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Solitons and other solutions of (3 + 1)-dimensional space–time fractional modified KdV–Zakharov–Kuznetsov equation

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Abstract

In the current paper, we carry out an investigation into the exact solutions of the (3+1)-dimensional space-time fractional modified KdV–Zakharov–Kuznetsov (fractional mKdV–ZK) equation. Based on the conformable fractional derivative and its properties, the fractional mKdV–ZK equation is reduced into an ordinary differential equation which has been solved analytically by the variable separated ODE method. Various types of analytic solutions in terms of hyperbolic functions, trigonometric functions and Jacobi elliptic functions are derived. All conditions for the validity of all obtained solutions are given.

Keywords: The (3+1)-dimensional space-time fractional mKdV–ZK; the variable separated ODE method; solitons and periodic wave solutions
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1 Introduction

Fractional differential equations (FDEs) have become under remarkable consideration as being the generalisation form of the differential equations of integer order and due to their roles in the modelling of several physical processes. Many nonlinear phenomena in physics can be described via FDEs such as chemical physics, optical fibers, plasma, electromagnetic waves, diffusion processes, vibrations in a nonlinear string and etc [1–5]. Seeking different types of solutions to FDEs for such phenomena has become the subject of interest for researchers. Thus, a lot of powerful mathematical methods have been applied to obtain exact analytic solutions of FDEs, namely, the extended tanh-function method [6, 7], the exp-function method [8, 9], the sub-equation method [10, 11], the improved $\tan(\phi/2)$-expansion method [12, 13], the (G'/G)-expansion method [14, 15],

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the modified trial equation method [16, 17], the new extended direct algebraic method [18, 19], the extended sinh-Gordon equation expansion method [20, 21], the unified method [22] and so on.

One of the important physical model is the (3+1)-dimensional space-time fractional modified KdV–Zakharov–Kuznetsov (fractional mKdV–ZK) equation [23–25] of the form

$$D_t^{\alpha}u+\delta u^2 D_x^{\alpha}u+D_x^{3\alpha}u+D_x^{\alpha}D_y^{2\alpha}u+D_x^{\alpha}D_z^{2\alpha}u=0, \tag{1}$$

where $0<\alpha\leq 1$ and δ is an arbitrary constant. This model is developed for a plasma comprised of cool and hot electrons and a species of fluid ions [26]. Recently, Sahoo and Ray [10] have applied the improved fractional sub equation method with the aid of Jumarie's modified Riemann–Liouville derivative for space-time fractional mKdV–ZK equation and they derived exact solutions. Then, the fractional derivatives in the sense of modified Riemann–Liouville derivative and three different solution methods have been employed by Guner et. al. [27] to obtain a variety of exact solutions to equation (1). Moreover, using the ansatz method and the functional variable method, Guner [28] examined exact analytic solutions for equation (1) and constructed singular soliton solutions. In the two latter studies, the fractional complex transform method is used to reduce equation (1) into ordinary differential equation (ODE). However, in a recent study, this transformation has been proved to be incorrect by Herzallah [29]. Subsequently, all obtained solutions by means of this method are wrong.

The present study focuses on investigating the exact analytic solutions to space-time fractional mKdV–ZK equation using the variable separated ODE method [30, 31]. The structure of this work is organised as follows. In Section 2, we introduce the definition of conformable fractional derivative [32] and its properties which will be utilised to reduce FDE into an ODE. Section 3 contains the description of variable separated ODE method and the technique of implementing it to ODEs. In Section 4, the proposed method will be applied to construct the solitons and periodic wave solutions of space-time fractional mKdV–ZK equation. Then, the behaviour of some obtained solutions is displayed graphically. Finally, our discussions and conclusions are presented in Section 5.

