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Some Properties Curvature of Lorentzian Kenmotsu Manifolds

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Abstract

In this paper different curvature tensors on Lorentzian Kenmotsu manifold are studied. We investigate constant ϕ -holomorphic sectional curvature and \mathcal{L} -sectional curvature of Lorentzian Kenmotsu manifolds, obtaining conditions for them to be constant of Lorentzian Kenmotsu manifolds in such condition. We calculate the Ricci tensor and scalar curvature for all the cases. Moreover we investigate some properties of semi invariant submanifolds of a Lorentzian Kenmotsu space form. We show that if a semi-invariant submanifold of a Lorentzian Kenmotsu space form M is totally geodesic, then M is an η -Einstein manifold. We consider sectional curvature of semi invariant product of a Lorentzian Kenmotsu manifolds.

Keywords: Lorentz Kenmotsu manifold, projective curvature tensor, \mathcal{L} -sectional curvature, semi invariant submanifold.

AMS 2010 codes: 53C15, 53C25, 53C40.

1 Introduction

Contact structure has most important applications in physics. Many authors gave their valuable and essential results on differential geometry [2], [7], [8]. Firstly contact manifolds were defined by Boothby and Wang [6]. In 1959, Gray defined almost contact manifold by the condition that the structural group of the tangent bundle is reducible to $U(n) \times 1$ [8]. Sasakian introduced Sasaki manifold, which is an almost contact manifold with a special kind a Riemannian metric [15]. Compared to that Sasakian manifolds have only recently become subject of deeper research in mathematics, mechanics and physics [3, 18]. To study manifolds with negative curvature, Bishop and O'Neill introduced the notion of warped product as a generalization of Riemannian product [4]. In 1960's and 1970's, when almost contact manifolds were studied as an odd dimensional counterpart of almost complex manifolds, the warped product was used to make examples of almost contact manifolds [18]. In addition, S. Tanno classified the connected $(2n + 1)$ dimensional almost contact manifold M whose automorphism group has maximum dimension $(n + 1)^2$ in [18]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c . Then there are three classes:

- i) $c > 0$, M is homogeneous Sasakian manifold of constant holomorphic sectional curvature.
- ii) $c = 0$, M is the global Riemannian product of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature.
- iii) $c < 0$, M is warped product space $\mathbb{R} \times_f \mathbb{C}^n$.

Kenmotsu obtained some tensorial equations to characterize manifolds of the third case.

In 1972, Kenmotsu abstracted the differential geometric properties of the third case. In [9], Kenmotsu studied a class of almost contact Riemannian manifold which satisfy the following two conditions,

$$\begin{aligned}(\nabla_X \varphi)Y &= -\eta(Y)\varphi X - g(X, \varphi Y)\xi \\ \nabla_X \xi &= X - \eta(X)\xi\end{aligned}\tag{1.1}$$

He showed normal an almost contact Riemannian manifold with (1.1) but not quasi Sasakian hence not Sasakian. He characterized warped product space $L \times_f \mathbb{C}E^n$ by an almost contact Riemannian manifold with (1.1). Moreover, he showed that every point of an almost contact Riemannian manifold with (1.1) has a neighborhood which is a warped $(-\varepsilon, \varepsilon) \times_f V$ where $f(t) = ce^t$ and V is Kähler.

In 1981, Janssens and Vanhecke [10], an almost contact metric manifold satisfying this (1.1) is called a Kenmotsu manifold. Some authors studied Kenmotsu manifold [1], [12], [13], [16], [19].

At the same time, in the year 1969, Takahashi [17] has introduced the Sasakian manifolds with Pseudo-Riemannian metric and prove that one can study the Lorentzian Sasakian structure with an indefinite metric. Furthermore, in 1990, K. L. Duggal [7] has initiated the space time manifolds with contact structure and analyzed the paper of Takahashi. In 1991, Roşca introduced Lorentzian Kenmotsu manifold [14].

Our aim in the present note is to extend the study of some properties curvature to the setting of a Lorentzian Kenmotsu manifold. We first review, in section 2, basic formula and definition of a Lorentzian Kenmotsu manifold. In section 3, we introduced \mathcal{L} -sectional curvature of Lorentzian Kenmotsu manifold. In section 4, we call semi invariant submanifold of Lorentzian Kenmotsu manifold. In section 5, we study semi invariant submanifold of Lorentzian Kenmotsu space form, In last section, we investigate semi invariant products of a Lorentzian Kenmotsu manifold.

