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sciendo Applied Mathematics and Nonlinear Sciences 5(1) (2020) 147–162

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

A study on certain properties of generalized special functions defined by Fox-Wright function

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Submission Info

Communicated by Hacı Mehmet Baskonus
Received March 26th 2019
Accepted April 8th 2019
Available online March 31st 2020

Abstract

In this study, motivated by the frequent use of Fox-Wright function in the theory of special functions, we first introduced new generalizations of gamma and beta functions with the help of Fox-Wright function. Then by using these functions, we defined generalized Gauss hypergeometric function and generalized confluent hypergeometric function. For all the generalized functions we have defined, we obtained their integral representations, summation formulas, transformation formulas, derivative formulas and difference formulas. Also, we calculated the Mellin transformations of these functions.

Keywords: Gamma function, Beta function, Fox-Wright function, Gauss hypergeometric function, Confluent hypergeometric function.
AMS 2010 codes: 33B15, 33C05, 33C015.

1 Introduction

On the last quarter century, some generalizations of special functions, which frequently used in applied mathematics, have been studied by many scientists [1, 7–14, 17, 19–29, 31–33]. Chaudhry and Zubair [10] defined the extended gamma function in 1994 as

$$\Gamma_p(x) = \int_0^\infty t^{x-1} \exp\left[-t - \frac{p}{t}\right] dt,$$

where $Re(p) > 0$. Three years later, Chaudhry et al. [7] defined the extended beta function as

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[-\frac{p}{t(1-t)}\right] dt,$$

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sciendo
ISSN 2444-8656 doi:10.2478/AMNS.2020.1.00014 sciendo

where $Re(p) > 0,\ Re(x) > 0,\ Re(y) > 0$. It clearly seems that, for $p = 0$, $\Gamma_0(x) = \Gamma(x)$ and $B_0(x,y) = B(x,y)$, where $\Gamma(x)$ and $B(x,y)$ are the classical gamma and beta functions [6].

In 2004, Chaudhry et al. [8] used $B_p(x,y)$ to extend the Gauss and confluent hypergeometric functions as follows:

$$\mathrm{F}_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \tag{1}$$

$$\Phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \tag{2}$$

where $p \geq 0,\ Re(c) > Re(b) > 0$. In the same paper, the authors also gave the integral representations of (1) and (2) as

$$\mathrm{F}_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[-\frac{p}{t(1-t)}\right] dt,$$

where $p > 0,\ p = 0$ and $|arg(1-z)| < \pi < p,\ Re(c) > Re(b) > 0$, and

$$\Phi_p(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[zt - \frac{p}{t(1-t)}\right] dt,$$

where $p > 0,\ p = 0$ and $Re(c) > Re(b) > 0$. Here $(a)_n$ is the Pochhammer symbol which defined as

$$(a)_\nu = \frac{\Gamma(a+\nu)}{\Gamma(a)},\ a,\nu \in \mathbb{C}$$

with the assume $(a)_0 \equiv 1$.

The Fox-Wright function is given in [18] as

$${}_\xi\Psi_\eta(z) = {}_\xi\Psi_\eta\left[\begin{matrix}(\beta_i,\alpha_i)_{1,\xi} \\ (\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{\xi}\Gamma(\alpha_i n+\beta_i)}{\prod_{j=1}^{\eta}\Gamma(\kappa_j n+\mu_j)} \frac{z^n}{n!}, \tag{3}$$

where $z, \beta_i, \mu_j \in \mathbb{C}, \alpha_i, \kappa_j \in \mathbb{R}, i = 1\ldots\xi$ and $j = 1\ldots\eta$. The asymptotic behaviour of the above function was studied by Fox [15, 16] and Wright [34–36] for the large values of z, considering the condition

$$\sum_{j=1}^{\eta}\kappa_j - \sum_{i=1}^{\xi}\alpha_i > -1.$$

If these conditions are met, for any $z \in \mathbb{C}$ the series (3) is convergent. For $\kappa, \mu, z \in \mathbb{C}, Re(\kappa) > -1$, the classic Wright function [18]

$${}_0\Psi_1(z) = {}_0\Psi_1\left[\begin{matrix}\overline{} \\ (\mu,\kappa)\end{matrix}\middle| z\right] = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\kappa n+\mu)} \frac{z^n}{n!}$$

can obtained by choosing $\xi = 0$ and $\eta = 1$ in equation (3).

Inspired by the aforementioned studies and motivated by the frequent use of Fox-Wright function in the theory of special functions, we defined two new functions as generalizations of gamma and beta functions.

2 Generalized functions and their properties

Throughout the study, we assume that $x,y,z \in \mathbb{C}$, $k,m,n \in \mathbb{N}$, $\alpha_i, \kappa_j \in \mathbb{R}$, $\beta_i, \mu_j, a, b, c, p \in \mathbb{C}$, $Re(p) > 0$, $Re(x) > 0$, $Re(y) > 0$, and $Re(c) > Re(b) > 0$. For the sake of shortness, we did not wrote these conditions for the rest of the article, unless otherwise stated.

Let us defined the new generalizations as

$$^{\Psi}\hat{\Gamma}_p(x) := {}^{\Psi}\Gamma_p \left[\begin{matrix} (\beta_i, \alpha_i)_{1,\xi} \\ (\mu_j, \kappa_j)_{1,\eta} \end{matrix} \middle| x \right] = \int_0^\infty t^{x-1} {}_\xi\Psi_\eta \left(-t - \frac{p}{t} \right) dt \tag{4}$$

and

$$^{\Psi}\hat{B}_p(x,y) := {}^{\Psi}B_p \left[\begin{matrix} (\beta_i, \alpha_i)_{1,\xi} \\ (\mu_j, \kappa_j)_{1,\eta} \end{matrix} \middle| x,y \right] = \int_0^1 t^{x-1}(1-t)^{y-1} {}_\xi\Psi_\eta \left(-\frac{p}{t(1-t)} \right) dt. \tag{5}$$

We called them as ${}_\xi\Psi_\eta$-gamma and ${}_\xi\Psi_\eta$-beta functions.

