln^2(x)$
$(MinMaxEnt)_4$	2.5010	0.7027	0.0349	0.0380	$1, x, \ln x, \ln^2(x), \ln(1+x^2)$

Table 5 Evaluation of performance of $(MaxMaxEnt)_m, m=1,2,3,4$ distributions for $\hat{X}(t_{16})$

$(MaxMaxEnt)_m$	entropy	R^2	RMSE	χ^2	moment functions
$(MaxMaxEnt)_1$	2.5572	0.2053	0.0573	0.1223	$1, x^2$
$(MaxMaxEnt)_2$	2.5538	0.2353	0.0562	0.1169	$1, \ln x, \ln^2(x)$
$(MaxMaxEnt)_3$	2.5537	0.2403	0.0560	0.1160	$1, \ln x, \ln^2(x), \ln(1+x^2)$
$(MaxMaxEnt)_4$	2.5026	0.7011	0.0350	0.0382	$1, x^2, \ln x, \ln^2(x), \ln(1+x^2)$

Table 6 Evaluation of performance of $(MinMaxEnt)_m, m=1,2,3,4$ distributions for $\hat{X}(t_{39})$

$(MinMaxEnt)_m$	entropy	R^2	RMSE	χ^2	moment functions
$(MinMaxEnt)_1$	2.5274	0.6133	0.0354	0.0601	$1, \ln(1+x^2)$
$(MinMaxEnt)_2$	2.5135	0.6897	0.0318	0.0452	$1, x, \ln x$
$(MinMaxEnt)_3$	2.5135	0.6897	0.0318	0.0452	$1, x, \ln x, \ln(1+x^2)$
$(MinMaxEnt)_4$	2.5243	0.6826	0.0320	0.0495	$1, x, \ln x, \ln^2(x), \ln(1+x^2)$

Table 7 Evaluation of performance of $(MaxMaxEnt)_m, m=1,2,3,4$ distributions for $\hat{X}(t_{39})$

$(MinMaxEnt)_m$	entropy	R^2	RMSE	χ^2	moment functions
$(MaxMaxEnt)_1$	2.5303	0.5900	0.0365	0.0630	$1, x^2$
$(MaxMaxEnt)_2$	2.5278	0.6174	0.0352	0.0595	$1, \ln x, \ln^2(x)$
$(MaxMaxEnt)_3$	2.5296	0.6678	0.0328	0.0519	$1, x, x^2, \ln(1+x^2)$
$(MaxMaxEnt)_4$	2.5299	0.6437	0.0339	0.0555	$1, x, x^2, \ln^2(x), \ln(1+x^2)$

In order to determine the performance of the $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$ distributions, it is used various criteria which are often used in statistics. These criteria are RMSE, determination coefficient (R^2)and chi-squared (χ^2) [17]. The optimum distribution function can be determined according to the lowest value RMSE, χ^2, MaxEnt measure and the highest values of R^2.

It can be deduced from Table 4 and 5 that $(MinMaxEnt)_m$ distributions perform better than the $(MaxMaxEnt)_m$ distributions in terms of modelling data. Also, $(MinMaxEnt)_4$ of $\hat{X}(t_{16})$ distribution is more suitable for given estimation data in the sense of R^2, RMSE, χ^2 criterias and MaxEnt measure.

The same reasoning for $\hat{X}(t_{16})$ is used for $\hat{X}(t_{39})$ as illustrated in Tables 6 and 7 . It can be shown from Tables

6 and 7 that $(MinMaxEnt)_m$ distributions perform better than $(MaxMaxEnt)_m$ distribution in terms of modelling data. Also, $(MinMaxEnt)_3$ of $\hat{X}(t_{39})$ distribution is more suitable for given estimated data in the sense of R^2, RMSE, χ^2 criterias and MaxEnt measure.

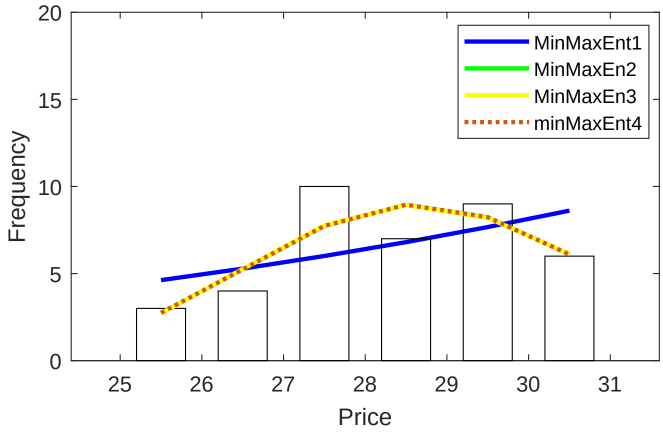

Fig. 3 Comparison of the $(MinMaxEnt)_m$, $m = 1,2,3,4$ distributions for stock data $\hat{X}(t_{16})$