2 Lorentzian Kenmotsu Manifolds

Let M be a real $(2n+1)$ -dimensional differentiable manifold endowed with an almost contact structure (φ, η, ξ) , where φ is a tensor field of type $(1, 1)$, η is a 1-form, and ξ is a vector field on M satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.\tag{2.1}$$

then M is called an *almost contact manifold*. It follows that $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, $\text{rank } \varphi = 2n$. If there exists a semi-Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y), \quad g(\xi, \xi) = \varepsilon = -1\tag{2.2}$$

then (φ, η, ξ, g) is called a *Lorentzian almost contact structure* and M is said to be a *Lorentzian almost contact manifold*.

For a Lorentzian almost contact manifold we also have $\eta(X) = \varepsilon g(X, \xi)$. We note that ξ is neither a light-like nor a spacelike vector fields on M . We note that ξ is the time-like vector field. We consider a local basis $\{e_1, \dots, e_{2n}, \xi\}$ in TM i.e.

$$g(e_i, e_j) = \delta_{ij} \text{ and } g(\xi, \xi) = -1$$

that is e_1, \dots, e_{2n} are spacelike vector fields.

Then a 2-form Φ is defined by $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(TM)$, called the *fundamental 2-form*. Moreover, a Lorentzian almost contact manifold is *normal* if

$$N = [\varphi, \varphi] + 2d\eta \otimes \xi = 0$$

where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ .

Definition 2.1. Let M be a Lorentzian almost contact manifold of dimension $(2n+1)$, with (φ, ξ, η, g) . M is said to be a Lorentzian almost Kenmotsu manifold if 1-form η is closed and $d\Phi = -2\eta \wedge \Phi$. A normal Lorentzian almost Kenmotsu manifold M is called a Lorentzian Kenmotsu manifold [14].

Theorem 2.1. Let $(M, \varphi, \xi, \eta, g)$ be a Lorentzian almost contact manifold. M is a Lorentzian Kenmotsu manifold if and only if

$$(\nabla_X \varphi)Y = -g(\varphi X, Y)\xi + \eta(Y)\varphi X \quad (2.3)$$

for all $X, Y \in \Gamma(TM)$, where ∇ is Levi-Civita connection on M [14].

Corollary 2.1. Let M be $(2n+1)$ -dimensional a Lorentzian Kenmotsu manifold with structure (φ, ξ, η, g) . Then we have

$$\nabla_X \xi = -\varphi^2 X \quad (2.4)$$

for all $X \in \Gamma(TM)$ [14].

Let $K(X_p, Y_p)$ be the sectional curvature for 2-plane spanned by X_p and Y_p , $p \in M$. M is said to have constant φ -holomorphic sectional curvature if $K(X_p, \varphi X_p)$ is constant for any point p and for any unit vector $X_p \neq 0$ such that $\eta(X_p) = 0$.

A Lorentzian Kenmotsu manifold is said to be a Lorentzian Kenmotsu space form if it has constant φ -holomorphic section curvature c and then, it is denoted by $M(c)$. The curvature tensor field R of $M(c)$ is given by,

$$\begin{aligned} R(X, Y, Z, W) = & \frac{c+3}{4} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ & + \frac{c-1}{4} \{g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) - 2g(\varphi X, Y)g(\varphi Z, W) \\ & + g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z)\}. \end{aligned} \quad (2.5)$$

where $X, Y, Z, W \in \Gamma(TM)$.

By virtue of (2.5), we have the following proposition.

Proposition 2.2. A Lorentzian Kenmotsu manifold of constant φ -holomorphic sectional curvature cannot be flat manifold.

Also, the Ricci curvature of M is given by

$$S(X, Y) = \sum_{i=1}^{2n+1} R(E_i, X, Y, E_i),$$

for $X, Y \in \Gamma(TM)$. Then from (2.5) on $M(c)$, we have,

$$S(X, Y) = \frac{(c-3)n + (c+1)}{2} g(\varphi X, \varphi Y) - 2n\eta(X)\eta(Y) \quad (2.6)$$

for all $X, Y \in \Gamma(TM)$.

Proposition 2.3. A Lorentzian Kenmotsu manifold of constant φ -holomorphic sectional curvature cannot be η -Einstein manifold.