Our first theorem is about the current relationship of the two ${}_\xi\Psi_\eta$-gamma functions.

Theorem 1. *The following equality holds true:*

$$\begin{aligned} ^{\Psi}\hat{\Gamma}_p(x)^{\Psi}\hat{\Gamma}_p(y) = 4 \int_0^{\frac{\pi}{2}} \int_0^\infty & r^{2(x+y)-1} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} \\ & \times {}_\xi\Psi_\eta \left(-r^2(\cos\theta)^2 - \frac{p}{r^2(\cos\theta)^2} \right) \\ & \times {}_\xi\Psi_\eta \left(-r^2(\sin\theta)^2 - \frac{p}{r^2(\sin\theta)^2} \right) dr d\theta. \end{aligned}$$

Proof. Substituting $t = u^2$ in (4), we get

$$^{\Psi}\hat{\Gamma}_p(x) = 2 \int_0^\infty u^{2x-1} {}_\xi\Psi_\eta \left(-u^2 - \frac{p}{u^2} \right) du.$$

Therefore,

$$^{\Psi}\hat{\Gamma}_p(x)^{\Psi}\hat{\Gamma}_p(y) = 4 \int_0^\infty \int_0^\infty u^{2x-1} v^{2y-1} {}_\xi\Psi_\eta \left(-u^2 - \frac{p}{u^2} \right) {}_\xi\Psi_\eta \left(-v^2 - \frac{p}{v^2} \right) du dv.$$

In the above equality, taking $u = r(\cos\theta)$ and $v = r(\sin\theta)$ yields

$$\begin{aligned} ^{\Psi}\hat{\Gamma}_p(x)^{\Psi}\hat{\Gamma}_p(y) = 4 \int_0^{\frac{\pi}{2}} \int_0^\infty & r^{2(x+y)-1} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} \\ & \times {}_\xi\Psi_\eta \left(-r^2(\cos\theta)^2 - \frac{p}{r^2(\cos\theta)^2} \right) \\ & \times {}_\xi\Psi_\eta \left(-r^2(\sin\theta)^2 - \frac{p}{r^2(\sin\theta)^2} \right) dr d\theta, \end{aligned}$$

which completes the proof.

Theorem 2. *The ${}_\xi\Psi_\eta$-beta function has the following integral representations:*

$$\begin{aligned} ^{\Psi}\hat{B}_p(x,y) &= 2 \int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} {}_\xi\Psi_\eta \left(-p(\sec\theta)^2 (\csc\theta)^2 \right) d\theta, \\ ^{\Psi}\hat{B}_p(x,y) &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} {}_\xi\Psi_\eta \left(-2p - p\left(u + \frac{1}{u} \right) \right) du, \\ ^{\Psi}\hat{B}_p(x,y) &= (c-a)^{1-x-y} \int_a^c (u-a)^{x-1} (c-u)^{y-1} {}_\xi\Psi_\eta \left(\frac{-p(c-a)^2}{(u-a)(c-u)} \right) du. \end{aligned}$$

Proof. Taking $t = (\sin\theta)^2$ in (5), we get

$$
\begin{aligned}
{}^{\Psi}\hat{B}_p(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\
&= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}{}_{\xi}\Psi_{\eta}\left(-p(\sec\theta)^2(\csc\theta)^2\right)d\theta.
\end{aligned}
$$

Taking $t = \frac{u}{1+u}$ in (5), we get

$$
\begin{aligned}
{}^{\Psi}\hat{B}_p(x,y) &= \int_0^{\infty}\left(\frac{u}{1+u}\right)^{x-1}\left(\frac{1}{1+u}\right)^{y-1}\left(\frac{1}{1+u}\right)^{2}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{\left(\frac{u}{1+u}\right)\left(\frac{1}{1+u}\right)}\right)du \\
&= \int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}\,{}_{\xi}\Psi_{\eta}\left(-2p-p\left(u+\frac{1}{u}\right)\right)du.
\end{aligned}
$$

Taking $t = \frac{u-a}{c-a}$ in (5), we get

$$
\begin{aligned}
{}^{\Psi}\hat{B}_p(x,y) &= \int_a^c\left(\frac{u-a}{c-a}\right)^{x-1}\left(1-\frac{u-a}{c-a}\right)^{y-1}\frac{1}{c-a}\,{}_{\xi}\Psi_{\eta}\left(\frac{-p(c-a)^2}{(u-a)(c-u)}\right)du \\
&= (c-a)^{1-x-y}\int_a^c (u-a)^{x-1}(c-u)^{y-1}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p(c-a)^2}{(u-a)(c-u)}\right)du,
\end{aligned}
$$

which gives the result.