2 Conformable fractional derivative

Khalil, et. al. [32] introduced a completely new definition of fractional calculus with the limit operator which is more natural and effective on satisfying some conventional properties than the existing fractional derivatives. The definition of conformable fractional derivative is given as follows:

Definition 1. Let $f:(0,\infty)\longrightarrow\mathbb{R}$, then the conformable fractional derivative of f of order α is defined as

$$D_t^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \tag{2}$$

for all $t>0$, $\alpha\in(0,1)$.

It is said that if the conformable fractional derivative of f of order α exists, then f is α-differentiable. The conformable fractional derivative satisfies the properties shown in the following theorems:

Theorem 1. *Let $\alpha\in(0,1]$ and $f=f(t), g=g(t)$ be α-differentiable at a point $t>0$, then:*

1. $D_t^{\alpha}(af+bg)=aD_t^{\alpha}f+bD_t^{\alpha}g$, *for all* $a,b\in\mathbb{R}$.
2. $D_t^{\alpha}(t^{\mu})=\mu t^{\mu-\alpha}$, *for all* $\mu\in\mathbb{R}$.
3. $D_t^{\alpha}(fg)=fD_t^{\alpha}g+gD_t^{\alpha}f$.
4. $D_t^{\alpha}\left(\frac{f}{g}\right)=\frac{gD_t^{\alpha}f-fD_t^{\alpha}g}{g^2}$.

Moreover, if f is differentiable, then $D_t^{\alpha}(f(t)) = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}$.

Theorem 2. *Let $f : (0,\infty) \longrightarrow \mathbb{R}$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable; then, one has the following rule:*

$$D_t^{\alpha}(fg)(t) = t^{1-\alpha}g'(t)f'(g(t)), \tag{3}$$

where prime denotes the classical derivatives with respect to t.

Remark 1. We may use the notation $\frac{\partial^{\alpha}}{\partial t^{\alpha}}f$ for $D_t^{\alpha}(f(t))$ to denote the conformable fractional derivatives of f with respect to the variable t of order α.

3 The variable separated ODE method

In this section we shall describe the technique of implementing the proposed method to FDEs in order to extract exact analytic solutions. Suppose that a nonlinear conformable fractional partial differential equation, say, in four independent variables x, y, z and t is given by

$$P(u, \frac{\partial^{\alpha}u}{\partial t^{\alpha}}, \frac{\partial^{\alpha}u}{\partial x^{\alpha}}, \frac{\partial^{\alpha}u}{\partial y^{\alpha}}, \frac{\partial^{\alpha}u}{\partial z^{\alpha}}, \frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}, \frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}, \frac{\partial^{2\alpha}u}{\partial y^{2\alpha}}, \frac{\partial^{2\alpha}u}{\partial z^{2\alpha}}, \dots) = 0, \tag{4}$$

where $u(x,y,z,t)$ is an unknown function, P is a polynomial in u and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. This method consists of the following steps:

Step 1. Using the wave transformation

$$u(x,y,z,t) = \phi(\xi), \qquad \xi = k_1\frac{x^{\alpha}}{\alpha} + k_2\frac{y^{\alpha}}{\alpha} + k_3\frac{z^{\alpha}}{\alpha} + \nu\frac{t^{\alpha}}{\alpha}, \tag{5}$$

where k_1, k_2, k_3 and ν are constants to be determined later, one can find

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \nu\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \quad \frac{\partial^{\alpha}u}{\partial x^{\alpha}} = k_1\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \quad \frac{\partial^{\alpha}u}{\partial y^{\alpha}} = k_2\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \quad \frac{\partial^{\alpha}u}{\partial z^{\alpha}} = k_3\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \quad \frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = \nu^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}, \quad \dots \tag{6}$$

Employing (5) and (6), equation (4) is reduced into the following ODE

$$P(\phi, \nu\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, k_1\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, k_2\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, k_3\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \nu^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}, k_1^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}, k_2^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}, k_3^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}, \dots) = 0. \tag{7}$$