3 \mathcal{L} -sectional curvature of Lorentzian Kenmotsu manifold

Let M be Lorentzian Kenmotsu manifold. Therefore, TM splits into two complementary subbundles $Im\phi$ (whose differentiable distribution is usually denoted by \mathcal{L}) and $\ker\phi$ (whose differentiable distribution is usually denoted by \mathcal{M}). The sectional curvature of planar sections spanned by vector fields of \mathcal{L} called \mathcal{L} -sectional curvature.

In what follows, we denote by \mathcal{M} the distribution spanned by the structure vector field ξ and by \mathcal{L} its orthogonal complementary distribution. Then we have,

$$TM = \mathcal{L} \oplus \mathcal{M}.$$

If $X \in \mathcal{M}$ we have $\phi X = 0$ and if $X \in \mathcal{L}$ we have $\eta(X) = 0$, that is, $\phi^2 X = -X$.

From (2.5) the \mathcal{L} -sectional curvature of Lorentzian Kenmotsu space form is given by

$$K_{\mathcal{L}}(X, Y) = \frac{c-3}{4} + 3\frac{c+1}{4}g(X, \phi Y)^2 \quad (3.1)$$

Corollary 3.1. *Let M be Lorentzian Kenmotsu space form. If \mathcal{L} -sectional curvature $K_{\mathcal{L}}$ is constant equal to c , then $c = -1$.*

Proof. We can chose X and Y such that $g(X, \phi Y) = 0$. Thus, from (3.1) we deduce

$$c = \frac{c-3}{4} \Rightarrow c = -1.$$

□

Corollary 3.2. *Let M be Lorentzian Kenmotsu manifold and $X, Y \in \mathcal{L}$. In this case, the scalar curvature of M is*

$$\tau = -n(2n+1).$$

Proposition 3.1. *Let M be Lorentzian Kenmotsu manifold and $X, Y \in \mathcal{L}$. Then M is Einstein manifold.*

Proof. For all $X, Y \in \mathcal{L}$, using (2.6), we can proof that M is Einstein manifold. □

4 Semi Invariant Submanifolds of a Lorentzian Kenmotsu Space Form

Definition 4.1. *An $(2m+1)$ -dimensional Riemannian submanifold M of Lorentzian Kenmotsu space form \overline{M} is called a semi invariant submanifold if ξ is tangent to M and there exists on M two differentiable distributions D and D^\perp on M satisfying:*

- (i) $TM = D \oplus D^\perp \oplus sp\{\xi\}$;
- (ii) The distribution D is invariant under ϕ , that is $\phi D_x = D_x$ for any $x \in M$;
- (iii) The distribution D^\perp is anti-invariant under ϕ , that is, $\phi D_x^\perp \subseteq T_x^\perp M$ for any $x \in M$, where $T_x M$ and $T_x^\perp M$ are the tangent space of M at x .

Now, we choose a local field of orthonormal frames $\{E_1, \dots, E_{2p}, E_{2p+1}, \dots, E_{2m}, \xi\}$ on M . Then we have,

$$D = sp\{E_1, \dots, E_{2p}\}, \quad D^\perp = sp\{E_{2p+1}, \dots, E_{2m}\} \quad (4.1)$$

where $\dim D = 2p$ and $\dim D^\perp = q$.

Then if $p = 0$ we have an anti-invariant submanifold tangent to ξ and if $q = 0$, we have an invariant submanifold. Now, we give the following example.

Example 4.1. In what follows, $(\mathbb{R}^{2n+1}, \varphi, \eta, \xi, g)$ will denote the manifold \mathbb{R}^{2n+1} with its usual Lorentzian Kenmotsu structure given by

$$\begin{aligned} \eta &= dz, \quad \xi = \frac{\partial}{\partial z} \\ \varphi\left(\sum_{i=1}^n \left(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right) + Z \frac{\partial}{\partial z}\right) &= \sum_{i=1}^n \left(Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}\right) + \sum_{i=1}^n Y_i Y_i \frac{\partial}{\partial z} \\ g &= e^{-2z} \left(\sum_{i=1}^n dx_i \otimes dx_i + dy_i \otimes dy_i\right) - \varepsilon dz \otimes dz \end{aligned}$$

