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Nonlinear sub-diffusion and nonlinear sub-diffusion dispersion equations and their proposed solutions

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## Abstract

Many investigations related to the analytical solutions of the nonlinear sub-diffusion equation exist. In this paper, we investigate the conditions under which the analytical and the approximate solutions of the nonlinear sub-diffusion equation and the nonlinear sub-advection dispersion equation exist. In other words, the problems of existence and uniqueness of the solutions the fractional diffusion equations have been addressed. We use the Banach fixed Theorem. After proving the existence and uniqueness, we propose the analytical and the approximate solutions of the nonlinear sub-diffusion, and the nonlinear sub-advection dispersion equations. We analyze the impact of the sub-diffusion coefficient, the advection coefficient and the dispersion coefficient in the diffusion processes. The homotopy perturbation Laplace transform method has been used in this paper. Some numerical examples are provided to illustrate the main results of the article.

**Keywords:** Nonlinear sub-advection dispersion equation; the nonlinear sub-diffusion equation; Approximate solution **AMS 2010 codes:** ???

## Lead paragraph

> The homotopy perturbation Laplace transform method has been described for solving the fractional differential equations,

▷ The existence and the uniqueness of the solution of the nonlinear sub-diffusion and the nonlinear sub-advection dispersion equations has been presented,

▷ The approximate solutions of the nonlinear sub-diffusion and the nonlinear sub-advection dispersion equations described by the integer order time derivative using homotopy perturbation Laplace transform method have been proposed,

▷ Some graphical representations of the approximate solutions of the nonlinear sub-diffusion and the nonlinear sub-advection dispersion equations have been provided to illustrate the main result of the paper.

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## 1 Introduction

In fluids mechanics, there exist two categories of processes. Transport process when the process moves substance through hydrosphere and atmosphere. Transformation process when the process changes the substance of interest into another substance. The diffusion process belongs to the first category, subject of research in this paper. The diffusion satisfies two properties: firstly, it is random in nature, and the transport is from the regions with high concentration to the areas with low concentrations. For a classic example, we have the diffusion of perfume into an empty room. The diffusion equations have attracted many researchers in these last two decades. The diffusion equation has been used in many areas of mathematics, statistics, probability and physics. The diffusion equations exist in the probability models, in the microscopic models and the mesoscopic models. There exist various type of diffusion equations: The Chapman Kolmogorov equation, the Fokker Plank equation (known as the diffusion dispersion equation or convection equation), and the particle diffusion equation obtained with Fickian law's.

There exist many investigations related to the diffusion and fractional diffusion equations. The first works were proposed by Robert Brown in 1827 [1]. The Brown works were extended later in 1905 by Einstein [4]. In 1905, Person modeled the Brownian motion as a random walk [11]. In [6], Henry et al. have modeled the diffusion equation in the context of the fractional order operators. In [14, 16], Sene propose the analytical solution of the fractional diffusion equation, the works make a connection between the Fourier sine and the Laplace transformations. In [15], Sene propose the approximate solution of the fractional diffusion reaction equation using the homotopy perturbation Laplace transform method. Statistical interpretation of the fractional diffusion equation is investigated by Santos in [2, 3]. The list of works is long, we summarize them in the following references [5, 8-10].

In this paper, we introduce a new model of the particle diffusion equation. We mainly use Fick's law to establish the diffusion equation. The Fokker Plank equation is defined by

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} - \eta \frac{\partial u}{\partial x},\tag{1}$$

where for instant  $\mu$  denotes the advection coefficient and  $\eta$  represents the dispersion coefficient. Our motivation is to study the behavior of the diffusion processes when the advection term  $\frac{\partial^2 u}{\partial x^2}$  is replaced by the nonlinear subdiffusion term  $\frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right)$  and the diffusion parameters vary. The obtained diffusion equation is called the nonlinear sub-diffusion equation dispersion equation expressed as

$$\frac{\partial u}{\partial t} = \mu \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right) - \eta \frac{\partial u}{\partial x},\tag{2}$$

where  $\mu$  denotes the sub-advection coefficient and  $\eta$  represents the dispersion coefficient. Firstly, we investigate the existence and the uniqueness of the solution of the introduced model. Secondly, we propose the homotopy perturbation Laplace transform method for getting the approximate solution of the proposed model. It will be helpful to observe the behaviors of the approximate solutions of the nonlinear sub-diffusion dispersion equation graphically.