Step 2. Assuming that equation (7) has the solution of the form

$$\phi(\xi) = \sum_{j=-N}^{N} a_j(d + F(\xi))^j, \tag{8}$$

where a_{-N} or a_N might be zero, but both of them could not be zero simultaneously. The coefficients a_j $(j = 0, \pm1, \pm2, \dots, \pm N)$ and d are constants to be determined, whereas $F(\xi)$ satisfies the general elliptic equation of the form

$$F'^2(\xi) = c_0 + c_1F(\xi) + c_2F^2(\xi) + c_3F^3(\xi) + c_4F^4(\xi), \tag{9}$$

where c_i $(i = 0,1,2,3,4)$ are constants. Recently, Sirendaoreji [31] presented five classifications of solutions for equation (9) in accordance with the presence of coefficients $c_i\,(i = 0,1,2,3,4)$ and they are given as: (1) $c_0 = c_1 = 0$, (2) $c_3 = c_4 = 0$, (3) $c_1 = c_3 = 0$, (4) $c_2 = c_4 = 0$, and (5) $c_0 = 0$. Herein, we only concentrate on two cases which are Case I $c_0 = c_1 = 0$ and Case II $c_1 = c_3 = 0$, for convenience. Solution structures of both cases are exhibited as follows.

Case I. $c_0 = c_1 = 0$:

$$F_1(\xi) = \frac{2c_2}{\varepsilon\sqrt{\Delta}\cosh(\sqrt{c_2}\xi) - c_3},\ \Delta > 0,\ c_2 > 0, \tag{10}$$

$$F_2(\xi) = \frac{2c_2}{\varepsilon\sqrt{-\Delta}\sinh(\sqrt{c_2}\xi) - c_3},\ \Delta < 0,\ c_2 > 0, \tag{11}$$

$$F_3(\xi) = \frac{2c_2}{\varepsilon\sqrt{\Delta}\cos(\sqrt{-c_2}\xi) - c_3},\ \Delta > 0,\ c_2 < 0, \tag{12}$$

$$F_4(\xi) = \frac{2c_2}{\varepsilon\sqrt{\Delta}\sin(\sqrt{-c_2}\xi) - c_3},\ \Delta > 0,\ c_2 < 0, \tag{13}$$

$$F_5(\xi) = -\frac{c_2}{c_3}\left[1 + \varepsilon\tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right],\ \Delta = 0, c_2 > 0, \tag{14}$$

$$F_6(\xi) = -\frac{c_2}{c_3}\left[1 + \varepsilon\coth\left(\frac{\sqrt{c_2}}{2}\xi\right)\right],\ \Delta = 0, c_2 > 0, \tag{15}$$

$$F_7(\xi) = \frac{\varepsilon}{\sqrt{c_4}\xi},\ c_2 = c_3 = 0, c_4 > 0, \tag{16}$$

$$F_8(\xi) = \frac{4c_3}{c_3^2\xi^2 - 4c_4},\ c_2 = 0, \tag{17}$$

where $\Delta = c_3^2 - 4c_2c_4,\ \varepsilon = \pm 1$.

Case II. $c_1 = c_3 = 0$:

$$F_9(\xi) = \varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right),\ \Delta_1 = 0, c_2 < 0,\ c_4 > 0, \tag{18}$$

$$F_{10}(\xi) = \varepsilon\sqrt{-\frac{c_2}{2c_4}}\coth\left(\sqrt{-\frac{c_2}{2}}\xi\right),\ \Delta_1 = 0, c_2 < 0,\ c_4 > 0, \tag{19}$$

$$F_{11}(\xi) = \varepsilon\sqrt{\frac{c_2}{2c_4}}\tan\left(\sqrt{\frac{c_2}{2}}\xi\right),\ \Delta_1 = 0, c_2 > 0,\ c_4 > 0, \tag{20}$$

$$F_{12}(\xi) = \varepsilon\sqrt{\frac{c_2}{2c_4}}\cot\left(\sqrt{\frac{c_2}{2}}\xi\right),\ \Delta_1 = 0, c_2 > 0,\ c_4 > 0, \tag{21}$$