$(x_1, \dots, x_n, y_1, \dots, y_n, z)$ denoting the Cartesian coordinates on \mathbb{R}^{2n+1} . The consider a submanifold of \mathbb{R}^7 defined by

$$M = X(u, v, k, l, t) = (u, k, 0, v, 0, l, t).$$

Then local frame of TM is given by

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial y_1}, \\ e_3 &= \frac{\partial}{\partial x_2}, \quad e_4 = \frac{\partial}{\partial y_2}, \\ e_5 &= \frac{\partial}{\partial z} = \xi \end{aligned}$$

and we have

$$e_1^* = \frac{\partial}{\partial x_3}, \quad e_2^* = \frac{\partial}{\partial y_2}$$

which are the a basis of $T^\perp M$. We determine $D_1 = sp\{e_1, e_2\}$ and $D_2 = sp\{e_3, e_4\}$. Then D_1, D_2 are invariant and anti-invariant distribution, respectively. Thus $TM = D_1 \oplus D_2 \oplus sp\{\xi\}$ is a semi invariant submanifold of \mathbb{R}^7 .

Let $\bar{\nabla}$ be the Levi-Civita connection of \bar{M} with respect to the g . Then Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (4.2)$$

$$\bar{\nabla}_X N = \nabla_X^\perp N - A_N X \quad (4.3)$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$. ∇^\perp is the connection in the normal bundle, h is the second fundamental form of \bar{M} and A_N is the Weingarten endomorphism associated with N . The second fundamental form h and the shape operator A are related with by

$$g(h(X, Y), N) = g(A_N X, Y). \quad (4.4)$$

Let M be semi invariant submanifold of \bar{M} . M is said to be totally geodesic if $h(X, Y) = 0$, for any $X, Y \in \Gamma(TM)$.

We denote by \bar{R} and R the curvature tensor fields associated with $\bar{\nabla}$ and ∇ respectively. The Gauss equation is given by

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \quad (4.5)$$

for all $X, Y, Z, W \in \Gamma(TM)$.

On the other hand, let M be a semi invariant submanifold of a Lorentzian Kenmotsu space form \bar{M} . Then using (2.5) and (4.5), a semi invariant submanifold M has constant ϕ -sectional curvature c if and only if the Riemannian curvature tensor \bar{R} satisfied

$$\begin{aligned} R(X, Y, Z, W) = & \frac{c+3}{4} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\ & + \frac{c-1}{4} \{g(\phi X, W)g(\phi Y, Z) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W) \\ & + g(X, Z)\eta(Y)\eta(W) - g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z)\} \\ & + g(h(X, W), h(Y, Z)) - g(h(Y, W), h(X, Z)). \end{aligned} \quad (4.6)$$

Theorem 4.1. Let M be a semi-invariant submanifold of a Lorentzian Kenmotsu space from $\bar{M}(c)$. Then we get Ricci tensor of M ,

$$\begin{aligned} S(X, Y) = & \left\{ \frac{c+3}{4}(p+q-3) + 3\frac{c-1}{2} \right\} g(X, Y) \\ & - \left\{ \frac{c-1}{4}(p+q-6) + \frac{c+1}{2} \right\} \eta(X)\eta(Y) \\ & + \sum_{i=1}^{p+q} \{g(h(X, Y), h(E_i, E_i)) - g(h(E_i, Y), h(X, E_i))\} \end{aligned} \quad (4.7)$$

for all $X, Y \in \Gamma(TM)$.

Proof. Let $\Gamma(TM) = sp\{e_1, \dots, e_p, e_{p+1}, \dots, e_q, e_{p+q+1}\}$ such that $\{e_1, \dots, e_p\}$ are tangent to D_1 and $\{e_{p+1}, \dots, e_q\}$ are tangent to D_2 . Then we have,

$$S(X, Y) = \sum_{i=1}^p R(X, E_i, E_i, Y) + \sum_{i=p+1}^q R(X, E_i, E_i, Y) + R(X, \xi, \xi, Y).$$

