Applied Mathematics and Nonlinear Sciences

Second Binormal Motions of Inextensible Curves in 4-dimensional Galilean Space

Fatma Bulut Korkmaz[1], Mehmet Bektaş[2]

[1] SDepartment of Mathematics, Faculty of Arts and Sciences, Bitlis Eren University, 13000 Bitlis, Turkiye,
E-mail: fbulut@beu.edu.tr

[2] Department of Mathematics, Faculty of Science, Fırat University, 23119 Elazığ, Turkiye,
E-mail: mbektas@firat.edu.tr

Abstract

In our study, we give the associated evolution equations for curvature and torsion as a system of partial differential equations. In addition, we study second binormal motions of inextensible curves in 4-dimensional Galilean space.

1 Introduction

Galilean 3-space G_3 is simply defined as a Klein geometry of the product space $\mathbb{R}x\mathbb{E}^2$ whose symmetry group is Galilean transformation group which has an important place in classical and modern physics. It is well known that the idea of world lines originates in physics and was pioneered by Einstein.

In the recent time, the study of the motion of inextensible curves has been arisen in a number of diverse engineering applications and has many applications in computer vision, snake-like, robots. The flow of a curve is said to be inextensible if the arc length is preserved.

Nowadays, many important and intensive studies are seen about inextensible curve flows in the different space. Physically, inextensible curve flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of a physical applications. For example, both Chirikjian and Burdick [1] used novel and efficient kinematic modeling techniques for "hyper-redundant" robots that to determining the time varying backbone curve behavior and Mochiyama et al.

Inextensible flows of curves are studied surfaces in $\mathbb{R}^3$ by Kwon in [2]. In addition, many researchers have studied on inextensible flows such as [3], [4] and [8]. In [7] and [19], the authors studied inextensible flows in

Minkowski space-time E_1^4. In [10] is obtained a theoretical framework for controlling a manipulator with hyper degrees of freedom in which the shape control of hyper-redundant, or snake-like robots.

In [15], the generalization of Bertrand curves in Galilean 4-space is introduced and the characterization of the generalized Bertrand curves is obtained. In [16], the author constructed Frenet-Serret frame of a curve in the Galilean 4-space and obtained the mentioned curve's Frenet-Serret equations. Inextensible curve and surface flows also arise in the context of many problems in computer vision [6], [12] and computer animation [9], and even structural mechanics [17]. Papers in [5], [11], [13], [14] and [18] are obtained some new characterizations of the inextensible curve flows curves. By drawing inspiration from them, in this paper, we consider the second binormal motions of inextensible curves in Galilean 4-space. We give the necessary and sufficient conditions to be inextensible flows and we find the evolution equations for the inextensible curve flows in 4-dimensional Galilean Space.

2 Geometric Preliminaries

The scalar product of two vectors $\vec{U} = (u_1, u_2, u_3, u_4)$ and $\vec{V} = (v_1, v_2, v_3, v_4)$ in G_4 is defined by

$$< \overrightarrow{U}, \overrightarrow{V} > = \begin{cases} u_1 v_1, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0 \\ u_2 v_2 + u_3 v_3 + u_4 v_4, & \text{if } u_1 = 0, v_1 = 0 \end{cases}. \tag{2.1}$$

The Galilean cross product in G_4 for the vectors $\vec{u} = (u_1, u_2, u_3, u_4), \vec{v} = (v_1, v_2, v_3, v_4)$, and $\vec{w} = (w_1, w_2, w_3, w_4)$ is defined by

$$\vec{u} \wedge \vec{v} \wedge \vec{w} = - \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 \end{vmatrix}, \tag{2.2}$$

where e_i $1 \leq i \leq 4$, are the standard basis vectors.

