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Important Notes for a Fuzzy Boundary Value Problem

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Abstract

In this paper is studied a fuzzy Sturm-Liouville problem with the eigenvalue parameter in the boundary condition. Important notes are given for the problem. Integral equations are found of the problem.

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1 Introduction

Firstly, Zadeh introduced the concept of fuzzy numbers and fuzzy arithmetic [22]. The major application of fuzzy arithmetic is fuzzy differential equations. Fuzzy differential equations are suitable models to model dynamic systems in which there exist uncertainties or vaguness. Fuzzy differential equations can be examined by several approach, such as Hukuhara differentiability, generalized differentiability, the concept of differential inclusion etc [1], [3], [4]- [5], [6]- [9], [11], [13]- [15] [17], [19]- [20].

In this paper is on a fuzzy Sturm-Liouville problem with the eigenvalue parameter in the boundary condition. Important notes are given for the problem. Integral equations are found of the problem.

2 Preliminaries

Definition 1. *[18] A fuzzy number is a function $u:\mathbb{R}\to[0,1]$ satisfying the properties:u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$, $cl\left\{x\varepsilon\mathbb{R}\mid u(x)>0\right\}$ is compact, where cl denotes the closure of a subset.*

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Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2. *[15] Let $u\varepsilon\mathbb{R}_F$. The α-level set of u, denoted , $[u]^\alpha$, $0<\alpha\leq 1$, is*

$$[u]^\alpha = \{x\varepsilon\mathbb{R} \mid u(x)\geq\alpha\}.$$

If $\alpha=0$, the support of u is defined

$$[u]^0 = cl\{x\varepsilon\mathbb{R} \mid u(x)>0\}.$$

The notation, $[u]^\alpha=[\underline{u}_\alpha,\overline{u}_\alpha]$ denotes explicitly the α-level set of u.We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u,respectively.

The following remark shows when $[\underline{u}_\alpha,\overline{u}_\alpha]$ is a valid α-level set.

Remark 1. *[10, 15] The sufficient and necessary conditions for $[\underline{u}_\alpha,\overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:*

$\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,
$\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha=0$,
$\underline{u}_\alpha\leq\overline{u}_\alpha$, $0\leq\alpha\leq 1$.

Definition 3. *[15] For $u,v\varepsilon\mathbb{R}_F$ and $\lambda\in\mathbb{R}$, the sum $u+v$ and the product λu are defined by $[u+v]^\alpha=[u]^\alpha+[v]^\alpha$, $[\lambda u]^\alpha=\lambda[u]^\alpha$ where means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.*
The metric structure is given by the Hausdorff distance

$$D:\mathbb{R}_F\times\mathbb{R}_F\to\mathbb{R}_+\cup\{0\},$$

by

$$D(u,v)=\sup_{\alpha\in[0,1]}\max\{|\underline{u}_\alpha-\underline{v}_\alpha|,|\overline{u}_\alpha-\overline{v}_\alpha|\}.$$

Definition 4. *[18] If A is a symmetric triangular numbers with supports $[\underline{a},\overline{a}]$, the $\alpha-$level sets of A is $[A]^\alpha=\left[\underline{a}+\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha,\overline{a}-\left(\frac{\overline{a}-\underline{a}}{2}\right)\alpha\right]$.*

Definition 5. *[16] $u,v\in\mathbb{R}_F$, $[u]^\alpha=[\underline{u}_\alpha,\overline{u}_\alpha]$, $[v]^\alpha=[\underline{v}_\alpha,\overline{v}_\alpha]$, the product uv is defined by*

$$[uv]^\alpha=[u]^\alpha[v]^\alpha,\ \forall\alpha\in[0,1],$$

where

$$[u]^\alpha[v]^\alpha=[\underline{u}_\alpha,\overline{u}_\alpha][\underline{v}_\alpha,\overline{v}_\alpha]=[\underline{w}_\alpha,\overline{w}_\alpha],$$
$$\underline{w}_\alpha=\min\{\underline{u}_\alpha\underline{v}_\alpha,\underline{u}_\alpha\overline{v}_\alpha,\overline{u}_\alpha\underline{v}_\alpha,\overline{u}_\alpha\overline{v}_\alpha\},$$
$$\overline{w}_\alpha=\max\{\underline{u}_\alpha\underline{v}_\alpha,\underline{u}_\alpha\overline{v}_\alpha,\overline{u}_\alpha\underline{v}_\alpha,\overline{u}_\alpha\overline{v}_\alpha\}.$$

