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Properties of a New Subclass of Analytic Functions With Negative Coefficients Defined by Using the Q-Derivative

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Abstract

In this paper we define a new class of analytic functions with negative coefficients involving the q- differential operator. Our main purpose is to determine coefficient inequalities and distortion theorems for functions belonging to this class. Connections with previous results are pointed out.

Keywords: analytic functions, Janowski functions, negative coefficients, q-calculus, q-differential operator.
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1 Introduction

The quantum calculus or $q-$calculus has called the attention of many great researchers from Geometric function theory field due to its numerous applications in mathematics and physics. In 1999 was defined the first q-analogue of a starlike function by Ismail et al. [1]. In fact, many results obtained for univalent functions can be extended by using q-analogues functions. Recently, analytic functions with negative coefficient were studied in papers [5, 8–10].

Let $U = \{z : |z| < 1\}$ be the open unit disk of the complex plane and let $\mathcal{A}$ represent the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U. \tag{1.1}$$

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For $0 < q < 1$, the q-derivative of a function $f \in \mathcal{A}$ is defined by (see [2])

$$d_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \qquad (z \neq 0), \tag{1.2}$$

with $d_q f(0) = f^{'}(0)$.

Motivated by the aforementionated works, we define the following class of functions associated with Janowski functions:

Definition 1.1. For $0 \leq \mu \leq 1$, $k \geq 0$ and $-1 \leq B < A \leq 1$, we let $ST(q,\mu,k;A,B)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the condition

$$\operatorname{Re}\left\{\frac{(B-1)zd_q f(z)/[(1-\mu)z+\mu f(z)]-(A-1)}{(B+1)zd_q f(z)/[(1-\mu)z+\mu f(z)]-(A+1)}\right\} > \tag{1.3}$$

$$k\left|\frac{(B-1)zd_q f(z)/[(1-\mu)z+\mu f(z)]-(A-1)}{(B+1)zd_q f(z)/[(1-\mu)z+\mu f(z)]-(A+1)}-1\right|, z \in U. \tag{1.4}$$

Let $\mathcal{T}$ denote the subclass of analytic functions $f \in \mathcal{A}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{1.5}$$

Further, let $\mathcal{T}ST(q,\mu,k;A,B) \equiv ST(q,\mu,k;A,B) \cap \mathcal{T}$.

Also, we remark that $ST(q,\mu,k;A,B)$ reduces to the following known classes:
(i) In case $A = 1-2\alpha$, $0 \leq \alpha < 1$, $B = -1$, $\mu = 1$ and $q \to 1^-$, we obtain the class $S_p(k,\alpha)$ of $k-$uniformly starlike functions of order α(see [4]);
(ii) In case $A = 1$, $B = -1$, $\mu = 1$, $k = 1$ and $q \to 1^-$, we get the class S_p of uniformly starlike functions (see [6]);
(iii) In case $\mu = 1$ and $q \to 1^-$, we obtain the class $k - ST[A,B]$ which was introduced and studied by Noor and Malik (see [3]);
(iv) In case $A = 1-2\alpha$, $0 \leq \alpha < 1$, $B = -1$, $\mu = 1$, $k = 0$ and $q \to 1^-$, we get the class of starlike functions of order α denoted $S^*(\alpha)$;
(v) In case $A = 1$, $B = -1$, $\mu = 0$, $k = 0$ and $q \to 1^-$, we obtain the class R of functions whose derivative has positive real part.

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \mu \leq 1$, $k \geq 0$ and $-1 \leq B < A \leq 1$.

2 Coefficient Estimates

We begin with a result that provides coefficient inequalities for functions in the class $ST(q,\mu,k;A,B)$.