The norm of vector $\vec{U} = (u_1, u_2, u_3, u_4)$ is defined by

$$\|\vec{U}\| = \sqrt{|<\vec{U}, \vec{U}>|} \text{ [16]}.$$

Let $\alpha : I \subset R \to G_4$, $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arclength s in G_4. .The first vector of the Frenet-Serret frame,that is,the tangent vector of α is defined by

$$\mathbf{t} = \alpha'(s) = (1, y'(s), z'(s), w'(s)). \tag{2.3}$$

Since $\mathbf{t}$ is a unit vector, we can express

$$< \mathbf{t}, \mathbf{t} > = 1. \tag{2.4}$$

Differentiating (2.4) with respect to s, we have

$$< \mathbf{t}', \mathbf{t} > = 0. \tag{2.5}$$

The vector function $\mathbf{t}'$ gives us the rotation measurement of the curve α. The real valued function

$$\kappa(s) = \|\mathbf{t}'(s)\| = \sqrt{(y'')^2 + (z'')^2 + (\omega'')^2} \tag{2.6}$$

is called the first curvature of the curve α. We assume that $\kappa(s) \neq 0$ for all $s \in l$. Similar to space G_3, the principal vector is defined by

$$\mathbf{n}(s) = \frac{\mathbf{t}'(s)}{\kappa(s)};$$

in other words

$$\mathbf{n}(s) = \frac{1}{\kappa(s)} \left(0, \ y''(s), \ z''(s), \ \omega''(s) \right). \tag{2.7}$$

By the aid of the differentiation of the principal normal vector given in (2.7), define the second curvature function that is defined by

$$\tau(s) = \|\mathbf{n}'(s)\|.$$

This real valued function is called torsion of the curve α. The third vector field, namely, binormal vector field of the curve α, is defined by

$$\mathbf{b}(s) = \frac{1}{\tau(s)} \left(0, \left(\frac{y''(s)}{\kappa(s)} \right)', \left(\frac{z''(s)}{\kappa(s)} \right)', \left(\frac{\omega''(s)}{\kappa(s)} \right)' \right).$$

Thus the vector $\mathbf{b}(s)$ is perpendicular to both $\mathbf{t}$ and $\mathbf{n}$. The fourth unit vector is defined by

$$\mathbf{e}(s) = \mu \mathbf{t}(s) \wedge \mathbf{n}(s) \wedge \mathbf{b}(s).$$

Here the coefficient μ is taken ± 1 to make $+1$ determinant of the matrix $[\mathbf{t}, \mathbf{n}, \mathbf{b}, \mathbf{e}]$. The third curvature of the curve α by the Galilean inner product is defined by

$$\sigma = \langle \mathbf{b}', \mathbf{e} \rangle. \tag{2.8}$$

Here, as well known, the set $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \mathbf{e}, \kappa, \tau, \sigma\}$ is called the Frenet-Serret apparatus of the curve α. We know that the vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}, \mathbf{e}\}$ are mutually orthogonal vectors satisfying

$$\langle \mathbf{t}, \mathbf{t} \rangle = \langle \mathbf{n}, \mathbf{n} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle = \langle \mathbf{e}, \mathbf{e} \rangle = 1,$$
$$\langle \mathbf{t}, \mathbf{n} \rangle = \langle \mathbf{t}, \mathbf{b} \rangle = \langle \mathbf{t}, \mathbf{e} \rangle = \langle \mathbf{n}, \mathbf{b} \rangle = \langle \mathbf{n}, \mathbf{e} \rangle = \langle \mathbf{b}, \mathbf{e} \rangle = 0.$$

For the curve α in G_4, we have the following Frenet-Serret equations [16]:

$$\mathbf{t}' = \kappa(s)\mathbf{n}(s),$$
$$\mathbf{n}' = \tau(s)\mathbf{b}(s),$$
$$\mathbf{b}' = -\tau(s)\mathbf{n}(s) + \sigma(s)\mathbf{e}(s),$$
$$\mathbf{e}' = -\sigma(s)\mathbf{b}(s).$$

3 Second Binormal motions of curves in the four-dimensional Galilean space G_4.

Let $\hat{\gamma}_0 : I \longrightarrow G_4$, be a regular curve in four-dimensional Galilean space G_4.