Definition 6. *[15, 21] Let $u,v\in\mathbb{R}_F$.If there exists $w\in\mathbb{R}_F$ such that $u=v+w$, then w is called the Hukuhara difference of fuzzy numbers u and v,and it is denoted by $w=u\ominus v$.*

Definition 7. *[2, 15] Let $f:[a,b]\to\mathbb{R}_F$ and $t_0\in[a,b]$.We say that f is Hukuhara differential at t_0, if there exists an element $f'(t_0)\in\mathbb{R}_F$ such that for all $h>0$ sufficiently small, $\exists f(t_0+h)\ominus f(t_0)$, $f(t_0)\ominus f(t_0-h)$ and the limits (in the metric D)*

$$\lim_{h\to 0}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to 0}\frac{f(t_0)\ominus f(t_0-h)}{h}=f'(t_0).$$

Definition 8. *[12] If $p^{'}(x) = 0$, $r(x) = 1$ and $Ly = p(x)y^{''} + q(x)y$ in the fuzzy differential equation $\left(p(x)y^{'}\right)^{'} + q(x)y + \lambda r(x)y = 0$, $p(x)$, $p^{'}(x)$, $q(x)$, $r(x)$, are continuous functions and positive, the fuzzy differential equation*

$$Ly + \lambda y = 0 \tag{2.1}$$

is called a fuzzy Sturm-Liouville equation.

Definition 9. *[12] $[y(x,\lambda_0)]^{\alpha} = \left[\underline{y}(x,\lambda_0), \overline{y}(x,\lambda_0)\right] \neq 0$, we say that $\lambda = \lambda_0$ is eigenvalue of (2.1) if the fuzzy differential equation (2.1) has the nontrivial solutions $\underline{y}(x,\lambda_0) \neq 0$, $\overline{y}(x,\lambda_0) \neq 0$.*

3 Findings and Main Results

Consider the fuzzy boundary value problem

$$\tau u = u^{''} + q(x)u,$$

$$\tau u + \lambda u = 0, x \in (-1,1) \tag{3.1}$$

$$u(-1) + u^{'}(-1) = 0, \tag{3.2}$$

$$\lambda \beta u(1) + \gamma u^{'}(1) = 0, \tag{3.3}$$

where q(x) is continuous function and positive on [-1,1] , $\lambda > 0$ and $\beta, \gamma > 0$.

Let $u_1(x,\lambda)$ and $u_2(x,\lambda)$ be linearly independent solutions of the classical differential equation $\tau u + \lambda u = 0$. Then, the general solution of the fuzzy differential equation (3.1) is

$$[u(x,\lambda)]^{\alpha} = [\underline{u}_{\alpha}(x,\lambda), \overline{u}_{\alpha}(x,\lambda)],$$

$$\underline{u}_{\alpha}(x,\lambda) = a_{\alpha}(\lambda)u_1(x,\lambda) + b_{\alpha}(\lambda)u_2(x,\lambda),$$

$$\overline{u}_{\alpha}(x,\lambda) = c_{\alpha}(\lambda)u_1(x,\lambda) + d_{\alpha}(\lambda)u_2(x,\lambda).$$

Also,

$$[\varphi(x,\lambda)]^{\alpha} = [\underline{\varphi}_{\alpha}(x,\lambda), \overline{\varphi}_{\alpha}(x,\lambda)]$$

be the solution of the equation (3.1) satisfying the conditions

$$u(-1) = 1, u^{'}(-1) = -1 \tag{3.4}$$

and

$$[\chi(x,\lambda)]^{\alpha} = [\underline{\chi}_{\alpha}(x,\lambda), \overline{\chi}_{\alpha}(x,\lambda)]$$