## 2 Constructive equations

In this section, we introduce new mathematical model in physics. We present the constructive equations related to the nonlinear sub-diffusion equation and nonlinear sub-advection dispersion equation. The Fick first [16] and second laws give the diffusion equation described by the following differential equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2},\tag{3}$$

where  $\mu$  denotes the diffusion coefficient. Let's the flux *F* of the diffusing material be nonlinear and represented by the following relation

$$F = -\mu u \frac{\partial u}{\partial x},\tag{4}$$

Applying the Fick second law to both sides of Eq. (4), we arrive at a known nonlinear sub-diffusion equation described by the equation

$$\frac{\partial u}{\partial t} = -\frac{\partial F}{\partial x} = \mu \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right),\tag{5}$$

where  $\mu$  denotes the sub-diffusion coefficient. We will in the next section investigate in the existence and the uniqueness of the solution of the nonlinear sub-diffusion equation described by Eq. (5). If the condition of the validity of this model exist, what is the analytical or the approximate solution? What is the method, we will use to get the solution? The questions which we will try to bring the answers in details in the next sections? Let's the advection-dispersion equation described by the following equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} - \eta \frac{\partial u}{\partial x},\tag{6}$$

where  $\mu$  denotes the advection coefficient and  $\eta$  represents the dispersion coefficient. In physics and the reallife problem; the advection coefficient is generally nonlinear. It proves the limits of the model described in Eq. (6). The novelty in our modeling is, we substitute the advection coefficient term in Eq. (6) by a subadvection coefficient term in the form of equation Eq. (4). Summarising, the new obtained model is called the nonlinear sub-advection dispersion equation. The following equation will represent the differential equation under consideration

$$\frac{\partial u}{\partial t} = \mu \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right) - \eta \frac{\partial u}{\partial x},\tag{7}$$

where  $\mu$  denotes the sub-advection coefficient, and  $\eta$  represents the dispersion coefficient. Does the solution of this model exist, it is unique? If the solution exists, what is the method to get this solution? Can we depict the solution to analyze the diffusion processes of the nonlinear sub-advection dispersion equation (7)? What are the impact of the diffusion parameters in the diffusion processes? This model will help for studying the diffusion processes of the fluid flow through the stationary porous landscape, and debris material introduced in the literature by Pudasaini, see in.

### **3** Homotopy perturbation Laplace transform method

In this section, we propose a new method for applying the homotopy perturbation method [7]. Let's the solutions of the diffusion Eq. (5) and Eq. (7) exist. The technique consists of introducing the usual Laplace transformation into the resolution of the differential equation. In other words, the method combines both the usual Laplace transform and the homotopy perturbation method. Let's the initial boundary condition defined by the following form

$$\frac{\partial^m u(x,0)}{\partial t^m} = u_m(x) \tag{8}$$

For the possible use of the usual Laplace transform, we rewrite Eq. (5) in the following form

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u^2}{\partial x^2} \tag{9}$$

Applying the Laplace transform into Eq. (9), and using the boundary condition (8), we obtain the following relationships

$$\bar{u}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^2}{\partial x^2} \right] \bar{u}^2(x,s) + \frac{1}{s^n} \left[ s^{n-1} u_0(x) + \dots + u_{n-1}(x) \right]$$
(10)

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where  $\bar{u}(x,s)$  denotes the Laplace transform of the function u(x,t). Using homotopy parameter, Eq. (10) can be rewritten in the following form

$$\bar{u}(x,s) = \frac{q}{s} \left[ \mu \frac{\partial^2}{\partial x^2} \right] \bar{u}^2(x,s) + \frac{1}{s^n} \left[ s^{n-1} u_0(x) + \dots + u_{n-1}(x) \right]$$
(11)

where q denotes the homotopy parameter taking it value into [0, 1]. Using homotopy procedure the solution of Eq. (11) is expressed in the following form

$$\bar{u}(x,s) = \sum_{i=0}^{\infty} q^i \bar{u}_i(x,s) \tag{12}$$

Replacing Eq. (12) into Eq. (11), it yields that

$$\sum_{i=0}^{\infty} q^{i} \bar{u}_{i}(x,s) = \frac{q}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \left[ \sum_{i=0}^{\infty} q^{i} \bar{u}_{i}(x,s) \right]^{2} + \frac{1}{s^{n}} \left[ s^{n-1} u_{0}(x) + \dots + u_{n-1}(x) \right]$$
(13)