Now, using (4.6), we get

$$\begin{aligned} S(X, Y) = & \left\{ \frac{c+3}{4}(p-1) + 3\frac{c-1}{4} \right\} g(X, Y) + \frac{c-1}{4}(3-p)\eta(X)\eta(Y) \\ & + \sum_{i=1}^p \{g(h(X, Y), h(E_i, E_i)) - g(h(E_i, Y), h(X, E_i))\} \\ & + \left\{ \frac{c+3}{4}(q-1) + 3\frac{c-1}{4} \right\} g(X, Y) + \frac{c-1}{4}(3-q)\eta(X)\eta(Y) \\ & + \sum_{i=p+1}^q \{g(h(X, Y), h(E_i, E_i)) - g(h(E_i, Y), h(X, E_i))\} \\ & - \frac{c+3}{4}g(X, Y) - \left\{ \frac{c+3}{4} + \frac{c-1}{4} \right\} \eta(X)\eta(Y) \end{aligned}$$

which gives proof. □

Corollary 4.1. Let M be a semi-invariant submanifold of a Lorentzian Kenmotsu space from $\bar{M}(c)$. If M is totally geodesic, then M is an η -Einstein manifold.

Proposition 4.2. Let M be a semi-invariant submanifold of a Lorentzian Kenmotsu space from $\bar{M}(c)$. Then we have scalar curvature

$$\begin{aligned} \tau = & \left\{ \frac{c+3}{4}(p+q-3) + 3\frac{c-1}{2} \right\} (p+q-1) \\ & + \frac{c-1}{4}(p+q-6) + \frac{c+1}{2} + \frac{1}{(p+q+1)^2} \|H\|^2 + \|h\|^2. \end{aligned}$$

Proof. From (4.7) by using $X = Y = e_k$ we get

$$\tau = \sum_{k=1}^{p+q+1} S(e_k, e_k).$$

The proof is completed. \square

Proposition 4.3. *Let M be semi invariant submanifold of Lorentzian Kenmotsu space form $\overline{M}(c)$. Then*

$$R(X, Y, Z, W) = \frac{c+3}{4} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} + g(h(X, W), h(Y, Z)) - g(h(Y, W), g(X, Z)) \quad (4.8)$$

for all $X, Y, Z, W \in \Gamma(D^\perp)$.

Proof. Using (4.6). For all $X, Y, Z, W \in \Gamma(D^\perp)$, since $\phi X, \phi Y, \phi Z, \phi W \in \phi D^\perp \subset TM^\perp$ we have (4.8). \square

Corollary 4.2. *Let M be semi invariant submanifold of Lorentzian Kenmotsu space form $\overline{M}(c)$. If D^\perp is totally geodesic, then D^\perp is flat if and only if $c = -3$.*

Proposition 4.4. *Let M be semi invariant submanifold of Lorentzian Kenmotsu space form $\overline{M}(c)$. Then*

$$S(X, Y) = \frac{c+3}{4}(q-1)g(X, Y) + \sum_{i=1}^q \{g(h(X, Y), h(E_i, E_i)) - g(h(E_i, Y), h(X, E_i))\} \quad (4.9)$$

for all $X, Y \in \Gamma(D^\perp)$, where S is Ricci tensor.

Proof. Using (4.8). From $S(X, Y) = \sum_{i=1}^q R(X, E_i, E_i, Y)$, for all $X, Y \in \Gamma(D^\perp)$, we have equation (4.9). \square

Corollary 4.3. *Let M be a semi-invariant submanifold of a Lorentzian Kenmotsu space from \overline{M} . If D^\perp is totally geodesic, then distribution D^\perp is Einstein.*

Corollary 4.4. *Let M be semi inavariant submanifold of Lorentzian Kenmotsu space form $\overline{M}(c)$. If D^\perp is totally geodesic, then*

$$\tau_{D^\perp} = \frac{c+3}{4}q(q-1)$$

where τ is the scalar curvature.

Proposition 4.5. *Let M be semi invariant submanifold of Lorentzian Kenmotsu space form $\overline{M}(c)$. Then the Ricci curvature determined by D*

$$S(X, Y) = \left\{ \frac{c+3}{4}(p-1) + 3\frac{c-1}{4} \right\} g(X, Y)$$

for all $X, Y \in \Gamma(D)$.

Proof. For all $X, Y \in \Gamma(D)$, from (4.8) we have

$$R(X, Y, Z, W) = \frac{c+3}{4} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} + \frac{c-1}{4} \{g(X, \phi W)g(Y, \phi Z) - g(X, \phi Z)g(Y, \phi W)\}$$

Then, from $S(X, Y) = \sum_{i=1}^p R(X, E_i, E_i, Y)$, using last equation, this complates the proof. \square

Corollary 4.5. Let M be semi invariant submanifold of Lorentzian Kenmotsu space form $\overline{M}(c)$. Then the scalar curvature determined by D is given

$$\tau_D = p \frac{(c+3)2p-1+3(c-1)}{4}.$$

Corollary 4.6. Let M be a semi-invariant submanifold of a Lorentzian Kenmotsu space from \overline{M} . If D is totally geodesic, then distribution D is Einstein.