be the solution of the equation (3.1) satisfying the conditions

$$u(1) = \gamma, u^{'}(1) = -\lambda\beta \tag{3.5}$$

where

$$\underline{\varphi}_{\alpha}(x,\lambda) = \underline{c}_{1\alpha}(\lambda)u_1(x,\lambda) + \underline{c}_{2\alpha}(\lambda)u_2(x,\lambda),$$
$$\overline{\varphi}_{\alpha}(x,\lambda) = \overline{c}_{1\alpha}(\lambda)u_1(x,\lambda) + \overline{c}_{2\alpha}(\lambda)u_2(x,\lambda)$$

$$\underline{\chi}_\alpha(x,\lambda)=\underline{c}_{3\alpha}(\lambda)u_1(x,\lambda)+\underline{c}_{4\alpha}(\lambda)u_2(x,\lambda)$$
$$\overline{\chi}_\alpha(x,\lambda)=\overline{c}_{3\alpha}(\lambda)u_1(x,\lambda)+\overline{c}_{4\alpha}(\lambda)u_2(x,\lambda).$$

From here, yields

$$W\left(\underline{\varphi}_\alpha,\underline{\chi}_\alpha\right)(x,\lambda)=\left(\underline{c}_{1\alpha}(\lambda)\underline{c}_{4\alpha}(\lambda)-\underline{c}_{2\alpha}(\lambda)\underline{c}_{3\alpha}(\lambda)\right)W(u_1,u_2)(x,\lambda)\tag{3.6}$$

$$W\left(\overline{\varphi}_\alpha,\overline{\chi}_\alpha\right)(x,\lambda)=\left(\overline{c}_{1\alpha}(\lambda)\overline{c}_{4\alpha}(\lambda)-\overline{c}_{2\alpha}(\lambda)\overline{c}_{3\alpha}(\lambda)\right)W(u_1,u_2)(x,\lambda)\tag{3.7}$$

Also, since $u_1(x,\lambda)$ and $u_2(x,\lambda)$ are linearly independent solutions of the classical differential equation $\tau u+\lambda u=0$,the solution of the equation is

$$u(x,\lambda)=a(\lambda)u_1(x,\lambda)+b(\lambda)u_2(x,\lambda).$$

$\varphi(x,\lambda)$ be the solution of the classical differential equation $\tau u+\lambda u=0$ satisfying the conditions $u(-1)=1, u'(-1)=-1$. Using boundary conditions, we have

$$a(\lambda)u_1(-1,\lambda)+b(\lambda)u_2(-1,\lambda)=1,$$

$$a(\lambda)u_1'(-1,\lambda)+b(\lambda)u_2'(-1,\lambda)=-1.$$

From this, $a(\lambda)$, $b(\lambda)$ are obtained as

$$a(\lambda)=\frac{u_2'(-1,\lambda)+u_2(-1,\lambda)}{W(u_1,u_2)(-1,\lambda)},b(\lambda)=\frac{-u_1(-1,\lambda)-u_1'(-1,\lambda)}{W(u_1,u_2)(-1,\lambda)}.$$

Then,

$$\begin{aligned}\varphi(x,\lambda)=\frac{1}{W(u_1,u_2)(-1,\lambda)}&\left\{\left(u_2'(-1,\lambda)+u_2(-1,\lambda)\right)u_1(x,\lambda)\right.\\&\left.-\left(u_1(-1,\lambda)+u_1'(-1,\lambda)u_2(x,\lambda)\right)\right\}.\end{aligned}$$

Again, $\chi(x,\lambda)$ be the solution of the classical differential equation $\tau u+\lambda u=0$ satisfying the conditions $u(1)=\gamma, u'(1)=-\lambda\beta$. Similarly, $\chi(x,\lambda)$ is obtained as

$$\begin{aligned}\chi(x,\lambda)=\frac{1}{W(u_1,u_2)(-1,\lambda)}&\left\{\left(\lambda\beta u_2(1,\lambda)+\gamma u_2'(1,\lambda)\right)u_1(x,\lambda)\right.\\&\left.-\left(\lambda\beta u_1(1,\lambda)+\gamma u_1'(1,\lambda)\right)u_2(x,\lambda)\right\}\end{aligned}$$