Theorem 3. *The following derivative formula is provided for $Re(x) > m, Re(y) > m$:*

$$
\frac{d^m}{dp^m}\left\{{}^{\Psi}\hat{B}_p(x,y)\right\} = (-1)^m\,{}^{\Psi}B_p\left[\begin{matrix}(\alpha_i m+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j m+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\,\middle|\, x-m, y-m\right].
$$

Proof. It is done by induction. The first order derivative of (5) is as follows:

$$
\begin{aligned}
\frac{d}{dp}\left\{{}^{\Psi}\hat{B}_p(x,y)\right\} &= \frac{d}{dp}\left\{\int_0^1 t^{x-1}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt\right\} \\
&= (-1)\,{}^{\Psi}B_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\,\middle|\, x-1, y-1\right].
\end{aligned}
$$

Let us assume that the k-order derivative of (5) is

$$
\frac{d^k}{dp^k}\left\{{}^{\Psi}\hat{B}_p(x,y)\right\} = (-1)^k\,{}^{\Psi}B_p\left[\begin{matrix}(\alpha_i k+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j k+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\,\middle|\, x-k, y-k\right]. \tag{6}
$$

From the first order derivative of (6), the $k+1$-order derivative is found as follows:

$$
\begin{aligned}
\frac{d^{k+1}}{dp^{k+1}}\left\{{}^{\Psi}\hat{B}_p(x,y)\right\} &= \frac{d}{dp}\left\{\frac{d^k}{dp^k}\left\{{}^{\Psi}\hat{B}_p(x,y)\right\}\right\} \\
&= (-1)^{k+1}\,{}^{\Psi}B_p\left[\begin{matrix}(\alpha_i(k+1)+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j(k+1)+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\,\middle|\, x-(k+1), y-(k+1)\right].
\end{aligned}
$$

This gives the result.

Theorem 4. *The following equality is provided for $Re(s) > 0$:*

$$
\mathscr{M}\left[{}^{\Psi}\hat{B}_p(x,y)\right] = B(x+s,y+s)\,{}^{\Psi}\hat{\Gamma}_p(s).
$$

Proof. If we apply Mellin transformation according to argument p in equation (5), we have

$$\begin{aligned}\mathscr{M}\left[{}^{\Psi}\hat{B}_p(x,y)\right] &= \int_0^\infty p^{s-1}\int_0^1 t^{x-1}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(\frac{-p}{t(1-t)}\right)dtdp \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\int_0^\infty p^{s-1}{}_{\xi}\Psi_{\eta}\left(\frac{-p}{t(1-t)}\right)dpdt. \end{aligned} \tag{7}$$

Letting $v = \frac{p}{t(1-t)}$ in (7), we get

$$\mathscr{M}\left[{}^{\Psi}\hat{B}_p(x,y)\right] = \int_0^1 t^{x+s-1}(1-t)^{y+s-1}dt\int_0^\infty v^{s-1}{}_{\xi}\Psi_{\eta}(-v)\,dv.$$

Thus, we have

$$\mathscr{M}\left[{}^{\Psi}\hat{B}_p(x,y)\right] = B(x+s,y+s)\,{}^{\Psi}\hat{\Gamma}_p(s),$$

which completes the proof.

Remark 1. By using the inverse Mellin transform, it is easy to see

$${}^{\Psi}\hat{B}_p(x,y) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} B(x+s,y+s)\,{}^{\Psi}\hat{\Gamma}_p(s)p^{-s}ds$$

for $Re(s) > 0$.

Theorem 5. *The following equality holds true:*

$${}^{\Psi}\hat{B}_p(x,y) = {}^{\Psi}\hat{B}_p(x+1,y) + {}^{\Psi}\hat{B}_p(x,y+1).$$

Proof. Direct calculation yields

$$\begin{aligned}{}^{\Psi}\hat{B}_p(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \int_0^1 t^{x}(1-t)^{y}\frac{1}{t(1-t)}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \int_0^1 t^{x}(1-t)^{y}\left[(1-t)^{-1}+t^{-1}\right]{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \int_0^1 \left[t^{x}(1-t)^{y-1}+t^{x-1}(1-t)^{y}\right]{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \int_0^1 t^{x}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt + \int_0^1 t^{x-1}(1-t)^{y}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= {}^{\Psi}\hat{B}_p(x+1,y) + {}^{\Psi}\hat{B}_p(x,y+1), \end{aligned}$$

which is the result.

Theorem 6. *The following summation formula is provided for $Re(y) < 1$:*

$${}^{\Psi}\hat{B}_p(x,1-y) = \sum_{n=0}^{\infty}\frac{(y)_n}{n!}\,{}^{\Psi}\hat{B}_p(x+n,1).$$

Proof. From the definition of the ${}_{\xi}\Psi_{\eta}$-beta function, we obtain

$$^{\Psi}\hat{B}_p(x,1-y) = \int_0^1 t^{x-1}(1-t)^{-y}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt.$$

With the help of the following series expression

$$(1-t)^{-y} = \sum_{n=0}^{\infty}(y)_n\frac{t^n}{n!}, \quad |t|<1,$$

we obtain

$$\begin{aligned} ^{\Psi}\hat{B}_p(x,1-y) &= \int_0^1 \sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^{x+n-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \sum_{n=0}^{\infty}\frac{(y)_n}{n!}\int_0^1 t^{x+n-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \sum_{n=0}^{\infty}\frac{(y)_n}{n!}{}^{\Psi}\hat{B}_p(x+n,1). \end{aligned}$$

This completes the proof.

Theorem 7. *The following equality holds true:*

$$^{\Psi}\hat{B}_p(x,y) = \sum_{n=0}^{\infty}{}^{\Psi}\hat{B}_p(x+n,y+1).$$

Proof. From the definition of the ${}_{\xi}\Psi_{\eta}$-beta function, we get

$$^{\Psi}\hat{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt.$$

With the help of the following series expression

$$(1-t)^{y-1} = (1-t)^y\sum_{n=0}^{\infty}t^n, \quad |t|<1,$$

we obtain

$$\begin{aligned} ^{\Psi}\hat{B}_p(x,y) &= \int_0^1 t^{x-1}(1-t)^y\sum_{n=0}^{\infty}t^n{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \sum_{n=0}^{\infty}\int_0^1 t^{x+n-1}(1-t)^y{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\ &= \sum_{n=0}^{\infty}{}^{\Psi}\hat{B}_p(x+n,y+1), \end{aligned}$$

which gives the result.