By comparing the homotopy parameter q, we get from equation Eq. (13), the following iterative differential equations

$$q^{0}: \bar{u}_{0}(x,s) = \frac{1}{s^{n}} \left[ s^{n-1} u_{0}(x) + ... + u_{n-1}(x) \right];$$

$$q^{1}: \bar{u}_{1}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{0}^{2}(x,s);$$

$$q^{2}: \bar{u}_{2}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{1}^{2}(x,s);$$

$$q^{3}: \bar{u}_{3}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{2}^{2}(x,s)$$

$$\vdots::$$

$$q^{n+1}: \bar{u}_{n+1}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{n-1}^{2}(x,s)$$
(14)

When the homotopy parameter q converges to 1 at each step, it follows from Eq. (14), the solution of equation Eq. (13) is expressed by

$$Q_n(x,s) = \sum_{i=0}^{\infty} \bar{u}_i(x,s)$$
(15)

Applying the inverse of the Laplace transform to both sides of Eq. (15), we get the approximate solution of the nonlinear sub-diffusion equation (5) given by

$$u(x,t) = \mathscr{L}^{-1}(Q_n(x,s)).$$
(16)

where  $\mathcal L$  represents the usual Laplace operator.

In this subsection, we apply the homotopy perturbation Laplace transform method on the nonlinear subadvection dispersion equation (7) under boundary condition defined by

$$\frac{\partial^m u(x,0)}{\partial t^m} = u_m(x) \tag{17}$$

We reapeat the same procedure as in the previous subsection. We rewritte Eq. (7) in the following form

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u^2}{\partial x^2} - \eta \frac{\partial u}{\partial x}$$
(18)

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Applying the Laplace transform into Eq. (18), and using the boundary condition Eq. (17), we obtain the following relationships

$$\bar{u}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^2}{\partial x^2} \right] \bar{u}^2(x,s) - \frac{1}{s} \left[ \eta \frac{\partial}{\partial x} \right] \bar{u}(x,s) + \frac{1}{s^n} \left[ s^{n-1} u_0(x) + \dots + u_{n-1}(x) \right], \tag{19}$$

where  $\bar{u}(x,s)$  denotes the Laplace transform of the function u(x,t). Using homotopy parameter, Eq. (19) can be expressed in the following form

$$\bar{u}(x,s) = \frac{q}{s} \left[ \mu \frac{\partial^2}{\partial x^2} \right] \bar{u}^2(x,s) - \frac{q}{s} \left[ \eta \frac{\partial}{\partial x} \right] \bar{u}(x,s) + \frac{1}{s^n} \left[ s^{n-1} u_0(x) + \dots + u_{n-1}(x) \right]$$
(20)

where q denotes the homotopy parameter taking values into [0,1]. Using the homotopy procedure the solution of Eq. (20) is expressed in the following form

$$\bar{u}(x,s) = \sum_{i=0}^{\infty} q^i \bar{u}_i(x,s) \tag{21}$$

Replacing Eq. (21) into Eq. (20), it's yields that the relation

$$\sum_{i=0}^{\infty} q^{i} \bar{u}_{i}(x,s) = \frac{q}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \left[ \sum_{i=0}^{\infty} q^{i} \bar{u}_{i}(x,s) \right]^{2} - \frac{q}{s} \left[ \eta \frac{\partial}{\partial x} \right] \left[ \sum_{i=0}^{\infty} q^{i} \bar{u}_{i}(x,s) \right] + \frac{1}{s^{n}} \left[ s^{n-1} u_{0}(x) + \dots + u_{n-1}(x) \right]$$
(22)

By comparing the homotopy parameter q, we get from equation Eq. (22), the following iterative differential equations

$$q^{0}: \bar{u}_{0}(x,s) = \frac{1}{s^{n}} \left[ s^{n-1}u_{0}(x) + \dots + u_{n-1}(x) \right];$$

$$q^{1}: \bar{u}_{1}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{0}^{2}(x,s) - \frac{1}{s} \left[ \eta \frac{\partial}{\partial x} \right] \bar{u}_{0}(x,s);$$

$$q^{2}: \bar{u}_{2}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{1}^{2}(x,s) - \frac{1}{s} \left[ \eta \frac{\partial}{\partial x} \right] \bar{u}_{1}(x,s);$$

$$q^{3}: \bar{u}_{3}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{2}^{2}(x,s) - \frac{1}{s} \left[ \eta \frac{\partial}{\partial x} \right] \bar{u}_{2}(x,s)$$

$$\vdots::$$

$$q^{n+1}: \bar{u}_{n+1}(x,s) = \frac{1}{s} \left[ \mu \frac{\partial^{2}}{\partial x^{2}} \right] \bar{u}_{n-1}^{2}(x,s) - \frac{1}{s} \left[ \eta \frac{\partial}{\partial x} \right] \bar{u}_{n-1}(x,s)$$
(23)