Thus,

$$[\varphi(x,\lambda)]^\alpha=\left[\underline{\varphi}_\alpha(x,\lambda),\overline{\varphi}_\alpha(x,\lambda)\right]=[c_1(\alpha),c_2(\alpha)]\varphi(x,\lambda)$$

is the solution of the equation (3.1) satisfying the conditions (3.4) and

$$[\chi(x,\lambda)]^\alpha=\left[\underline{\chi}_\alpha(x,\lambda),\overline{\chi}_\alpha(x,\lambda)\right]=[c_1(\alpha),c_2(\alpha)]\chi(x,\lambda)$$

is the solution of the equation (3.1) satisfying the conditions (3.5), where $[c_1(\alpha),c_2(\alpha)]=[1]^\alpha$. We take $[1]^\alpha=[\alpha,2-\alpha]$. From here,

$$\underline{W}_{\alpha}(x,\lambda)=\alpha^{2}\left(\varphi(x,\lambda)\chi^{'}(x,\lambda)-\chi(x,\lambda)\varphi^{'}(x,\lambda)\right) \tag{3.8}$$

$$\overline{W}_{\alpha}(x,\lambda)=(2-\alpha)^{2}\left(\varphi(x,\lambda)\chi^{'}(x,\lambda)-\chi(x,\lambda)\varphi^{'}(x,\lambda)\right) \tag{3.9}$$

are obtained. Computing the value $\varphi(x,\lambda)\chi^{'}(x,\lambda)-\chi(x,\lambda)\varphi^{'}(x,\lambda)$, we have

$$\frac{W(u_1,u_2)(x,\lambda)}{W(u_1,u_2)(-1,\lambda)W(u_1,u_2)(1,\lambda)}\left\{\left(u_1(-1,\lambda)+u_1^{'}(-1,\lambda)\right)\left(\lambda\beta u_2(1,\lambda)+\gamma u_2^{'}(1,\lambda)\right)\right.$$
$$\left.-\left(u_2(-1,\lambda)+u_2^{'}(-1,\lambda)\right)\left(\lambda\beta u_1(1,\lambda)+\gamma u_1^{'}(1,\lambda)\right)\right\}$$

Considering the equations (3.6) and (3.7), the value $\underline{c}_{1\alpha}(\lambda)\underline{c}_{4\alpha}(\lambda)-\underline{c}_{2\alpha}(\lambda)\underline{c}_{3\alpha}(\lambda)$ is

$$\frac{\alpha^{2}}{W(u_1,u_2)(-1,\lambda)W(u_1,u_2)(1,\lambda)}\left\{\left(u_1(-1,\lambda)+u_1^{'}(-1,\lambda)\right)\left(\lambda\beta u_2(1,\lambda)+\gamma u_2^{'}(1,\lambda)\right)\right.$$
$$\left.-\left(u_2(-1,\lambda)+u_2^{'}(-1,\lambda)\right)\left(\lambda\beta u_1(1,\lambda)+\gamma u_1^{'}(1,\lambda)\right)\right\}$$

and the value $\overline{c}_{1\alpha}(\lambda)\overline{c}_{4\alpha}(\lambda)-\overline{c}_{2\alpha}(\lambda)\overline{c}_{3\alpha}(\lambda)$ is

$$\frac{(2-\alpha)^{2}}{W(u_1,u_2)(-1,\lambda)W(u_1,u_2)(1,\lambda)}\left\{\left(u_1(-1,\lambda)+u_1^{'}(-1,\lambda)\right)\left(\lambda\beta u_2(1,\lambda)+\gamma u_2^{'}(1,\lambda)\right)\right.$$
$$\left.-\left(u_2(-1,\lambda)+u_2^{'}(-1,\lambda)\right)\left(\lambda\beta u_1(1,\lambda)+\gamma u_1^{'}(1,\lambda)\right)\right\}$$

Consequently,

$$W\left(\underline{\varphi}_{\alpha},\underline{\chi}_{\alpha}\right)(x,\lambda)=\frac{\alpha^{2}}{(2-\alpha)^{2}}W\left(\overline{\varphi}_{\alpha},\overline{\chi}_{\alpha}\right)(x,\lambda).$$