Theorem 8. *The following relation is provided for* $Re(x)>1, Re(y)>1$*:*

$$\begin{aligned} x^{\Psi}\hat{B}_p(x,y+1) &= y^{\Psi}\hat{B}_p(x+1,y) \\ &\quad + 2p^{\Psi}B_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi} \\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| x,y-1\right] \\ &\quad - p^{\Psi}B_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi} \\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| x-1,y-1\right]. \end{aligned} \tag{8}$$

Proof. (5) equality provides the following equation

$$ {}^{\Psi}\hat{B}_p(x,y) = \mathscr{M}\left[\hat{f}(t:y;p):x\right] $$

where

$$ \hat{f}(t:y;p) = (1-t)^{y-1} H(1-t)\ {}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right) $$

and

$$ H(1-t) = \begin{cases} 0, & t>1, \\ 1, & t<1. \end{cases} $$

The derivative of $\hat{f}(t:y;p)$ according to the parameter t provides the following equation:

$$ \begin{aligned} \frac{d}{dt}\left\{\hat{f}(t:y;p)\right\} = & -\delta(1-t)(1-t)^{y-1}\,{}_{\xi}\Psi_{\eta}\left(\frac{-p}{t(1-t)}\right) \\ & -(y-1)(1-t)^{y-2}H(1-t)\ {}_{\xi}\Psi_{\eta}\left(\frac{-p}{t(1-t)}\right) \\ & +\frac{p(1-2t)}{t^2(1-t)^2}(1-t)^{y-1}H(1-t)\ {}_{\xi}\Psi_{\eta}\left(\frac{-p}{t(1-t)}\right), \end{aligned} $$

where $\frac{d}{dt}H(1-t) = -\delta(1-t)$ and δ represents the Dirac delta $\delta(1-t) = \delta(t-1) = 0$ for $t \neq 1$. The relationship between the derivative of a function and the Mellin transformation is as follows:

$$ \mathscr{M}[f(x):s] = F(s) \Rightarrow \mathscr{M}[f'(x):s] = -(s-1)F(s-1). $$

From here, by arranging, we find that

$$ \begin{aligned} -(x-1)\,{}^{\Psi}\hat{B}_p(x-1,y) = & -(y-1)\,{}^{\Psi}\hat{B}_p(x,y-1) \\ & + p\,{}^{\Psi}B_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi} \\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| x-2,y-2\right] \\ & -2p\,{}^{\Psi}B_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi} \\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| x-1,y-2\right]. \end{aligned} $$

Finally if x replaced by $x+1$ and y replaced by $y+1$ we get (8).

3 ${}_{\xi}\Psi_{\eta}$-generalization of Gauss and confluent hypergeometric functions

We used the ${}_{\xi}\Psi_{\eta}$-beta function (5) to define the generalizations of Gauss and confluent hypergeometric functions as

$$ {}^{\Psi}\hat{F}_p(a,b;c;z) := {}^{\Psi}F_p\left[\begin{matrix}(\beta_i,\alpha_i)_{1,\xi} \\ (\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| a,b;c;z\right] = \sum_{n=0}^{\infty}(a)_n \frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} $$

and

$$ {}^{\Psi}\hat{\Phi}_p(b;c;z) := {}^{\Psi}\Phi_p\left[\begin{matrix}(\beta_i,\alpha_i)_{1,\xi} \\ (\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| b;c;z\right] = \sum_{n=0}^{\infty}\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, $$

respectively. We call ${}^{\Psi}\hat{F}_p(a,b;c;z)$ as ${}_{\xi}\Psi_{\eta}$-Gauss hypergeometric function and ${}^{\Psi}\hat{\Phi}_p(b;c;z)$ as ${}_{\xi}\Psi_{\eta}$-confluent hypergeometric function.

The following two theorems are about the integral representations of ${}_{\xi}\Psi_{\eta}$-Gauss and ${}_{\xi}\Psi_{\eta}$-confluent hypergeometric functions.

Theorem 9. *The ${}_{\xi}\Psi_{\eta}$-Gauss hypergeometric function has the following integral representations:*

$$\begin{aligned}
{}^{\Psi}\hat{F}_p(a,b;c;z) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt, \\
{}^{\Psi}\hat{F}_p(a,b;c;z) &= \frac{1}{B(b,c-b)}\int_0^\infty u^{b-1}(1+u)^{a-c}\left[1+u(1-z)\right]^{-a}\,{}_{\xi}\Psi_{\eta}\left(-2p-p\left(u+\frac{1}{u}\right)\right)du, \\
{}^{\Psi}\hat{F}_p(a,b;c;z) &= \frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}}(\sin\theta)^{2b-1}(\cos\theta)^{2c-2b-1}\left(1-z(\sin\theta)^2\right)^{-a}\,{}_{\xi}\Psi_{\eta}\left(-p(\sec\theta)^2(\csc\theta)^2\right)d\theta.
\end{aligned} \tag{9}$$

Proof. Direct calculation yields

$$\begin{aligned}
{}^{\Psi}\hat{F}_p(a,b;c;z) &= \sum_{n=0}^{\infty}(a)_n\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \\
&= \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)\frac{z^n}{n!}dt \\
&= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}dt \\
&= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)(1-zt)^{-a}dt.
\end{aligned}$$

Setting $u=\frac{t}{1-t}$ in (9), we get

$${}^{\Psi}\hat{F}_p(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^\infty u^{b-1}(1+u)^{a-c}\left[1+u(1-z)\right]^{-a}\,{}_{\xi}\Psi_{\eta}\left(-2p-p\left(u+\frac{1}{u}\right)\right)du.$$

Besides, substituting $t=(\sin\theta)^2$ in (9), we have

$${}^{\Psi}\hat{F}_p(a,b;c;z) = \frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}}(\sin\theta)^{2b-1}(\cos\theta)^{2c-2b-1}\left(1-z(\sin\theta)^2\right)^{-a}\,{}_{\xi}\Psi_{\eta}\left(-p(\sec\theta)^2(\csc\theta)^2\right)d\theta.$$

Similarly, the ${}_{\xi}\Psi_{\eta}$-confluent hypergeometric function is also performed.