Let consider the homotopy parameter q converging to 1 at each step, the solution of Eq. (20) is represented in the following form

$$Q_n(x,s) = \sum_{i=0}^{\infty} \bar{u}_i(x,s)$$
(24)

Applying the inverse of Laplace transform to both sides of Eq. (24), we get the approximate solution of the nonlinear sub-diffusion dispersion equation (7) represented by

$$u(x,t) = \mathscr{L}^{-1}(Q_n(x,s)).$$
<sup>(25)</sup>

where  ${\mathscr L}$  denotes the Laplace transform operator.

## 4 Existence and uniqueness of the nonlinear sub-diffusion equation

In this section, we prove the existence and the uniqueness of the solution of the nonlinear sub-diffusion equation using Banach fixed Theorem. Let's the function defined by

$$\Omega(x,u) = \mu \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right) = \frac{\mu}{2} \frac{\partial^2 u^2}{\partial x^2}.$$
(26)

We begin by proving the function  $\Omega$  is Lipchitz continuous. Let's the function

$$\Omega(x,u) - \Omega(x,v) = \frac{\mu}{2} \frac{\partial^2 u^2}{\partial x^2} - \frac{\mu}{2} \frac{\partial^2 v^2}{\partial x^2}$$
(27)

Applying the Euclidean norm, we obtain the following relationships

$$\begin{aligned} |\Omega(x,u) - \Omega(x,v)|| &= \left\| \frac{\mu}{2} \frac{\partial^2 u^2}{\partial x^2} - \frac{\mu}{2} \frac{\partial^2 v^2}{\partial x^2} \right\| \\ &\leq \frac{\mu}{2} \left\| \frac{\partial^2 u^2}{\partial x^2} - \frac{\partial^2 v^2}{\partial x^2} \right\| \\ &\leq \frac{\mu}{2} \left\| u^2 - v^2 \right\| \end{aligned}$$
(28)

Using the fact u and v bounded, we can find a constant b such that we have the following relationships

$$\|\Omega(x,u) - \Omega(x,v)\| \le \frac{\mu}{2} \|u^2 - v^2\| \le k \|u - v\|$$
<sup>(29)</sup>

where the constant  $k = \frac{b\mu}{2}$  is called Lipchitz constant. In the second step, we define an operator  $Tu: H \to H$ where *H* is a Banach space. Let's the operator *T* expressed as the following form

$$Tu(x,t) = \int_0^t \Omega(x(s), u) \, ds \tag{30}$$

We first prove the operator posed in Eq.(30) is well definite. We apply the Euclidean norm again to the following equation

$$\|Tu(x,t) - u(x,0)\| = \left\| \int_0^t (\Omega(x(s),u)) ds \right\|$$
  

$$\leq \int_0^t \|\Omega(x(s),u)\| ds$$
  

$$\leq \|\Omega(x(s),u)\| \int_0^t ds$$
  

$$\leq Ma$$
(31)

where  $t \le a$  and the Lipchitz constant *M* comes from the fact,  $\Omega$  is Lipchitz continuous. Thus, the operator *T* is well defined. The next step is to prove the operator *T* is a contraction. We apply the Euclidean norm; we have the following relationships

$$\|Tu(x,t) - Tv(x,t)\| = \left\| \int_0^t (\Omega(x(s),u) - \Omega(x(s),v)) ds \right\|$$
  

$$\leq \int_0^t \|\Omega(x(s),u) - \Omega(x(s),v)\| ds$$
  

$$\leq \|\Omega(x(s),u) - \Omega(x(s),v)\| \int_0^t ds$$
  

$$\leq ka \|u - v\|$$
(32)

Thus, by imposing  $ka \le 1$ , the operator *T* defines a contraction. Recalling the classical Banach fixed Theorem, we conclude the solution of the nonlinear sub-diffusion equation (5) exists and is unique. Note that, the problem consisting of finding the analytical or approximate solution of Eq. (5) is well posed.