Consider the family of curves $C_t : \hat{\gamma}(\hat{s}, t)$, where $\hat{\gamma}(\hat{s}, t) : I \subset x \in [0, \infty) \to G_4$, with intial curve $\hat{\gamma}_0 = \hat{\gamma}(\hat{s}, 0)$. Let $\hat{\gamma}(\hat{s}, t)$ be the position vector of a point on the curve at the time t and at the arc length $\hat{s}$. The time parameter t is the parameter for the deformation C_t of the curve. The arc-length of the curve is defined by

$$\hat{s}(\hat{u}, t) = \int_0^{\hat{u}} \sqrt{g(\hat{\sigma}, t)} \, d\hat{\sigma}.$$

where $\sqrt{g} = \|\hat{\gamma}(\hat{\sigma}, t)\|$. Then the element of the arc-length is $d\hat{s} = \sqrt{g(\hat{u}, t)} d\hat{u}$, and the operator $\frac{\partial}{\partial \hat{s}}$ satisfies the following

$$\frac{\partial}{\partial \hat{s}} = \frac{1}{\sqrt{g}} \frac{1}{\partial \hat{u}} \ , \quad \frac{\partial \hat{s}}{\partial \hat{u}} = \sqrt{g}.$$

Definition 3.1. The curve $\hat{\gamma}(\hat{s},t)$ and its flow $\frac{\partial \hat{\gamma}(\hat{s},t)}{\partial t}$ in four-dimensional Galilean space G_4 are said to be inextensible if

$$\dot{s} = \frac{\partial}{\partial t}\|\hat{\gamma}(\dot{\hat{s}},t)\| = 0, \ i.e. \ g_t \ = 0 \ [14].$$
(3.1)

Hence, the arclength of curve $\hat{\gamma}(\hat{s},t)$ is preserved.

The second binormal motions of the curves can be expressed by the velocity vector field

$$\hat{\gamma_t} = \frac{\partial \hat{\gamma}}{\partial t} = v\,\mathbf{e}.$$
(3.2)

where $\{\mathbf{t},\mathbf{n},\mathbf{b},\mathbf{e}\}$ is the orthonormal Frenet Frame to the curve C_t and v is the velocity vector in the direction of second binormal vector $\mathbf{e}$ and it is a function of curvature $\hat{k}(\hat{s},t)$, torsion $\hat{\tau}(\hat{s},t)$.

Theorem 3.1 The time evolution equations of the curvature and torsion for the inextensible timelike curve $\hat{C}_t$ are given by:

$$k_t = v\sigma\tau$$
(3.3)

and

$$\tau_t = \psi_s - \phi\sigma$$
(3.4)

where $\varphi = \frac{[-2v_s\sigma - v\sigma]}{k}$ and $\phi = \frac{[v_{ss} - v\sigma^2]}{k}$.

Proof. Take the $\hat{u}$ derivative of (3.2), then

$$\hat{\gamma_{tu}} = \frac{\partial v}{\partial s}\frac{\partial s}{\partial u}\mathbf{e} + v\frac{\partial \mathbf{e}}{\partial s}\frac{\partial s}{\partial u} = \sqrt{g}(v_s\mathbf{e} - v\sigma\mathbf{b}).$$
(3.5)

Since

$$\hat{\gamma_u} = \frac{\partial \hat{\gamma}}{\partial s}\frac{\partial s}{\partial u} = \sqrt{g}\mathbf{t}$$

then

$$\hat{\gamma_{ut}} = \mathbf{t}_t\sqrt{g} + \frac{g_t}{2\sqrt{g}}\mathbf{t}.$$
(3.6)

Since the derivatives with respect to u and t commute, then

$$\hat{\gamma_{ut}} = \hat{\gamma_{tu}}.$$
(3.7)

Substituting from (3.7) and (3.5) into (3.6), then

$$\mathbf{g}_t = u,$$
$$\mathbf{t}_t = v_s\mathbf{e} - v\sigma\mathbf{b}.$$
(3.8)

Take the u derivative of the second equation of (3.8), then

$$\mathbf{t}_{tu} == \sqrt{g}(v\sigma\tau\mathbf{n} + (-2v_s\sigma - v\sigma)\mathbf{b} + (v_{ss} - v\sigma^2)\mathbf{e}).$$
(3.9)

Since

$$\mathbf{t}_u = \frac{\partial \mathbf{t}}{\partial s}\frac{\partial s}{\partial u} = k\mathbf{n}\sqrt{g}.$$
(3.9)

Taking the t derivative of (3.9), then we have

$$\mathbf{t}_{ut} = \sqrt{g}(k_t\mathbf{n} + k\mathbf{n}_t).$$
(3.10)