Theorem 1. *The Wronskian functions $W\left(\underline{\varphi}_{\alpha},\underline{\chi}_{\alpha}\right)(x,\lambda)$ and $W\left(\overline{\varphi}_{\alpha},\overline{\chi}_{\alpha}\right)(x,\lambda)$ are independent of variable x for $x\in(-1,1)$, where functions $\underline{\varphi}_{\alpha},\underline{\chi}_{\alpha},\overline{\varphi}_{\alpha},\overline{\chi}_{\alpha}$ are the solution of the fuzzy boundary value problem (3.1)-(3.3).*

Proof. Derivating of equations $W\left(\underline{\varphi}_{\alpha},\underline{\chi}_{\alpha}\right)(x,\lambda)$ and $W\left(\overline{\varphi}_{\alpha},\overline{\chi}_{\alpha}\right)(x,\lambda)$ according to variable x and using the functions $[\varphi(x,\lambda)]^{\alpha}$, $[\chi(x,\lambda)]^{\alpha}$ are the solutions of the equation (3.1)

$$W^{'}\left(\underline{\varphi}_{\alpha},\underline{\chi}_{\alpha}\right)(x,\lambda)=0 \textit{ and } W^{'}\left(\overline{\varphi}_{\alpha},\overline{\chi}_{\alpha}\right)(x,\lambda)=0.$$

are obtained. The proof is complete.

$$\underline{W}_{\alpha}(\lambda)=W\left(\underline{\varphi}_{\alpha},\underline{\chi}_{\alpha}\right)(x,\lambda),\ \overline{W}_{\alpha}(\lambda)=W\left(\overline{\varphi}_{\alpha},\overline{\chi}_{\alpha}\right)(x,\lambda). \tag{3.10}$$

□

Theorem 2. *The eigenvalues of the fuzzy boundary value problem (3.1)-(3.3) if and only if are consist of the zeros of functions $\underline{W}_{\alpha}(\lambda)$ and $\overline{W}_{\alpha}(\lambda)$.*

Proof. Let be $\lambda = \lambda_0$ is the eigenvalue. We show that $\underline{W}_\alpha(\lambda_0) = 0$ and $\overline{W}_\alpha(\lambda_0) = 0$. We assume that $\underline{W}_\alpha(\lambda_0) \neq 0$ or $\overline{W}_\alpha(\lambda_0) \neq 0$. Let be $\underline{W}_\alpha(\lambda_0) \neq 0$. Then, the functions $\underline{\varphi}_\alpha(x,\lambda_0)$ and $\underline{\chi}_\alpha(x,\lambda_0)$ are linearly independent. So, the general solution of the equation (3.1)

$$[u(x,\lambda_0)]^\alpha = [\underline{u}_\alpha(x,\lambda_0), \overline{u}_\alpha(x,\lambda_0)], \tag{3.11}$$

$$\underline{u}_\alpha(x,\lambda_0) = a_\alpha(\lambda_0)\,\underline{\varphi}_\alpha(x,\lambda_0) + b_\alpha(\lambda_0)\,\underline{\chi}_\alpha(x,\lambda_0), \tag{3.12}$$

$$\overline{u}_\alpha(x,\lambda_0) = c_\alpha(\lambda_0)\;\overline{\varphi}_\alpha(x,\lambda_0) + d_\alpha(\lambda_0)\,\overline{\chi}_\alpha(x,\lambda_0). \tag{3.13}$$

Using the boundary condition (3.2) and using the solution function $[\varphi(x,\lambda_0)]^\alpha = [\underline{\varphi}_\alpha(x,\lambda_0), \overline{\varphi}_\alpha(x,\lambda_0)]$ satisfies the boundary condition (3.2),

$$b_\alpha(\lambda_0)\left(\underline{\chi}_\alpha(-1,\lambda_0) + \underline{\chi}'_\alpha(-1,\lambda_0)\right) = 0,$$

$$d_\alpha(\lambda_0)\left(\overline{\chi}_\alpha(-1,\lambda_0) + \overline{\chi}'_\alpha(-1,\lambda_0)\right) = 0$$

are obtained. Again, using (3.4), (3.10), we have

$$b_\alpha(\lambda_0)\underline{W}_\alpha(\lambda_0) = 0,\ d_\alpha(\lambda_0)\,\overline{W}_\alpha(\lambda_0) = 0.$$

From this, since $\underline{W}_\alpha(\lambda_0) \neq 0$, $b_\alpha(\lambda_0) = 0$ is obtained. Similarly, using the boundary condition (3.3), we obtained $a_\alpha(\lambda_0) = 0$. Thus, $\underline{u}_\alpha(x,\lambda_0) = 0$, λ_0 is not an eigenvalue. That is, we have a contradiction. Similarly, if $\overline{W}_\alpha(\lambda_0) \neq 0$, $\overline{u}_\alpha(x,\lambda_0) = 0$ is obtained. λ_0 is not an eigenvalue.