$$F_{13}(\xi) = \sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\,\mathrm{sn}\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right),\ c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, c_2 < 0, c_4 > 0, \tag{22}$$

$$F_{14}(\xi)=\sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right),\ c_0=\frac{c_2^2m^2(m^2-1)}{c_4(2m^2-1)^2},\ c_2>0,\ c_4<0, \tag{23}$$

$$F_{15}(\xi)=\sqrt{\frac{-c_2}{c_4(2-m^2)}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right),\ c_0=\frac{c_2^2(1-m^2)}{c_4(2-m^2)^2},\ c_2>0,\ c_4<0, \tag{24}$$

$$F_{16}(\xi)=\frac{m^2}{2}\sqrt{\frac{-2c_2}{c_4(2-m^2)}}\frac{\mathrm{sn}\left(\sqrt{\frac{-2c_2}{2-m^2}}\xi\right)}{1\pm\mathrm{dn}\left(\sqrt{\frac{-2c_2}{2-m^2}}\xi\right)},\ c_0=\frac{c_2^2m^4}{4c_4(2-m^2)^2},\ c_2<0,\ c_4>0, \tag{25}$$

$$F_{17}(\xi)=\frac{1}{2}\sqrt{\frac{2c_2}{c_4(1-2m^2)}}\frac{\mathrm{sn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)}{1\pm\mathrm{cn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)},\ c_0=\frac{c_2^2}{4c_4(1-2m^2)^2},\ c_2>0,\ c_4>0, \tag{26}$$

$$F_{18}(\xi)=\varepsilon\left(-\frac{4c_0}{c_4}\right)^{\frac{1}{4}}\mathrm{ds}\left((-4c_0c_4)^{\frac{1}{4}}\xi,\frac{\sqrt{2}}{2}\right),\ c_2=0,\ c_0c_4<0, \tag{27}$$

$$F_{19}(\xi)=\varepsilon\left(\frac{c_0}{c_4}\right)^{\frac{1}{4}}\left[\mathrm{ns}\left(2(c_0c_4)^{\frac{1}{4}}\xi,\frac{\sqrt{2}}{2}\right)+\mathrm{cs}\left(2(c_0c_4)^{\frac{1}{4}}\xi,\frac{\sqrt{2}}{2}\right)\right],\ c_2=0,\ c_0c_4>0, \tag{28}$$

where $\Delta_1=c_2^2-4c_0c_4$, $\varepsilon=\pm1$. Here, $\mathrm{sn}(\xi)=\mathrm{sn}(\xi,m),\mathrm{cn}(\xi)=\mathrm{cn}(\xi,m),\mathrm{dn}(\xi)=\mathrm{dn}(\xi,m)$ are called the Jacobian elliptic sine function, the Jacobian elliptic cosine function and the Jacobian elliptic function of the third kind and $0<m<1$ is the modulus of Jacobian elliptic function. The other Jacobian functions are generated by these three kinds of functions as follows:

$$\begin{aligned}&\mathrm{ns}(\xi)=1/\mathrm{sn}(\xi),\ \mathrm{nc}(\xi)=1/\mathrm{cn}(\xi),\ \mathrm{nd}(\xi)=1/\mathrm{dn}(\xi),\\&\mathrm{sc}(\xi)=\mathrm{sn}(\xi)/\mathrm{cn}(\xi),\ \mathrm{sd}(\xi)=\mathrm{sn}(\xi)/\mathrm{dn}(\xi),\ \mathrm{cd}(\xi)=\mathrm{cn}(\xi)/\mathrm{dn}(\xi),\\&\mathrm{cs}(\xi)=\mathrm{cn}(\xi)/\mathrm{sn}(\xi),\ \mathrm{ds}(\xi)=\mathrm{dn}(\xi)/\mathrm{sn}(\xi),\ \mathrm{dc}(\xi)=\mathrm{dn}(\xi)/\mathrm{cn}(\xi).\end{aligned} \tag{29}$$

Step 3. The value of the positive integer N can be determined by balancing the highest order linear terms with the nonlinear terms of the highest order emerging in equation (7).