Theorem 10. *The ${}_{\xi}\Psi_{\eta}$-confluent hypergeometric function has the following integral representations:*

$$\begin{aligned}
{}^{\Psi}\hat{\Phi}_p(b;c;z) &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt, \\
{}^{\Psi}\hat{\Phi}_p(b;c;z) &= \frac{1}{B(b,c-b)}\int_0^1 u^{c-b-1}(1-u)^{b-1}e^{z(1-u)}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{u(1-u)}\right)du.
\end{aligned} \tag{10}$$

In the following theorems, we obtained the derivative formulas of ${}_{\xi}\Psi_{\eta}$-Gauss and ${}_{\xi}\Psi_{\eta}$-confluent hypergeometric functions with the help of the following equations:

$$\begin{aligned}
B(b,c-b) &= \frac{c}{b}B(b+1,c-b), \\
(a)_{n+1} &= a(a+1)_n.
\end{aligned} \tag{11}$$

Theorem 11. *The following equality holds true:*

$$\frac{d^n}{dz^n}\left\{{}^{\Psi}\hat{F}_p(a,b;c;z)\right\} = \frac{(a)_n(b)_n}{(c)_n}\left[{}^{\Psi}\hat{F}_p(a+n,b+n;c+n;z)\right].$$

Proof. The derivative of the ${}^{\Psi}\hat{F}_p(a,b;c;z)$ according to the argument z is as follows:

$$\frac{d}{dz}\left\{{}^{\Psi}\hat{F}_p(a,b;c;z)\right\} = \frac{d}{dz}\left\{\sum_{n=0}^{\infty}(a)_n\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\}$$
$$= \sum_{n=1}^{\infty}(a)_n\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}.$$

Replacing $n \to n+1$, we get

$$\frac{d}{dz}\left\{{}^{\Psi}\hat{F}_p(a,b;c;z)\right\} = \frac{(a)(b)}{(c)}\sum_{n=0}^{\infty}(a+1)_n\frac{{}^{\Psi}\hat{B}_p(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!}$$
$$= \frac{(a)(b)}{(c)}\left[{}^{\Psi}\hat{F}_p(a+1,b+1;c+1;z)\right]. \tag{12}$$

Thus, the general form of the above equation is

$$\frac{d^n}{dz^n}\left\{{}^{\Psi}\hat{F}_p(a,b;c;z)\right\} = \frac{(a)_n(b)_n}{(c)_n}\left[{}^{\Psi}\hat{F}_p(a+n,b+n;c+n;z)\right].$$

This completes the proof.

Theorem 12. *The following equality is provided for $Re(b) > 2, Re(c) > Re(b+2)$:*

$$\begin{aligned}
&(b-1)B(b-1,c-b+1)^{\Psi}\hat{F}_p(a,b-1;c;z)\\
&=(c-b-1)B(b,c-b-1)^{\Psi}\hat{F}_p(a,b;c-1;z)\\
&\quad - azB(b,c-b)^{\Psi}\hat{F}_p(a+1,b;c;z)\\
&\quad - pB(b-2,c-b-2)^{\Psi}F_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\, a,b-2;c-4;z\right]\\
&\quad + 2pB(b-1,c-b-2)^{\Psi}F_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\, a,b-1;c-3;z\right].
\end{aligned}$$

Proof. Since $B(b,c-b)^{\Psi}\hat{F}_p(a,b;c;z)$ is the Mellin transform of

$$\hat{f}_{a,b,c}(t:z;p) = (1-t)^{c-b-1}(1-zt)^{-a}H(1-t)\ {}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right),$$

$B(b,c-b)^{\Psi}\hat{F}_p(a,b;c;z)$ has the Mellin transform formula

$$B(b,c-b)^{\Psi}\hat{F}_p(a,b;c;z) = \mathscr{M}\left[\hat{f}_{a,b,c}(t:z;p):b\right].$$

Differentiating $\hat{f}_{a,b,c}(t:z;p)$ with respect to t we obtain

$$\begin{aligned}
\frac{d}{dt}\left\{\hat{f}_{a,b,c}(t:z;p)\right\} =& -(c-b-1)(1-t)^{c-b-2}(1-zt)^{-a}H(1-t)_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)\\
&+az(1-t)^{c-b-1}(1-zt)^{-(a+1)}H(1-t)\ {}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)\\
&+p\frac{1}{t^2}(1-t)^{c-b-3}(1-zt)^{-a}H(1-t)\ {}_{\xi}\Psi_{\eta}\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\,\frac{-p}{t(1-t)}\right]\\
&-2p\frac{1}{t}(1-t)^{c-b-3}(1-zt)^{-a}H(1-t)\ {}_{\xi}\Psi_{\eta}\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\,\frac{-p}{t(1-t)}\right].
\end{aligned}$$

Since

$$\mathscr{M}\left\{f^{'}(t):b\right\} = -(b-1)\mathscr{M}\left\{f(t):b-1\right\},$$

we get

$$\begin{aligned}(b-1)&B(b-1,c-b+1)^{\Psi}\hat{F}_p(a,b-1;c;z)\\ &= (c-b-1)B(b,c-b-1)^{\Psi}\hat{F}_p(a,b;c-1;z)\\ &\quad - azB(b,c-b)^{\Psi}\hat{F}_p(a+1,b;c;z)\\ &\quad - pB(b-2,c-b-2)^{\Psi}F_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| a,b-2;c-4;z\right]\\ &\quad + 2pB(b-1,c-b-2)^{\Psi}F_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| a,b-1;c-3;z\right],\end{aligned}$$

which gives the result.