Let λ_0 be zero of $\underline{W}_\alpha(\lambda)$ and $\overline{W}_\alpha(\lambda)$. Then,

$$\underline{\varphi}_\alpha(x,\lambda_0) = k_1\underline{\chi}_\alpha(x,\lambda_0),\ \overline{\varphi}_\alpha(x,\lambda_0) = k_2\overline{\chi}_\alpha(x,\lambda_0). \tag{3.14}$$

That is, the functions $\underline{\varphi}_\alpha$, $\underline{\chi}_\alpha$ and $\overline{\varphi}_\alpha$, $\overline{\chi}_\alpha$ are linearly dependent. Also, since $[\chi(x,\lambda)]^\alpha$ satisfies the boundary condition (3.3), $\underline{\chi}_\alpha(x,\lambda_0)$ and $\overline{\chi}_\alpha(x,\lambda_0)$ satisfy the boundary condition (3.3). In addition, from (3.14) the functions $\underline{\varphi}_\alpha(x,\lambda_0)$ and $\overline{\varphi}_\alpha(x,\lambda_0)$ satisfy the boundary condition (3.3). So, $[\varphi(x,\lambda_0)]^\alpha$ satisfies the boundary condition (3.3). Hence, $[\varphi(x,\lambda_0)]^\alpha$ is the solution of the boundary value problem (3.1)-(3.3) for $\lambda = \lambda_0$. Thus, $\lambda = \lambda_0$ is the eigenvalue. The proof is complete.

Lemma 1. *Let $\lambda = s^2$. The lower and the upper solutions $\underline{\varphi}_\alpha(x,\lambda), \overline{\varphi}_\alpha(x,\lambda)$ satisfy the following integral equations for k=0 and k=1:*

$$\left(\underline{\varphi}_\alpha(x,\lambda)\right)^{(k)} = \left(Cos\left(s\left(x+1\right)\right)\right)^{(k)} - \frac{1}{s}\left(\mathrm{S}in\left(s\left(x+1\right)\right)\right)^{(k)} + \frac{1}{s}\int_{-1}^{x}\left(\mathrm{S}in\left(s\left(x-y\right)\right)\right)^{(k)} q(y)\left(\underline{\varphi}_\alpha(y,\lambda)\right)^{(k)} dy, \tag{3.15}$$

$$\left(\overline{\varphi}_\alpha(x,\lambda)\right)^{(k)} = \left(Cos\left(s\left(x+1\right)\right)\right)^{(k)} - \frac{1}{s}\left(\mathrm{S}in\left(s\left(x+1\right)\right)\right)^{(k)} + \frac{1}{s}\int_{-1}^{x}\left(\mathrm{S}in\left(s\left(x-y\right)\right)\right)^{(k)} q(y)\left(\overline{\varphi}_\alpha(y,\lambda)\right)^{(k)} dy. \quad \square \tag{3.16}$$

Proof. Since

$$[\varphi(x,\lambda)]^{\alpha} = [\underline{\varphi}_{\alpha}(x,\lambda), \overline{\varphi}_{\alpha}(x,\lambda)]$$

is the solution of the equation (3.1), the equation

$$[\underline{\varphi}_{\alpha}(y,\lambda), \overline{\varphi}_{\alpha}(y,\lambda)]'' + q(y)[\underline{\varphi}_{\alpha}(y,\lambda), \overline{\varphi}_{\alpha}(y,\lambda)] + \lambda[\underline{\varphi}_{\alpha}(y,\lambda), \overline{\varphi}_{\alpha}(y,\lambda)] = 0$$