Theorem 2.1. *A function $f \in \mathcal{A}$ of the form (1.1) is in the class $ST(q,\mu,k;A,B)$ if*

$$\sum_{n=2}^{\infty}\{2(k+1)([n]_q-\mu)+|[n]_q(B+1)-\mu(A+1)|\}\cdot|a_n| \leq |B-A|. \tag{2.1}$$

Proof. It is suffices to prove that

$$k|g(z)-1| - \operatorname{Re}[g(z)-1] < 1, \quad z \in U, \tag{2.2}$$

where the function g is defined by

$$g(z) := \frac{(B-1)zd_q f(z)/[(1-\mu)z+\mu f(z)]-(A-1)}{(B+1)zd_q f(z)/[(1-\mu)z+\mu f(z)]-(A+1)}, \quad z \in U. \tag{2.3}$$

First of all,

$$k|g(z)-1|-\mathrm{Re}[g(z)-1]\leq (k+1)|g(z)-1|,\quad z\in U. \tag{2.4}$$

Because

$$\begin{aligned}(k+1)|g(z)-1| &= \frac{2(k+1)|\sum_{n=2}^{\infty}(\mu-[n]_q)a_n z^{n-1}|}{|(B-A)+\sum_{n=2}^{\infty}\{(B+1)[n]_q-\mu(A+1)\}a_n z^{n-1}|}\\ &\leq \frac{2(k+1)\sum_{n=2}^{\infty}(\mu-[n]_q)|a_n|}{|B-A|-\sum_{n=2}^{\infty}\{(B+1)[n]_q-\mu(A+1)\}|a_n|}\end{aligned} \tag{2.5}$$

and (2.1) take place, then $(k+1)|g(z)-1|$ is bounded above by 1. Hence $f\in ST(q,\mu,k;A,B)$ and the theorem is proved.

Corollary 2.1. *(see [3], Theorem 2.1) Let function $f\in\mathcal{A}$ be of the form (1.1). If*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)+|n(B+1)-(A+1)|\}\cdot|a_n|\leq|B-A|, \tag{2.6}$$

then $f\in k-ST[A,B]$.

Corollary 2.2. *(see [7], Theorem 1) Let function $f\in\mathcal{A}$ be of the form (1.1). If*

$$\sum_{n=2}^{\infty}(n-\alpha)\cdot|a_n|\leq 1-\alpha, \tag{2.7}$$

then function f is in the class $S^*(\alpha)$.

For $f\in\mathcal{T}ST(q,\mu,k;A,B)$ the converse of Theorem 2.1 is also true.

Theorem 2.2. *A function $f\in\mathcal{T}$ given by (1.5) is in the class $\mathcal{T}ST(q,\mu,k;A,B)$ if and only if*

$$\sum_{n=2}^{\infty}\{2(k+1)([n]_q-\mu)+|[n]_q(B+1)-\mu(A+1)|\}\cdot a_n\leq|B-A|. \tag{2.8}$$

The result is sharp with the extremal function f given by

$$f(z)=z-\frac{|B-A|}{2(k+1)([n]_q-\mu)+|[n]_q(B+1)-\mu(A+1)|}z^n,\quad z\in U. \tag{2.9}$$

Proof. In view of Theorem 2.1, we need only to prove that (2.8) holds if $f\in\mathcal{T}ST(q,\mu,k;A,B)$. Conversely, assuming that $f\in\mathcal{T}ST(q,\mu,k;A,B)$ and z is real, we find that $g(z)\geq k|g(z)-1|$, where g is given in (2.3). Therefore,

$$\begin{aligned}&\frac{(B-A)-\sum_{n=2}^{\infty}\{(B-1)[n]_q-\mu(A-1)\}a_n z^{n-1}}{(B-A)-\sum_{n=2}^{\infty}\{(B+1)[n]_q-\mu(A+1)\}a_n z^{n-1}}\geq\\ &2k\left|\frac{\sum_{n=2}^{\infty}([n]_q-\mu)a_n z^{n-1}}{(B-A)-\sum_{n=2}^{\infty}\{(B+1)[n]_q-\mu(A+1)\}a_n z^{n-1}}\right|.\end{aligned} \tag{2.10}$$

Next, letting $z\to 1^-$ through real values, we obtain the inequality (2.8). Also, the equality in (2.8) take place for the function f given in (2.9).