#### 5 Existence and uniqueness of the nonlinear sub-advection dispersion equation

As in the previous section, we prove the existence and the uniqueness of the solution of the nonlinear subadvection dispersion equation using the Banach fixed Theorem. Let's the function defined by

$$\Phi(x,u) = \mu \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right) - \eta \frac{\partial u}{\partial x} = \frac{\mu}{2} \frac{\partial^2 u^2}{\partial x^2} - \eta \frac{\partial u}{\partial x}$$
(33)

Let's prove the function  $\Phi$  is Lipchitz continuous with a Lipschitz constant to be determined. Let's the function

$$\Phi(x,u) - \Phi(x,v) = \frac{\mu}{2} \frac{\partial^2 u^2}{\partial x^2} - \eta \frac{\partial u}{\partial x} - \frac{\mu}{2} \frac{\partial^2 v^2}{\partial x^2} + \eta \frac{\partial v}{\partial x}$$
(34)

Applying the Euclidean norm and triangular inequality, we obtain the following relationships

$$\|\Phi(x,u) - \Phi(x,v)\| = \left\| \frac{\mu}{2} \frac{\partial^2 u^2}{\partial x^2} - \eta \frac{\partial u}{\partial x} - \frac{\mu}{2} \frac{\partial^2 v^2}{\partial x^2} + \eta \frac{\partial v}{\partial x} \right\|$$
$$\leq \frac{\mu}{2} \left\| \frac{\partial^2 u^2}{\partial x^2} - \frac{\partial^2 v^2}{\partial x^2} \right\| + \eta \left\| \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right\|$$
$$\leq \frac{\mu}{2} \left\| u^2 - v^2 \right\| + \eta \left\| u - v \right\|$$
(35)

We assume the functions u and v are bounded; it follows that we can find a constant b such that we have the following relationships

$$\|\Phi(x,u) - \Phi(x,v)\| \le \frac{\mu}{2} \|u^2 - v^2\| + \eta \|u - v\| \le k \|u - v\|$$
(36)

where the constant  $k = \frac{b\mu}{2} + \eta$  represents the Lipchitz constant. Let's define a operator  $Zu : H \to H$  where *H* is Banach space. We define the operator *Z* as follows

$$Zu(x,t) = \int_0^t \Phi(x(s),u) \, ds \tag{37}$$

We prove the operator posed in Eq.(37) is well definite. We apply the Euclidean norm again to the following equation

$$\begin{aligned} \|Zu(x,t) - u(x,0)\| &= \left\| \int_0^t \Phi(x(s),u) \, ds \right\| \\ &\leq \int_0^t \|\Phi(x(s),u)\| \, ds \\ &\leq \|\Phi(x(s),u)\| \int_0^t ds \\ &\leq Ma \end{aligned}$$
(38)

where  $t \le a$  and the Lipchitz constant *M* comes from the fact,  $\Phi$  is Lipchitz continuous. Thus, the operator *Z* is well defined. The next step is to prove the operator *T* is a contraction. We apply the Euclidean norm; we have

the following relationships

$$\|Zu(x,t) - Zv(x,t)\| = \left\| \int_0^t (\Phi(x(s),u) - \Phi(x(s),v)) ds \right\|$$
  

$$\leq \int_0^t \|\Phi(x(s),u) - \Phi(x(s),v)\| ds$$
  

$$\leq \|\Phi(x(s),u) - \Phi(x(s),v)\| \int_0^t ds$$
  

$$\leq ka \|u - v\|$$
(39)

Thus, by imposing  $ka \le 1$ , the operator Z defines a contraction. Recalling the classical Banach fixed Theorem, we conclude the solution of the nonlinear sub-diffusion equation (7) exists and is unique. Note that, the problem consisting of finding the analytical or approximate solution of Eq. (7) is well posed.