is provided. Using the Hukuhara differentiability and fuzzy arithmetic,

$$[\underline{\varphi}''_{\alpha}(y,\lambda), \overline{\varphi}''_{\alpha}(y,\lambda)] + [q(y)\underline{\varphi}_{\alpha}(y,\lambda), q(y)\overline{\varphi}_{\alpha}(y,\lambda)] + [\lambda\underline{\varphi}_{\alpha}(y,\lambda), \lambda\overline{\varphi}_{\alpha}(y,\lambda)] = 0$$

is obtained. From here, yields

$$\underline{\varphi}''_{\alpha}(y,\lambda) + q(y)\underline{\varphi}_{\alpha}(y,\lambda) + \lambda\underline{\varphi}_{\alpha}(y,\lambda) = 0,$$

$$\overline{\varphi}''_{\alpha}(y,\lambda) + q(y)\overline{\varphi}_{\alpha}(y,\lambda) + \lambda\overline{\varphi}_{\alpha}(y,\lambda) = 0.$$

Substituing the identity $q(y)\underline{\varphi}_{\alpha}(y,\lambda) = -\lambda\underline{\varphi}_{\alpha}(y,\lambda) - \underline{\varphi}''_{\alpha}(y,\lambda)$ in the right side of (3.15)

$$\begin{aligned}\int_{-1}^{x} \mathrm{Sin}(s(x-y))\,q(y)\,\underline{\varphi}_{\alpha}(y,\lambda)\,dy = &-s^2\int_{-1}^{x} \mathrm{Sin}(s(x-y))\,\underline{\varphi}_{\alpha}(y,\lambda)\,dy \\ &-\int_{-1}^{x} \mathrm{Sin}(s(x-y))\,\underline{\varphi}''_{\alpha}(x,\lambda)\,dy\end{aligned}$$

On integrating by parts twice and using (3.4)

$$\begin{aligned}\int_{-1}^{x} \mathrm{Sin}(s(x-y))\,\underline{\varphi}''_{\alpha}(x,\lambda)\,dy = &\,\mathrm{Sin}(s(x+1)) - s\underline{\varphi}_{\alpha}(x,\lambda) + sCos(s(x+1)) \\ &+s^2\int_{-1}^{x} \mathrm{Sin}(s(x-y))\,\underline{\varphi}_{\alpha}(y,\lambda)\,dy\end{aligned}$$

is obtained. Substituing this back into the previous equality yields

$$\int_{-1}^{x} \mathrm{Sin}(s(x-y))\,q(y)\,\underline{\varphi}_{\alpha}(y,\lambda)\,dy = \mathrm{Sin}(s(x+1)) - s\underline{\varphi}_{\alpha}(x,\lambda) + sCos(s(x+1))$$

From here, we have

$$\underline{\varphi}_{\alpha}(x,\lambda) = Cos(s(x+1)) - \frac{1}{s}\mathrm{Sin}(s(x+1)) + \frac{1}{s}\int_{-1}^{x} \mathrm{Sin}(s(x-y))\,q(y)\,\underline{\varphi}_{\alpha}(y,\lambda)\,dy.$$

Similarly $\overline{\varphi}_{\alpha}(x,\lambda)$ is found. Derivating in these equations according to x, the derivative equations are obtained.

Lemma 2. *Let $\lambda = s^2$. The lower and the upper solutions $\underline{\chi}_\alpha(x,\lambda), \overline{\chi}_\alpha(x,\lambda)$ satisfy the following integral equations for k=0 and k=1:*

$$\left(\underline{\chi}_\alpha(x,\lambda)\right)^{(k)} = -\beta\left(Cos(s(x+1))\right)^{(k)} + \frac{\lambda\alpha}{s}\left(\mathrm{Si}n(s(x+1))\right)^{(k)} - \frac{1}{s}\int_x^1 \left(\mathrm{Si}n(s(x-y))\right)^{(k)} q(y)\left(\underline{\chi}_\alpha(y,\lambda)\right)^{(k)} dy \tag{3.17}$$

$$\left(\overline{\chi}_\alpha(x,\lambda)\right)^{(k)} = -\beta\left(Cos(s(x+1))\right)^{(k)} + \frac{\lambda\alpha}{s}\left(\mathrm{Si}n(s(x+1))\right)^{(k)} - \frac{1}{s}\int_x^1 \left(\mathrm{Si}n(s(x-y))\right)^{(k)} q(y)\left(\overline{\chi}_\alpha(y,\lambda)\right)^{(k)} dy. \tag{3.18}$$