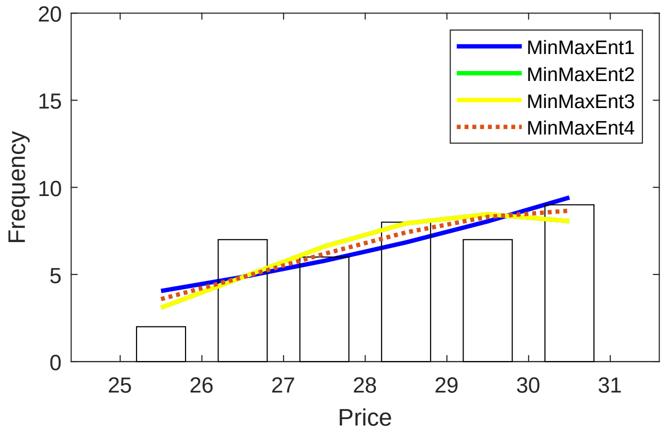

Fig. 4 Comparison of the $(MinMaxEnt)_m$, $m = 1,2,3,4$ distributions for stock data $\hat{X}(t_{39})$

In this application Tables 4–7 indicate that $(MinMaxEnt)_m$ distributions are better than $(MaxMaxEnt)_m$ distributions in the sense of almost all criteria for random variables $\hat{X}(t_{16})$ and $\hat{X}(t_{39})$. These results are also supported by the illustrations in Figures 3–6 for each data. It can be observed in all the figures that the $(MinMaxEnt)_m$ distributions are more suitable than $(MaxMaxEnt)_m$ distributions in modelling of random variables $\hat{X}(t_{16})$ and $\hat{X}(t_{39})$.

Up to now, all analyses were performed for random variables $\hat{X}(t_{16})$ and $\hat{X}(t_{39})$. As given in Table 6, the $(MinMaxEnt)_m$ distributions show even better fitting than the $(MaxMaxEnt)_m$ distributions for random variable $\hat{X}(t_{39})$ according to all criteria.

Finally, it can also be concluded that the $(MinMaxEnt)_m$ distributions fit better to different random variables of data than the $(MaxMaxEnt)_m$ distributions. These results are also supported by the illustrations in Figures 3–6 for each data. It can be observed in all the figures that the $(MinMaxEnt)_m$ distributions closely match the measured data much better than the $(MaxMaxEnt)_m$ distribution, particularly for random variable $\hat{X}(t_{39})$.

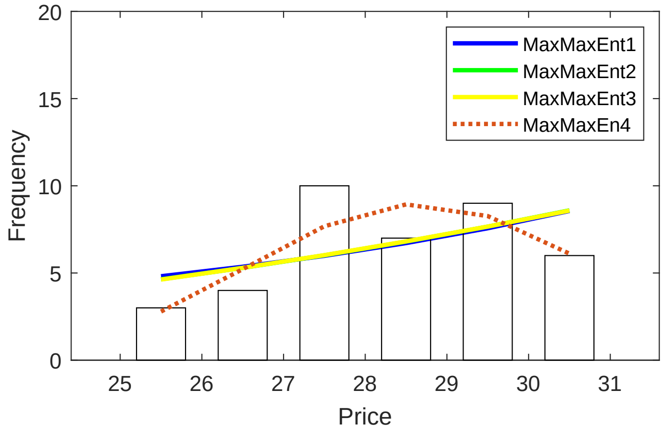

Fig. 5 Comparison of the $(MaxMaxEnt)_m$, $m = 1,2,3,4$ distributions for stock data $\hat{X}(t_{16})$

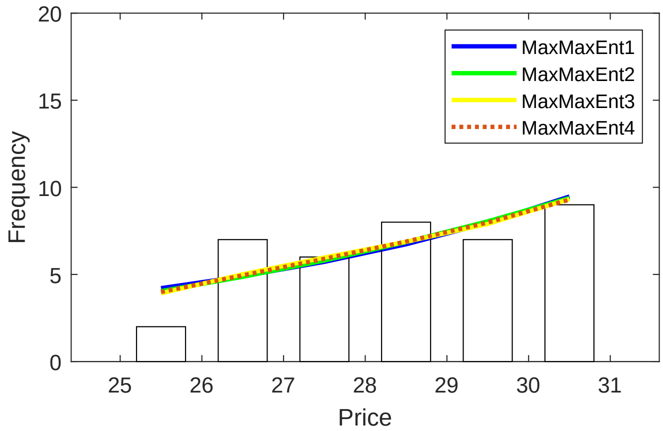

Fig. 6 Comparison of the $(MaxMaxEnt)_m$, $m = 1,2,3,4$ distributions for stock data $\hat{X}(t_{39})$

5 Conclusion

The main results obtained from the present study can be summarized as follows:

- A new method is illustrated to evaluate approximate PDF of random variable of solution of SDE model using GEOM.
- The use of the developed method on HMC stock data and the SDE model fitting is shown. It is obtained that HMC stock data and approximate EM values of $\hat{X}(t_i)$, $i = 16,39$ are fit to selected SDE model.
- For HMC stock data, $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$ distributions are compared in terms of modelling data. The results of the comparison indicate that the obtained $(MinMaxEnt)_m$ distributions give better results than the $(MaxMaxEnt)_m$ distributions. The results of the comparison indicate that the obtained $(MinMaxEnt)_3$ and $(MinMaxEnt)_4$ distributions give better results in the sense of R^2, RMSE, χ^2 criteria and MaxEnt measure in application.
- GEOM can be used for assessment of the stock potential and the performance of stock systems. It should be noted that distributions of $(MinMaxEnt)_m$ and $(MaxMaxEnt)_m$ generated by GEOM are applied first to the stock field.

- The present study may give different and useful insights to economists and scientists dealing with stock systems.

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