#### 6 Approximate solutions of the nonlinear sub-diffusion equation

This section contributes to describing briefly, the possible method to get the approximate solution of the nonlinear sub-diffusion equation. We also analyze the behavior of the approximate solution in some particular cases. We introduce the homotopy perturbation Laplace transform method for getting the approximate solution of the nonlinear sub-diffusion equation. The following equation describes the nonlinear sub-diffusion equation under consideration.

$$\frac{\partial u}{\partial t} = \mu \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right) \tag{40}$$

with initial Dirichlet boundary condition defined as

• 
$$u(x,0) = e^x$$
 for  $x > 0$ 

Let's the first iteration, and the initial boundary condition  $u_0(x,0) = e^x$ , using Eq. (14), we have the following equation defined by

$$\bar{u}_0(x,s) = \frac{e^x}{s} \tag{41}$$

Applying the inverse of Laplace transform to both sides of Eq. (41), we obtain the analytical solution of Eq. (40) given by

$$u_0(x,t) = u(x,0) = e^x$$
(42)

Let's the second iteration, and the initial boundary condition  $u_1(x,0) = 0$ , using Eq. (14), we have the following equation defined by

$$\bar{u}_1(x,s) = \frac{2\mu e^{2x}}{s^2}$$
(43)

Applying the inverse of the Laplace transform to both sides of equation (43), we obtain the following analytical solution

$$u_1(x,t) = 2t\mu e^{2x}.$$
(44)

Let's the third iteration, and the initial boundary condition  $u_2(x,0) = 0$ , using Eq. (14), we have the following equation

$$\bar{u}_2(x,s) = \frac{64\mu^3 e^{4x}}{s^4} \tag{45}$$

Applying the inverse of Laplace transform to both sides of Eq. (45), we obtain the analytical solution of Eq. (45) given by

$$u_2(x,t) = \frac{32}{3}t^3\mu^3 e^{4x}.$$
(46)

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Let's the fourth iteration, and the initial boundary condition  $u_3(x,0) = 0$ , using Eq. (14), we have the following equation

$$\bar{u}_3(x,s) = \frac{2621440\mu^7 e^{8x}}{s^8} \tag{47}$$

Applying the inverse of Laplace transform to both sides of Eq. (47) and inverting it, we obtain the analytical solution of Eq. (47) given by

$$u_3(x,t) = \frac{32768}{63} t^7 \mu^7 e^{8x}.$$
(48)

Let's the fifth iteration, and the initial boundary condition  $u_4(x,0) = 0$ , using Eq. (14), we have the following equation

$$\bar{u}_4(x,s) = \frac{137438953472 \times 15! \mu^{15} e^{16x}}{59535s^{16}}$$
(49)

Applying the inverse of Laplace transform to both sides of equation (49), we obtain the analytical solution of Eq. (49) given by

$$u_4(x,t) = \frac{137438953472}{59535} t^{15} \mu^{15} e^{16x}.$$
(50)

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and so on, we use the same manner in other steps.

Finally, according to homotopy perturbation procedure, the approximate solution of the nonlinear subdiffusion equation (40) is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots$$
  
=  $e^x + 2t\mu e^{2x} + \frac{32}{3}t^3\mu^3 e^{4x} + \frac{32768}{63}t^7\mu^7 e^{8x} + \dots$  (51)

The four-term approximate solution of the nonlinear sub-diffusion equation (40) is given by the following expression:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)$$
  
=  $e^x + 2t\mu e^{2x} + \frac{32}{3}t^3\mu^3 e^{4x} + \frac{32768}{63}t^7\mu^7 e^{8x}$  (52)

Let's analyze the behavior of the approximate solution of the nonlinear sub-diffusion equation (5). For the interpretation, we suppose the sub-diffusion coefficient  $\mu = 1$ . In Figure 6, we depict the behavior of the approximate solution in two-dimensional spaces.

We first fixe the time to different increasing values, and we depict the figure 6 regarding space coordinates x. The diffusion process follows the direction of the arrow. We note when the time increases the density of the material increase too.

We first fixe the times to different decreasing values, and we depict the figure regarding space coordinates *x*. The diffusion process follows the direction of the arrow. We note when the time increases the density of the material increase (decrease) too.

Let's fixe t = 0.5, and the sub-diffusion coefficient  $\mu$  take different increasing values, we depict the figure 6 regarding space coordinates x. The diffusion process follows the direction of the arrow. We note, when the sub-diffusion coefficient  $\mu$  increase the density of the material increase too. We note same behaviour as above.

Let's fixe t = 0.5, and the sub-diffusion coefficient  $\mu$  take different decreasing values, we depict the figure 6 regarding space coordinates x. The diffusion process follows the direction of the arrow. We note, when the sub-diffusion coefficient  $\mu$  decrease the density of the material increase (decrease ) too.

Thus the sub-diffusion coefficient  $\mu$  has a retardation effect in the diffusion processes, when it decreases and acceleration effect when it increases.