□

Proof. Substituing the identity $q(y)\underline{\chi}_\alpha(y,\lambda) = -\lambda\underline{\chi}_\alpha(y,\lambda) - \underline{\chi}_\alpha''(y,\lambda)$ in the right side of (3.17), integrating by parts twice and using (3.5) yields (3.17) for k=0. Similarly, the equation (3.18) is found for k=0. Derivating in these equations according to x, the derivative equations are obtained. □

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References

[1] B. Bede, A note on "two-point boundary value problems associated with non-linear fuzzy differential equations", Fuzzy Sets and Systems, 157 (2006), 986-989.
[2] B. Bede, Note on "Numerical solutions of fuzzy differential equations by predictor-corrector method", Inf. Sci., 178 (2008) 1917-1922.
[3] B. Bede, S.G. Gal, Almost periodic fuzzy-number-valued functions, Fuzzy Sets and Systems 147 (2004), 385-403.
[4] B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems, 151 (2005) 581-599.
[5] B. Bede, I.J. Rudas, A.L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Inform. Sci. 177 (2007), 1648-1662.
[6] J.J. Buckley, T. Feuring, Fuzzy differential equations, Fuzzy Sets and Systems, 110 (2000) 43-54.
[7] J.J. Buckley, T. Feuring, Fuzzy initial value problem for Nth-order linear differential equations, Fuzzy Sets and Systems, 121 (2001) 247-255.
[8] Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations, Chaos, Solutions and Fractals 38 (2008), 112-119.
[9] Y. Chalco-Cano, H. Roman-Flores, Comparation between some approaches to solve fuzzy differential equations, Fuzzy Sets and Systems 160 (2009), 1517-1527.
[10] D. Dubois, H. Prade, Operations on fuzzy numbers, International Journal of Systems Sciences, 9 (1978) 613-626.
[11] N. Gasilov, Ş.E. Amrahov, A.G. Fatullayev, A geometric approach to solve fuzzy linear systems of differential equations, Appl. Math. & Inf. Sci., 5 (2011) 484-495.
[12] H. Gültekin Çitil, N. Altınışık, On the eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy boundary value problem, Journal of Mathematical and Computational Science, 7(4) (2017) 786-805.
[13] E. Hüllermeir, An approach to modeling and simulation of uncertain dynamical systems, International Journal of Uncertanity Fuzziness and Knowledge-Based Systems, 5 (1997), 117-137.
[14] A. Khastan, F. Bahrami, K. Ivaz, New Results on Multiple Solutions for Nth-order Fuzzy Differential Equations under Generalized Differentiability, Boundary Value Problems, doi:10.1155/2009/395714, (2009), 1-13.

[15] A. Khastan, J.J. Nieto, A boundary value problem for second order fuzzy differential equations, Nonlinear Analysis, 72 (2010) 3583-3593.
[16] V. Lakshmikantham, R. N. Mohapatra, Theory of Fuzzy Differential Equations and Inclusions, Taylor&Francis, London, New York, 2003.
[17] V. Lakshmikantham, J.J. Nieto, Differential Equations in Metric Spaces: An introduction and application to fuzzy differential equations, Dynamics of Continuous., Discrete and Impulsive Systems Series A: Mathematical Analysis 10 (2003), 991-1000.
[18] H.-K. Liu, Comparison results of two-point fuzzy boundary value problems, International Journal of Computational and Mathematical Sciences, 5(1) (2011), 1-7.
[19] J.J. Nieto, R. Rodriguez-Lopez, Euler polygonal method for metric dynamical systems, Inform. Sci. 177 (2007), 4256-4270.
[20] J.J. Nieto, A. Khastan, K. Ivaz, Numerical solution of fuzzy differential equations under generalized differentiability, Nonlinear Analysis: Hybrid Syst. 3 (2009), 700-707.
[21] M.L. Puri, D.A. Ralescu, Differentials for fuzzy functions, Journal of Mathematical Analysis and Applications, 91 (1983) 552-558.
[22] L.A. Zadeh, Fuzzy Sets, Information and Control, 8 (1965), 338-353.