Theorem 13. *The following equality holds true:*

$$\frac{d^n}{dz^n}\left\{{}^{\Psi}\hat{\Phi}_p(b;c;z)\right\} = \frac{(b)_n}{(c)_n}\left[{}^{\Psi}\hat{\Phi}_p(b+n;c+n;z)\right]. \tag{13}$$

Proof. The derivative of the ${}^{\Psi}\hat{\Phi}_p(b;c;z)$ according to argument z is

$$\begin{aligned}\frac{d}{dz}\left\{{}^{\Psi}\hat{\Phi}_p(b;c;z)\right\} &= \frac{d}{dz}\left\{\sum_{n=0}^{\infty}\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\}\\ &= \sum_{n=1}^{\infty}\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}.\end{aligned}$$

Replacing $n \to n+1$, we get

$$\begin{aligned}\frac{d}{dz}\left\{{}^{\Psi}\hat{\Phi}_p(b;c;z)\right\} &= \frac{(b)}{(c)}\sum_{n=0}^{\infty}\frac{{}^{\Psi}\hat{B}_p(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!}\\ &= \frac{(b)}{(c)}\left[{}^{\Psi}\hat{\Phi}_p(b+1;c+1;z)\right].\end{aligned}$$

Thus, the general form of the above equation gives

$$\frac{d^n}{dz^n}\left\{{}^{\Psi}\hat{\Phi}_p(b;c;z)\right\} = \frac{(b)_n}{(c)_n}\left[{}^{\Psi}\hat{\Phi}_p(b+n;c+n;z)\right],$$

which is the result.

Theorem 14. *The following equality is provided for $Re(b) > 2, Re(c) > Re(b+2)$:*

$$\begin{aligned}(b-1)&B(b-1,c-b+1)^{\Psi}\hat{\Phi}_p(b-1;c;z)\\ &= (c-b-1)B(b,c-b-1)^{\Psi}\hat{\Phi}_p(b;c-1;z)\\ &\quad - zB(b,c-b)^{\Psi}\hat{\Phi}_p(b;c;z)\\ &\quad - pB(b-2,c-b-2)^{\Psi}\Phi_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| b-2;c-4;z\right]\\ &\quad + 2pB(b-1,c-b-2)^{\Psi}\Phi_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\ (\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| b-1;c-3;z\right].\end{aligned}$$

Proof. Since $B(b,c-b)^{\Psi}\hat{\Phi}_p(b;c;z)$ is the Mellin transform of

$$\hat{f}_{b,c}(t:z;p)=(1-t)^{c-b-1}e^{zt}H(1-t)\ {}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right),$$

$B(b,c-b)^{\Psi}\hat{\Phi}_p(b;c;z)$ has the Mellin transform formula

$$B(b,c-b)^{\Psi}\hat{\Phi}_p(b;c;z)=\mathscr{M}\left[\hat{f}_{b,c}(t:z;p):b\right].$$

Differentiating $\hat{f}_{b,c}(t:z;p)$ with regard to t obtain

$$\begin{aligned}\frac{d}{dt}\left\{\hat{f}_{b,c}(t:z;p)\right\}=&-(c-b-1)(1-t)^{c-b-2}e^{zt}H(1-t)\ {}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)\\ &+z(1-t)^{c-b-1}e^{zt}H(1-t)\ {}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)\\ &+p\frac{1}{t^2}(1-t)^{c-b-3}e^{zt}H(1-t)\ {}_{\xi}\Psi_{\eta}\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\frac{-p}{t(1-t)}\right]\\ &-2p\frac{1}{t}(1-t)^{c-b-3}e^{zt}H(1-t)\ {}_{\xi}\Psi_{\eta}\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\frac{-p}{t(1-t)}\right].\end{aligned}$$

Since

$$\mathscr{M}\left\{f^{'}(t):b\right\}=-(b-1)\mathscr{M}\left\{f(t):b-1\right\},$$

we get

$$\begin{aligned}&(b-1)B(b-1,c-b+1)^{\Psi}\hat{\Phi}_p(b-1;c;z)\\ &\quad=(c-b-1)B(b,c-b-1)^{\Psi}\hat{\Phi}_p(b;c-1;z)\\ &\qquad-zB(b,c-b)^{\Psi}\hat{\Phi}_p(b;c;z)\\ &\qquad-pB(b-2,c-b-2)^{\Psi}\Phi_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\,b-2;c-4;z\right]\\ &\qquad+2pB(b-1,c-b-2)^{\Psi}\Phi_p\left[\begin{matrix}(\alpha_i+\beta_i,\alpha_i)_{1,\xi}\\(\kappa_j+\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle|\,b-1;c-3;z\right],\end{aligned}$$

which completes the proof.

In the following theorems we obtain the Mellin transform formulas of the ${}_{\xi}\Psi_{\eta}$-Gauss and ${}_{\xi}\Psi_{\eta}$-confluent hypergeometric functions.