Fig. 1 Profil in 3 dimensions



Fig. 2 Profil in 3 dimensions

# 7 Approximate solution of the nonlinear sub-advection dispersion equation

In this section, we present the approximate solution of the nonlinear sub-advection dispersion equation. We introduce the homotopy perturbation Laplace transform method for getting the approximate solution of the nonlinear sub-advection dispersion equation. The following equation describes the nonlinear sub-advection dispersion equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right) - \eta \frac{\partial u}{\partial x}$$
(53)

\$ sciendo



Fig. 3 Profil in 3 dimensions



Fig. 4 Profil in 3 dimensions

with initial Dirichlet boundary conditions defined as

•  $u(x,0) = e^x$  for x > 0

Let's the first iteration equation obtained by Eq. (23), and let's the initial boundary condition  $u_0(x,0) = e^x$ . We have the following equation defined by

$$\bar{u}_0(x,s) = \frac{e^x}{s} \tag{54}$$

Applying the inverse of Laplace transform to both sides of Eq. (54), we obtain the analytical solution of Eq. (54) given by

$$u_0(x,t) = u(x,0) = e^x$$
(55)

Let's the second iteration, and the initial boundary condition  $u_1(x, 0) = 0$ , using Eq. (23), we have to invert the following equation

$$\bar{u}_1(x,s) = \frac{2\mu e^{2x}}{s^2} - \frac{\eta e^x}{s^2}$$
(56)

Applying the inverse of Laplace transform to both sides of Eq. (56), we obtain the analytical solution of Eq. (56) given by

$$u_1(x,t) = 2\mu t e^{2x} - \eta t e^x$$
(57)

For simplification, we continue the rest of the resolution by solving with induction, the nonlinear sub-advection dispersion equation defined for all  $i \ge 1$  by

$$\frac{\partial u_i}{\partial t} = \mu \frac{\partial}{\partial x} \left( u_{i-1} \frac{\partial u_{i-1}}{\partial x} \right) - \eta \frac{\partial u_{i-1}}{\partial x}$$
(58)

under initial boundary condition defined by  $u_i(x,0) = 0$ .

Let's the third iteration, we solve the differential equation with initial boundary condition  $u_2(x,0) = 0$  defined by

$$\frac{\partial u_2}{\partial t} = \mu \frac{\partial}{\partial x} \left( u_1 \frac{\partial u_1}{\partial x} \right) - \eta \frac{\partial u_1}{\partial x}$$
  
=  $32t^2 \mu^3 e^{4x} - 18t^2 \mu^2 \eta e^{3x} + 2t^2 \mu \eta^2 e^{2x} - 4t \mu \eta e^{2x} + t \eta^2 e^x$  (59)

Applying the Laplace transform to both sides of Eq. (59), and inverting it, we obtain the analytical solution of the differential Eq. (59) given by

$$u_2(x,t) = \frac{32}{3}t^3\mu^3 e^{4x} - 6t^3\mu^2\eta e^{3x} + \frac{2}{3}t^3\mu\eta^2 e^{2x} - 2t^2\mu\eta e^{2x} + \frac{1}{2}t^2\eta^2 e^x$$
(60)

Let's the fourth itteration, we solve the differential equation with initial boundary condition  $u_3(x,0) = 0$  defined by

$$\begin{aligned} \frac{\partial u_3}{\partial t} &= \mu \frac{\partial}{\partial x} \left( u_2 \frac{\partial u_2}{\partial x} \right) - \eta \frac{\partial u_2}{\partial x} \\ &= \frac{32768}{9} t^6 \mu^7 e^{8x} - 3136t^6 \mu^6 \eta e^{7x} + 904t^6 \mu^5 \eta^2 e^{6x} - 100t^6 \mu^4 \eta^3 e^{5x} \\ &+ \frac{32}{9} t^6 \mu^3 \eta^4 e^{4x} - 768t^5 \mu^5 \eta e^{6x} + \frac{1300}{3} t^5 \mu^4 \eta^2 e^{5x} - \frac{208}{3} t^5 \mu^3 \eta^3 e^{4x} + 3t^5 \mu^2 \eta^4 e^{3x} \\ &+ 32t^4 \mu^3 \eta^2 e^{4x} - 9t^4 \mu^2 \eta^3 e^{3x} + \frac{1}{2} t^4 \mu \eta^4 e^{2x} - \frac{128}{3} t^3 \mu^3 \eta e^{4x} + 18t^3 \mu^2 \eta^2 e^{3x} \\ &- \frac{4}{3} t^3 \mu \eta^3 e^{2x} + 4t^2 \mu \eta^2 e^{2x} - \frac{1}{2} t^2 \eta^3 e^x. \end{aligned}$$
(61)