Step 4. Substituting (8) and (9) into equation (7), we collect all terms with the same power of $(d+F(\xi))$. Equating each coefficient of the resulting polynomial to zero, yields a set of algebraic equations for a_j $(j=0,\pm1,\pm2,\ldots,\pm N)$, c_i $(i=0,1,2,3,4)$, k_l $(l=1,2,3)$, d and ν.

Step 5. Substituting the values of the constants obtained by solving the algebraic equations extracted in Step 4 together with the solutions of equation (9) into (8), we arrive at different types of solutions for equation (4).

4 Application of the method to the (3 + 1)-dimensional space–time fractional mKdV–ZK equation

Now, we aim to solve the fractional mKdV–ZK equation (1) by applying the variable separated ODE method described above. Thus, to deal with the complex form of equation (1) we will reduce it to an ODE using the transformation (5). The substitution of the transformation (5) into equation (1) leads to

$$\nu\phi'+\delta k_1\phi^2\phi'+k_1(k_1^2+k_2^2+k_3^2)\phi'''=0, \tag{30}$$

where prime denotes the derivative with respect to ξ. Integrating equation (30) once with respect to ξ, we obtain

$$\nu\phi+A\phi^3+B\phi''=0, \tag{31}$$

where $A=\delta k_1/3$, $B=k_1(k_1^2+k_2^2+k_3^2)$ and the integration constant is taken to be zero. Now, we assume that equation (31) has a solution in the form of equation (8). The homogeneous balance between the term ϕ'' and the term ϕ^3 in equation (31) gives rise to $N=1$. Hence, the solution to equation (31) is given in the form

$$\phi(\xi)=\frac{a_{-1}}{(d+F(\xi))}+a_0+a_1(d+F(\xi)), \tag{32}$$

where a_{-1}, a_0, a_1 are unknown constants to be computed. Substituting equation (32) into equation (31) and using equation (9), we obtain polynomials in $(d+F(\xi))^j$ and $(d+F(\xi))^{-j}$, $(j=0,1,2,3)$. Collecting all coefficients of identical power of the resulting polynomials and equating each coefficient to zero, yield the following set of algebraic equations

$$B(2a_{-1}c_4d^4-2a_{-1}c_3d^3+2a_{-1}c_2d^2-2a_{-1}c_1d+2a_{-1}c_0)+Aa_{-1}^3=0, \tag{33}$$

$$B(-6a_{-1}c_4d^3+\frac{9}{2}a_{-1}c_3d^2-3a_{-1}c_2d+\frac{3}{2}a_{-1}c_1)+3Aa_{-1}^2a_0=0, \tag{34}$$

$$B(6a_{-1}c_4d^2-3a_{-1}c_3d+a_{-1}c_2)+A(3a_{-1}^2a_1+3a_{-1}a_0^2)+\nu a_{-1}=0, \tag{35}$$

$$B(-2a_{-1}c_4d+\frac{1}{2}a_{-1}c_3-a_1c_2d+\frac{3}{2}a_1c_3d^2-2a_1c_4d^3+\frac{1}{2}a_1c_1)+A(6a_{-1}a_0a_1+a_0^3)+\nu a_0=0, \tag{36}$$

$$B(-3a_1c_3d+6a_1c_4d^2+a_1c_2)+A(3a_{-1}a_1^2+3a_0^2a_1)+\nu a_1=0, \tag{37}$$

$$B(-6a_1c_4d+\frac{3}{2}a_1c_3)+3Aa_0a_1^2=0, \tag{38}$$

$$2Ba_1c_4+Aa_1^3=0. \tag{39}$$

Solving equations (33)–(39) gives the following cases of solutions.

Case I. In this case, $c_0=c_1=0$. This case has seven different sets of coefficients for the solution of equation (32) displayed as follows.