Theorem 15. *The following equality is provided for $Re(s)>0$:*

$$\mathscr{M}\left[{}^{\Psi}\hat{F}_p(a,b;c;z):s\right]=\frac{{}^{\Psi}\hat{\Gamma}(s)B(b+s,c+s-b)}{B(b,c-b)}{}_2F_1(a,b+s;c+2s;z).$$

Proof. By applying Mellin transformation to equality (9), we get

$$\begin{aligned}\mathscr{M}\left[{}^{\Psi}\hat{F}_p(a,b;c;z):s\right]&=\int_0^{\infty}p^{s-1}\left[{}^{\Psi}\hat{F}_p(a,b;c;z)\right]dp\\ &=\int_0^{\infty}p^{s-1}\sum_{n=0}^{\infty}\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}(a)_n\frac{z^n}{n!}dp\\ &=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\int_0^{\infty}p^{s-1}{}_{\xi}\Psi_{\eta}\left(\frac{-p}{t(1-t)}\right)dpdt.\end{aligned}$$

Substituting $u = \frac{p}{t(1-t)}$ in the above equation gives us

$$\int_0^\infty p^{s-1} {}_\xi\Psi_\eta \left(-\frac{p}{t(1-t)}\right) dp = t^s (1-t)^s \, {}^\Psi\hat{\Gamma}(s).$$

Thus, we get

$$\mathscr{M}\left[{}^\Psi\hat{F}_p(a,b;c;z) : s\right] = \frac{{}^\Psi\hat{\Gamma}(s) B(b+s, c+s-b)}{B(b,c-b)} {}_2F_1(a, b+s; c+2s; z).$$

Corollary 16. *The following equality is provided for $Re(s) > 0$:*

$${}^\Psi\hat{F}_p(a,b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{{}^\Psi\hat{\Gamma}(s) B(b+s, c+s-b)}{B(b,c-b)} {}_2F_1(a, b+s; c+2s; z) p^{-s} ds.$$

Theorem 17. *The following equality is provided for $Re(s) > 0$:*

$$\mathscr{M}\left[{}^\Psi\hat{\Phi}_p(b;c;z) : s\right] = \frac{{}^\Psi\hat{\Gamma}(s) B(b+s, c+s-b)}{B(b,c-b)} \Phi(b+s; c+2s; z).$$

Proof. By applying Mellin transformation to equality (10), we get

$$\begin{aligned}
\mathscr{M}\left[{}^\Psi\hat{\Phi}_p(b;c;z) : s\right] &= \int_0^\infty p^{s-1}\left[{}^\Psi\hat{\Phi}_p(b;c;z)\right] dp \\
&= \int_0^\infty p^{s-1} \sum_{n=0}^\infty \frac{{}^\Psi\hat{B}_p(b+n, c-b)}{B(b,c-b)} \frac{z^n}{n!} dp \\
&= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} \left[\int_0^\infty p^{s-1} {}_\xi\Psi_\eta \left(\frac{-p}{t(1-t)}\right) dp\right] dt.
\end{aligned}$$

Substituting $u = \frac{p}{t(1-t)}$ in the above equation we get

$$\int_0^\infty p^{s-1} {}_\xi\Psi_\eta \left(-\frac{p}{t(1-t)}\right) dp = t^s (1-t)^s \, {}^\Psi\hat{\Gamma}(s).$$

Thus, we have

$$\mathscr{M}\left[{}^\Psi\hat{\Phi}_p(b;c;z) : s\right] = \frac{{}^\Psi\hat{\Gamma}(s) B(b+s, c+s-b)}{B(b,c-b)} \Phi(b+s; c+2s; z).$$

Corollary 18. *For $Re(s) > 0$, we have the following equality:*

$${}^\Psi\hat{\Phi}_p(b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{{}^\Psi\hat{\Gamma}(s) B(b+s, c+s-b)}{B(b,c-b)} \Phi(b+s; c+2s; z) p^{-s} ds.$$

The following two theorems are about the transformation formulas of ${}_\xi\Psi_\eta$-Gauss and ${}_\xi\Psi_\eta$-confluent hypergeometric functions.

Theorem 19. *The following equality holds true:*

$${}^\Psi\hat{F}_p(a,b;c;z) = (1-z)^{-a}\left[{}^\Psi\hat{F}_p\left(a, c-b; b; \frac{z}{z-1}\right)\right].$$

Proof. By writing

$$[1-z(1-t)]^{-a} = (1-z)^{-a}\left(1+\frac{zt}{1-z}\right)^{-a}$$

and replacing $t \to 1-t$ in (9), we obtain

$${}^{\Psi}\hat{F}_p(a,b;c;z) = \frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}\left(1-\frac{zt}{z-1}\right)^{-a} {}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt.$$

Then we have

$${}^{\Psi}\hat{F}_p(a,b;c;z) = (1-z)^{-a}\left[{}^{\Psi}\hat{F}_p\left(a,c-b;b;\frac{z}{z-1}\right)\right],$$

which is the result.

Theorem 20. *The following equality holds true:*

$${}^{\Psi}\hat{\Phi}_p(b;c;z) = e^{z}\left[{}^{\Psi}\hat{\Phi}_p(c-b;b;-z)\right].$$

Proof. From the definition of confluent hypergeometric function, we have

$$\begin{aligned}{}^{\Psi}\hat{\Phi}_p(b;c;z) =& \sum_{n=0}^{\infty}\frac{{}^{\Psi}\hat{B}_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\\ =& \frac{1}{B(b,c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}e^{zt}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt. \qquad (14)\end{aligned}$$

Replacing $t = 1-u$ in (14), we obtain

$$\begin{aligned}{}^{\Psi}\hat{\Phi}_p(b;c;z) =& \frac{1}{B(b,c-b)}\int_0^1 u^{c-b-1}(1-u)^{b+n-1}e^{z(1-u)}{}_{\xi}\Psi_{\eta}\left(\frac{-p}{u(1-u)}\right)du\\ =& e^{z}\left[{}^{\Psi}\hat{\Phi}_p(c-b;b;-z)\right],\end{aligned}$$

which gives the result.