Applying the Laplace transform to both sides of Eq. (61), and inverting it, we obtain the analytical solution of differential Eq. (61) given by

$$u_{3}(x,t) = \frac{32768}{63}t^{7}\mu^{7}e^{8x} - 448t^{7}\mu^{6}\eta e^{7x} + \frac{904}{7}t^{7}\mu^{5}\eta^{2}e^{6x} - \frac{100}{7}t^{7}\mu^{4}\eta^{3}e^{5x} + \frac{32}{63}t^{7}\mu^{3}\eta^{4}e^{4x} - 128t^{6}\mu^{5}\eta e^{6x} + \frac{650}{9}t^{6}\mu^{4}\eta^{2}e^{5x} - \frac{104}{9}t^{6}\mu^{3}\eta^{3}e^{4x} + \frac{1}{2}t^{6}\mu^{2}\eta^{4}e^{3x} + \frac{32}{5}t^{5}\mu^{3}\eta^{2}e^{4x} - \frac{9}{5}t^{5}\mu^{2}\eta^{3}e^{3x} + \frac{1}{10}t^{5}\mu\eta^{4}e^{2x} - \frac{32}{3}t^{4}\mu^{3}\eta e^{4x} + \frac{9}{2}t^{4}\mu^{2}\eta^{2}e^{3x} - \frac{1}{3}t^{4}\mu\eta^{3}e^{2x} + \frac{4}{3}t^{3}\mu\eta^{2}e^{2x} - \frac{1}{6}t^{3}\eta^{3}e^{x}.$$
(62)

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and so on, We use the same manner in other steps.

Finaly, according to the homotopy perturbation procedure, the approximate solution of the nonlinear subdiffusion dispersion Eq. (53) is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots$$
(63)

Let's analyze the impact of the advection coefficient  $\mu$  and the dispersion coefficient  $\eta$  is the diffusion processes. Let's two-terms approximate solution of the nonlinear sub-advection dispersion Eq. (53). It is given by

$$u(x,t) = u_0(x,t) + u_1(x,t)$$
(64)

In Figure 7, we depict the approximate solution of the nonlinear sub-advection dispersion equation in twodimensional space with  $\mu = \eta = 1$ .



Fig. 5 Profil in 3 dimensions

In Figure 7, we depict the approximate solution of the nonlinear sub-advection dispersion equation with t = 0.6. We consider both  $\mu$  and  $\eta$  take increase values. We note the profile of the nonlinear sub-advection dispersion equation increase (decrease). The profiles follow the direction of the arrow in Figure 7. In Figure 7, we depict the approximate solution of the nonlinear sub-advection dispersion equation with t = 0.6. We consider



Fig. 6 Profil in 3 dimensions

the values of the advection coefficient  $\mu$  decrease and the values of the dispersion coefficient  $\eta$  increase. We note the profile of the nonlinear sub-advection dispersion equation increase (decrease). The profiles follow the direction of the arrow in Figure 7.



Fig. 7 Profil in 3 dimensions

In Figure 7, we depict the approximate solution of the nonlinear sub-advection dispersion equation with t = 0.6. We consider the values of the advection coefficient  $\mu$  increase and the values dispersion coefficient  $\eta$  decrease. We note the profile of the nonlinear sub-advection dispersion equation increase (decrease). The profiles follow the direction of the arrow in Figure 7.



Fig. 8 Profil in 3 dimensions

In Figure 7, we depict the approximate solution of the nonlinear sub-advection dispersion equation with t = 0.6. We consider the values of the parameters  $\mu$  and  $\eta$  both decrease. We note the profile of the nonlinear sub-advection dispersion equation increase (decrease). The profiles follow the direction of the arrow in Figure 7.



Fig. 9 Profil in 3 dimensions

## 8 Conclusion

In this paper, we have proposed a new model in diffusion equations: namely the nonlinear sub-diffusion equation and the nonlinear sub-diffusion dispersion equation. We analyze the condition of the existence and the uniqueness of the solutions of the proposed model. We also recommend a novel method for getting the approximate solution. An important question can be considered for future word, does the solution exist when the dispersion term is a sub-dispersion term.

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