Set I. $a_0=-d\sqrt{-\frac{2c_4B}{A}}$, $a_1=\sqrt{-\frac{2c_4B}{A}}$, $a_{-1}=0$, $\nu=-c_2B$, $c_3=0$

$$u(x,y,z,t)=\varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\text{sech}\left(\sqrt{c_2}\xi\right), \tag{40}$$

where $c_2>0$ and $\delta>0$.

$$u(x,y,z,t)=\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\text{csch}\left(\sqrt{c_2}\xi\right), \tag{41}$$

where $c_2>0$ and $\delta<0$.

$$u(x,y,z,t)=\varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\sec\left(\sqrt{-c_2}\xi\right), \tag{42}$$

$$u(x,y,z,t)=\varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\csc\left(\sqrt{-c_2}\xi\right), \tag{43}$$

where $c_2<0$ and $\delta<0$. In solutions (40)–(43), $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}-c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set II. $a_0=\frac{(c_3-4c_4d)}{4c_4}\sqrt{-\frac{2c_4B}{A}}$, $a_1=\sqrt{-\frac{2c_4B}{A}}$, $a_{-1}=0$, $\nu=\frac{c_2B}{2}$, $c_4=\frac{c_3^2}{4c_2}$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right), \tag{44}$$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\coth\left(\frac{1}{2}\sqrt{c_2}\xi\right), \tag{45}$$

where $c_2>0,\ \delta<0$ and $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}+\frac{c_2k_1}{2}(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set III. $a_0=-\frac{(c_3-4c_4d)}{4c_4}\sqrt{-\frac{2c_4B}{A}},\ a_1=0,\ a_{-1}=d\frac{(c_3-2c_4d)}{2c_4}\sqrt{-\frac{2c_4B}{A}},\ \nu=\frac{c_2B}{2},\ c_4=\frac{c_3^2}{4c_2}$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{c_2+(c_2-c_3d)\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{(c_2-c_3d)+c_2\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)},\tag{46}$$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{c_2+(c_2-c_3d)\coth\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{(c_2-c_3d)+c_2\coth\left(\frac{1}{2}\sqrt{c_2}\xi\right)},\tag{47}$$

where $c_2>0,\ \delta<0$ and $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}+\frac{c_2k_1}{2}(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set IV. $a_0=-\sqrt{\frac{2B}{A}(c_2-c_4d^2)},\ a_1=0,\ a_{-1}=d\sqrt{\frac{2B}{A}(c_2-c_4d^2)},\ \nu=-c_2B,\ c_3=\frac{2c_2}{d}$
This gives us the solutions (40)–(43).

Set V. $a_0=0,\ a_1=\sqrt{-\frac{2c_4B}{A}},\ a_{-1}=d^2\sqrt{-\frac{2c_4B}{A}},\ \nu=2c_2B,\ c_3=\frac{c_2}{d},\ c_4=\frac{c_2}{4d^2}$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{1+\tanh^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)},\tag{48}$$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{1+\coth^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{\coth\left(\frac{1}{2}\sqrt{c_2}\xi\right)},\tag{49}$$

where $c_2>0,\ \delta<0$ and $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}+2c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set VI. $a_0=0,\ a_1=-\sqrt{-\frac{2c_4B}{A}},\ a_{-1}=d^2\sqrt{-\frac{2c_4B}{A}},\ \nu=-c_2B,\ c_3=\frac{c_2}{d},\ c_4=\frac{c_2}{4d^2}$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{\operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)},\tag{50}$$

$$u(x,y,z,t)=\frac{1}{2}\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{\operatorname{csch}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{\coth\left(\frac{1}{2}\sqrt{c_2}\xi\right)},\tag{51}$$

where $c_2>0,\ \delta<0$ and $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}-c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set VII. $a_0=-2d\sqrt{-\frac{2c_4B}{A}},\ a_1=\sqrt{-\frac{2c_4B}{A}},\ a_{-1}=d^2\sqrt{-\frac{2c_4B}{A}},\ \nu=-4c_2B,\ c_3=\frac{c_2}{d},\ c_4=-\frac{c_2}{4d^2}$

$$u(x,y,z,t)=2\varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{1}{2\cosh^2\left(\sqrt{c_2}\xi\right)-1},\tag{52}$$

where $c_2>0$ and $\delta>0$.