The following theorems are about the differential and difference relations for ${}_{\xi}\Psi_{\eta}$-Gauss hypergeometric and ${}_{\xi}\Psi_{\eta}$-confluent hypergeometric functions.

Theorem 21. *The following relations hold true:*

$$\Delta_a\left[{}^{\Psi}\hat{F}_p(a,b;c;z)\right] = z\frac{b}{c}\,{}^{\Psi}\hat{F}_p(a+1,b+1;c+1;z) \qquad (15)$$

$$a\Delta_a\left[{}^{\Psi}\hat{F}_p(a,b;c;z)\right] = z\frac{d}{dz}\left\{{}^{\Psi}\hat{F}_p(a,b;c;z)\right\} \qquad (16)$$

$$b\Delta_b\left[{}^{\Psi}\hat{\Phi}_p(b;c+1;z)\right] = -c\Delta_c\left[{}^{\Psi}\hat{\Phi}_p(b;c;z)\right] \qquad (17)$$

$$\frac{d}{dz}\left\{{}^{\Psi}\hat{\Phi}_p(b;c;z)\right\} = \frac{b}{c}\,{}^{\Psi}\hat{\Phi}_p(b;c+1;z)-\Delta_c\left[{}^{\Psi}\hat{\Phi}_p(b;c;z)\right] \qquad (18)$$

where Δ_α denotes the difference operator defined by

$$\Delta_\alpha f(\alpha,\ldots) = f(\alpha+1,\ldots)-f(\alpha,\ldots).$$

Proof. It is seen from (9) and the difference operator Δ_a that

$$\begin{aligned}\Delta_a {}^{\Psi}\hat{F}(a,b;c;z) =& {}^{\Psi}\hat{F}(a+1,b;c;z) - {}^{\Psi}\hat{F}(a,b;c;z)\\ =& \frac{z}{B(b,c-b)}\int_0^1 t^b(1-t)^{c-b-1}(1-zt)^{-a-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt.\end{aligned} \tag{19}$$

If we write $a+1, b+1$ and $c+1$ instead of a, b and c in equation (9), we get the following equation:

$${}^{\Psi}\hat{F}(a+1,b+1;c+1;z) = \frac{1}{B(b+1,c-b)}\int_0^1 t^b(1-t)^{c-b-1}(1-zt)^{-a-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt. \tag{20}$$

Now using (11) and (20) in (19) we get (15). Using differentiation formula (12) proves (16). Using the difference operator and (10), we obtain (17). Using differentiation formula (13) with $n = 1$, and considering (17) gives us (18).

4 Results and Recommendations

In this study, we introduced new generalizations of gamma, beta, Gauss and confluent hypergeometric functions with the help of Fox-Wright function. We also obtained some of their integral representations, Mellin transformations, derivative formulas, transformation formulas and reduction relations.

When the special cases of these functions are examined, it is seen that these functions are the generalizations of the following predefined functions which can be found in the literature:

For $p = 0$:

$$\begin{aligned}\Gamma(x) =& {}^{\Psi}\Gamma_0\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| x\right],\\ B(x,y) =& {}^{\Psi}B_0\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| x,y\right],\\ F(a,b;c;z) =& {}^{\Psi}F_0\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| a,b;c;z\right],\\ \Phi(b;c;z) =& {}^{\Psi}\Phi_0\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| b;c;z\right].\end{aligned}$$

For $p \neq 0$:

$$\begin{aligned}\Gamma_p(x) =& {}^{\Psi}\Gamma_p\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| x\right],\\ B_p(x,y) =& {}^{\Psi}B_p\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| x,y\right],\\ F_p(a,b;c;z) =& {}^{\Psi}F_p\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| a,b;c;z\right],\\ \Phi_p(b;c;z) =& {}^{\Psi}\Phi_p\left[\begin{matrix}(1,0)_{1,1}\\(1,0)_{1,1}\end{matrix}\middle| b;c;z\right].\end{aligned}$$

And also for $p \neq 0$:

$$\Gamma_p^{(\alpha,\beta)}(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} {}^{\Psi}\Gamma_p \left[\begin{matrix} (\alpha,1)_{1,1} \\ (\beta,1)_{1,1} \end{matrix} \middle| x \right],$$

$$B_p^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} {}^{\Psi}B_p \left[\begin{matrix} (\alpha,1)_{1,1} \\ (\beta,1)_{1,1} \end{matrix} \middle| x,y \right],$$

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} {}^{\Psi}F_p \left[\begin{matrix} (\alpha,1)_{1,1} \\ (\beta,1)_{1,1} \end{matrix} \middle| a,b;c;z \right],$$

$$\Phi_p^{(\alpha,\beta)}(b;c;z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} {}^{\Psi}\Phi_p \left[\begin{matrix} (\alpha,1)_{1,1} \\ (\beta,1)_{1,1} \end{matrix} \middle| b;c;z \right],$$

where Γ, B, F and Φ are the classic gamma, beta, Gauss and confluent hypergeometric functions; Γ_p, B_p, F_p and Φ_p are the functions defined in [7, 8, 10]; $\Gamma_p^{(\alpha,\beta)}$, $B_p^{(\alpha,\beta)}$, $F_p^{(\alpha,\beta)}$ and $\Phi_p^{(\alpha,\beta)}$ are the functions defined in [25].

Besides, the generalized beta function described in this study can be used to define similar generalizations of multivariate hypergeometric functions, which also known as Appell, Lauricella, Horn and Srivastava functions (see [3, 5] and the references therein). Further properties of these functions can be examined and they can also be used in fractional theory (see for example [2, 4, 30] and the references therein).

Acknowledgements

This work was partly presented in the 4th International Conference on Computational Mathematics and Engineering Sciences which organized by Akdeniz University on April 20-22, 2019 in Antalya-Turkey.

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