Since $\mathbf{t}_{ut} = \mathbf{t}_{tu}$, we obtain

$$k_t = v\sigma\tau.$$

On the other hand, we write

$$k\mathbf{n}_t = (-2v_s\sigma - v\sigma)\mathbf{b} + (v_{ss} - v\sigma^2)\mathbf{e}$$

or

$$\mathbf{n}_t = \frac{1}{k}\left[(-2v_s\sigma - v\sigma)\mathbf{b} + (v_{ss} - v\sigma^2)\mathbf{e}\right]. \tag{3.11}$$

If we choose $\varphi = \dfrac{[-2v_s\sigma - v\sigma]}{k}$ and $\phi = \dfrac{[v_{ss} - v\sigma^2]}{k}$, then we rewrite the equation of (3.11) as following

$$\mathbf{n}_t = \psi\mathbf{b} + \phi\mathbf{e}. \tag{3.12}$$

Taking the u derivative of (3.12), then we have

$$\mathbf{n}_{tu} = \sqrt{g}\left((\psi_s - \phi\sigma)\mathbf{b} + (-\tau\psi)n + (\psi\sigma + \phi_s)\mathbf{e}\right). \tag{3.13}$$

Since

$$\mathbf{n}_u = \frac{\partial\mathbf{n}}{\partial s}\frac{\partial s}{\partial u} = \sqrt{g}\tau\mathbf{b} \tag{3.14}$$

Taking the t derivative of (3.14), then we have

$$\mathbf{n}_{ut} = \sqrt{g}(\tau_t\mathbf{b} + \tau\mathbf{b}_t). \tag{3.15}$$

From (3.13) and (3.15), then we have

$$\tau_t = \psi_s - \phi\sigma. \tag{3.16}$$

Thus

$$\tau\mathbf{b}_t = (-\tau\psi)\mathbf{n} + (\psi\sigma + \phi_s)\mathbf{e}. \tag{3.17}$$

Since

$$\mathbf{b}_t = -\psi\mathbf{n} + \left(\frac{\psi\sigma + \phi_s}{\tau}\right)\mathbf{e}. \tag{3.18}$$

If we take $K = \dfrac{\psi\sigma + \phi_s}{\tau}$, then the time evolution equation for the first binormal vector $\mathbf{b}$ to the curve $\hat{C}_t$ is given as follows:

$$b_t = -\psi\mathbf{n} + K\mathbf{e}. \tag{3.19}$$

Taking the u derivative of (3.19), then we have

$$\mathbf{b}_{tu} = \sqrt{g}(-(\psi + K)\tau\mathbf{b} - \psi_s\mathbf{n} + K_s\mathbf{e}). \tag{3.20}$$

Since

$$\mathbf{b}_u = \frac{\partial\mathbf{b}}{\partial s}\frac{\partial s}{\partial u} = \sqrt{g}(-\tau\mathbf{n} + \sigma\mathbf{e}). \tag{3.21}$$

Taking the t derivative of (3.21), then we have

$$\mathbf{b}_{ut} = \sqrt{g}(-\tau_t\mathbf{n} - \tau\mathbf{n}_t + \sigma_t\mathbf{e} + \sigma\mathbf{e}_t). \tag{3.22}$$

Since $b_{tu} = b_{ut}$, then by substituting from (3.20) and from (3.22) into this equation, then we have the time evolution equation for the torsion τ_t:

$$\tau_t = \psi_s \text{ and } K_s = \sigma_t. \tag{3.23}$$

Thus

$$-(\psi + K)\tau\mathbf{b} = -\tau\mathbf{n}_t + \sigma\mathbf{e}_t$$

also

$$\sigma \mathbf{e}_t = \tau \mathbf{n}_t - (\psi + K)\tau \mathbf{b}.$$

or

$$\mathbf{e}_t = -\frac{\tau}{\sigma} K \mathbf{b} + \frac{\tau}{\sigma} \phi \mathbf{e}$$

where τ second curvature function, σ third curvature function, $K = \dfrac{\psi\sigma + \phi_s}{\tau}$, $\psi = \dfrac{-2v_s\sigma - v\sigma}{K}$ and $\phi = \dfrac{v_{ss} - v\sigma^2}{K}$. Hence, the theorem holds.