Theorem 4.6. Let M be semi invariant submanifold of Lorentzian Kenmotsu space form $\overline{M}(c)$. Then, ϕ -sectional curvature of D is $-c$ if and only if D is totally geodesic.

Proof. Using (4.6). For all $X \in \Gamma(D)$,

$$\begin{aligned} R(X, \phi X, X, \phi X) &= \frac{c+3}{4} \{g(X, \phi X)g(\phi X, X) - g(X, X)g(\phi X, \phi X)\} \\ &\quad + \frac{c-1}{4} \{g(\phi X, \phi X)g(\phi^2 X, X) - g(\phi X, X)g(\phi^2 X, \phi X) \\ &\quad - 2g(\phi X, \phi X)g(\phi X, \phi X) \\ &\quad + g(h(X, \phi X), h(\phi X, X)) - g(h(\phi X, \phi X), h(X, X))\}. \end{aligned}$$

Then,

$$R(X, \phi X, X, \phi X) = -c - 2 \|h(X, X)\|^2.$$

□

5 Semi Invariant Product in a Lorentzian Kenmotsu Space Form

Let M be a semi invariant submanifold of a Lorentzian Kenmotsu space form \overline{M} . We say that M is a semi invariant product if the distribution $D \oplus sp\{\xi\}$ is integrable and locally M is a Riemannian product $M_1 \times M_2$, where M_1 (resp. M_2) is leaf of $D \oplus sp\{\xi\}$ (resp. D^\perp). If we have $pq \neq 0$, we say that M is a proper semi invariant product.

Theorem 5.1. Let M be a proper semi invariant product of a Lorentzian Kenmotsu space form $\overline{M}(c)$. Then

$$R(X, \phi X, Z, \phi Z) = 2(\|h(X, Z)\|^2 - \frac{c-1}{4}) \quad (5.1)$$

for any unit vector fields $X \in D$ and $Z \in D^\perp$.

Proof. Using (4.6) and $\phi Z \in \Gamma(\phi D^\perp) \subset TM^\perp$ this completes the proof. □

Theorem 5.2. Let M be a proper semi invariant product of a Lorentzian Kenmotsu space form $\overline{M}(c)$. Then,

$$\|h\|^2 \geq pq(1-c) + 2qp \quad (5.2)$$

Proof. Since h is fundamental form, we have

$$\begin{aligned} \|h\|^2 &= \sum_{i,j=1}^{2p} \|h(E_i, E_j)\|^2 + \sum_{k,l=2p+1}^{2m} \|h(E_k, E_l)\|^2 \\ &\quad + 2 \sum_{i=1}^{2p} \sum_{k=2p+1}^{2m} \|h(E_i, E_k)\|^2 + 2 \sum_{k=2p+1}^{2m} \|h(E_k, \xi)\|^2 \end{aligned}$$

from (4.1)

$$\|h\|^2 = pq(1-c) + 2qp + \sum_{i,j=1}^{2p} \|h(E_i, E_j)\|^2 + \sum_{k,l=2p+1}^{2m} \|h(E_k, E_l)\|^2$$

which gives (5.2). \square

Proposition 5.3. *Let M be a proper semi invariant product of an a Lorentzian Kenmotsu space form $\overline{M}(c)$. Then,*

$$R(X, Y, Z, W) = 0$$

for all $X, Y \in \Gamma(D \oplus sp\{\xi\})$ and $Z, W \in \Gamma(D^\perp)$.

Proof. Let M be semi invariant submanifold of Lorentzian Kenmotsu manifold \overline{M} . Then for all $Z, W \in \Gamma(D^\perp)$,

$$\phi Z, \phi W \in \phi D^\perp \subset TM^\perp.$$

Using (4.6), which completes the proof. \square

Proposition 5.4. *Let M be a proper semi invariant product of a Lorentzian Kenmotsu \overline{M} . Then*

$$R(X, Y, Z, W) = 0$$

for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp \oplus sp\{\xi\})$.

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