Corollary 2.3. *Let function f be of the form (1.5). Then $f\in k-ST[A,B]\cap\mathcal{T}$ if and only if*

$$\sum_{n=2}^{\infty}\{2(k+1)(n-1)+|n(B+1)-(A+1)|\}\cdot a_n\leq|B-A|. \tag{2.11}$$

Corollary 2.4. *(see [7], Theorem 2) Let function $f\in\mathcal{T}$ be of the form (1.5). Then f is a starlike function of order α if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)\cdot a_n\leq 1-\alpha. \tag{2.12}$$

3 Distortion Theorems

Theorem 3.1. *Let the function f of the form (1.5) be in the class $\mathcal{T}ST(q,\mu,k;A,B)$. Then, for $|z|=r$,*

$$|f(z)| \le r + \frac{|B-A|}{2(k+1)([2]_q-\mu)+|[2]_q(B+1)-\mu(A+1)|}r^2 \tag{3.1}$$

and

$$|f(z)| \ge r - \frac{|B-A|}{2(k+1)([2]_q-\mu)+|[2]_q(B+1)-\mu(A+1)|}r^2, \tag{3.2}$$

with equality for

$$f(z) = z - \frac{|B-A|}{2(k+1)([2]_q-\mu)+|[2]_q(B+1)-\mu(A+1)|}z^2. \tag{3.3}$$

Proof. By using the hypothesis and the triangle inequality, we find that

$$r - r^2\sum_{n=2}^{\infty} a_n \le |f(z)| \le r + r^2\sum_{n=2}^{\infty} a_n, \tag{3.4}$$

which, in conjunction with (2.8) gives us the inequalities (3.1) and (3.2).

Corollary 3.1. *Let the function f given by (1.5) be in the class $k-ST[A,B]\cap\mathcal{T}$. Then, for $|z|=r$,*

$$r - \frac{|B-A|}{2(k+1)+|2B-A+1|}r^2 \le |f(z)| \le r + \frac{|B-A|}{2(k+1)+|2B-A+1|}r^2,$$

with equality for

$$f(z) = z - \frac{|B-A|}{2(k+1)+|2B-A+1|}z^2. \tag{3.5}$$

Corollary 3.2. *(see [7], Theorem 4) Let the function $f \in \mathcal{T}$ given by (1.5) be in the class $S^*(\alpha)\cap\mathcal{T}$. Then, for $|z|=r$,*

$$r - \frac{1-\alpha}{2-\alpha}r^2 \le |f(z)| \le r + \frac{1-\alpha}{2-\alpha}r^2,$$

with equality for

$$f(z) = z - \frac{1-\alpha}{2-\alpha}z^2. \tag{3.6}$$

The next theorem can be proven by employing similar techniques as in the demonstration of Theorem 3.1, so we will omit the details of our proof.

Theorem 3.2. *Let the function f of the form (1.5) be in the class $\mathcal{T}ST(q,\mu,k;A,B)$. Then, for $|z|=r$,*

$$|f'(z)| \le 1 + \frac{2|B-A|}{2(k+1)([2]_q-\mu)+|[2]_q(B+1)-\mu(A+1)|}r \tag{3.7}$$

and

$$|f'(z)| \ge 1 - \frac{2|B-A|}{2(k+1)([2]_q-\mu)+|[2]_q(B+1)-\mu(A+1)|}r, \tag{3.8}$$

with equality for f given by (3.3).

Corollary 3.3. *Let the function f given by (1.5) be in the class $k-ST[A,B]\cap\mathfrak{T}$. Then, for $|z|=r$,*

$$1-\frac{2|B-A|}{2(k+1)+|2B-A+1|}r\leq|f^{'}(z)|\leq1+\frac{2|B-A|}{2(k+1)+|2B-A+1|}r,$$

with equality for f given by (3.5).

Corollary 3.4. *(see [7], Theorem 4) Let the function $f\in\mathfrak{T}$ given by (1.5) be in the class $S^{*}(\alpha)\cap\mathfrak{T}$. Then, for $|z|=r$,*

$$1-\frac{2(1-\alpha)}{2-\alpha)}r\leq|f^{'}(z)|\leq1+\frac{2(1-\alpha)}{2-\alpha}r,$$

with equality for f given by (3.6).

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