Theorem 3.2 The time evolution of the Serret Frenet frame for the curve can be given in matrix form:

$$\begin{pmatrix} t \\ n \\ b \\ e \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & -v\sigma & v_s \\ 0 & 0 & \psi & \phi \\ 0 & -\psi & 0 & K \\ 0 & 0 & -\dfrac{\tau}{\sigma}K & \dfrac{\tau}{\sigma}\phi \end{pmatrix} \begin{pmatrix} t \\ n \\ b \\ e \end{pmatrix}.$$

References

[1] G. Chirikjian, J. Burdick, A modal approach to hyper-redundant manipulator kinematics,IEEE Trans. Robot. Autom. 10, 343–354,1994.

[2] D.Y. Kwon , F.C. Park and D.P.Chi, Inextensible flows of curves and developable surfaces, Appl. Math. Lett. 2005, 18, 1156–1162.

[3] M.Ergut, E.Turhan, T.Körpinar, Characterization of inextensible flows of spacelike curves with Sabban frame in S_1^2. Bol. Soc. Parana. Mat. Vol. 31 No. 2 (2013), 47–53.

[4] O.,Gökmen, M.Tosun and S.Ozkaldı Karakus , A note on inextensible flows of curves in E^n. Int. Electron. J. Geom. Vol. 6 No. 2 (2013), 118–124.

[5] R. A. Hussien and S. G. Mohamed, Generated Surfaces via Inextensible Flows of Curves in R^3, J. Appl. Math. Volume 2016, Article ID 6178961, 8 pages (2016), Article ID 6178961, 10.1155/2016/6178961.

[6] M. Kass, A. Witkin, D. Terzopoulos, Snakes: active contour models, in: Proc. 1st Int. Conference on Computer Vision, 259–268,1987.

[7] T. Körpinar, and E.Turhan A New Version of Inextensible Flows of Spacelike Curves with Timelike B2 in Minkowski Space-Time E_1^4, Differ. Equ. Dyn. Syst., 2013, 21 (3), 281–290.

[8] T. Körpinar, and E.Turhan, Approximation for inextensible flows of curves in E^3. Bol. Soc. Parana. Mat. Vol. (3) 32 No. 2 (2014), 45–54.

[9] M. Desbrun, M.P. Cani-Gascuel, Active implicit surface for animation, in: Proc. Graphics Interface Canadian Inf. Process. Soc., 143–150, 1998.

[10] H. Mochiyama, E. Shimemura, H. Kobayashi, Shape control of manipulators with hyper degrees of freedom, Int. J. Robot.Res., 18, 584–600, 1999.

[11] D.Latifi, A.Razavi A , Inextensible Flows of Curves in Minkowskian Space, Adv. Studies Theor. Phys. 2(16): 761-768, 2008.

[12] H.Q. Lu, J.S. Todhunter, T.W. Sze, Congruence conditions for nonplanar developable surfaces and their application to surface recognition, CVGIP, Image Underst. 56, 265–285, 1993.

[13] S. G. Mohamed, Inextensible Flows of Spacelike Curves in De-Sitter Space S_1^2, Appl. Math. Inf. Sci. Lett. 6, No. 2, 75-83 (2018).

[14] S. G. Mohamed, Binormal motions of inextensible Curves in de-sitter space S_1^2, Journal of Egyptian Mathematical Society 25 313-318, 2017.

[15] H. Öztekin, Special Bertrand Curves in 4D Galilean Space, Mathematical Problems in Engineering Volume 2014, Article ID 318458, 7 pages http://dx.doi.org/10.1155/2014/318458.

[16] S. Yilmaz,"Construction of the Frenet-Serret frame of a curve in 4D-Galilean space and some applications,"International Journal of Physical Sciences,vol.5,no.8,pp.1284–1289,2010.

[17] D.J. Unger, Developable surfaces in elastoplastic fracture mechanics, Int. J. Fract. 50, 33–38, 1991.

[18] M. Yeneroğlu, On new characterization of inextensible flows of space-like curves in de Sitter space, Open Mathematics, Vol14(1) , 2016.

[19] Z. Yüzbaşı and M. Bektaş , A Note on İnextensible Flows of Partially and Pseudo Null Curves In E_1^4. arXiv.org math arXiv:1303.2956v1