$$u(x,y,z,t)=2\varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{1}{2\cos^2\left(\sqrt{-c_2}\xi\right)-1},\tag{53}$$

where $c_2<0$ and $\delta<0$. In solutions (52) and (53), $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}-4c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Case II. In the second case, $c_1 = c_3 = 0$. This case has five different sets of coefficients for the solution of equation (32) shown as follows.

Set I. $a_0 = -d\sqrt{-\frac{2c_4B}{A}}$, $a_1 = \sqrt{-\frac{2c_4B}{A}}$, $a_{-1} = 0$, $\nu = -c_2B$

$$u(x,y,z,t) = \varepsilon m\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(m^2+1)}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right), \tag{54}$$

where $c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}$, $c_2 < 0$ and $\delta < 0$. As $m \to 1$, solution (54) reduces to

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right). \tag{55}$$

$$u(x,y,z,t) = \varepsilon m\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(2m^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right), \tag{56}$$

where $c_0 = \frac{c_2^2m^2(m^2-1)}{c_4(2m^2-1)^2}$, $c_2 > 0$ and $\delta > 0$. As $m \to 1$, solution (56) reduces to solution (40).

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(2-m^2)}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right), \tag{57}$$

where $c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}$, $c_2 > 0$ and $\delta > 0$. As $m \to 1$, solution (57) reduces to solution (40).

$$u(x,y,z,t) = \varepsilon m^2\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta(2-m^2)}}\frac{\mathrm{sn}\left(\sqrt{-\frac{2c_2}{2-m^2}}\xi\right)}{1+\varepsilon\,\mathrm{dn}\left(\sqrt{-\frac{2c_2}{2-m^2}}\xi\right)}, \tag{58}$$

where $c_0 = \frac{c_2^2m^4}{4c_4(2-m^2)^2}$, $c_2 < 0$ and $\delta < 0$. As $m \to 1$, solution (58) becomes

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{\tanh\left(\sqrt{-2c_2}\xi\right)}{1+\varepsilon\,\mathrm{sech}\left(\sqrt{-2c_2}\xi\right)}. \tag{59}$$

$$u(x,y,z,t) = \varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta(1-2m^2)}}\frac{\mathrm{sn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)}{1+\varepsilon\,\mathrm{cn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)}, \tag{60}$$

where $c_0 = \frac{c_2^2}{4c_4(1-2m^2)^2}$, $c_2 > 0$ and $\delta < 0$. As $m \to 0$, solution (60) degenerates to

$$u(x,y,z,t) = \varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{\sin\left(\sqrt{2c_2}\xi\right)}{1+\varepsilon\cos\left(\sqrt{2c_2}\xi\right)}, \tag{61}$$

and as $m \to 1$, solution (60) reduces to solution (59). In solutions (54)–(61), $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} - c_2k_1(k_1^2 + k_2^2 + k_3^2)\frac{t^\alpha}{\alpha}$.

Set II. $a_0 = 0$, $a_1 = 0$, $a_{-1} = \sqrt{-\frac{2c_0B}{A}}$, $\nu = -c_2B$, $d = 0$

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(m^2+1)}}\,\mathrm{ns}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right), \tag{62}$$

where $c_0 = \frac{c_2^2 m^2}{c_4(m^2+1)^2}$, $c_2 < 0$ and $\delta < 0$. As $m \to 0$, solution (62) degenerates to solution (43) while as $m \to 1$, solution (62) reduces to

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\coth\left(\sqrt{-\frac{c_2}{2}}\xi\right). \tag{63}$$

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(m^2-1)(k_1^2+k_2^2+k_3^2)}{\delta(2m^2-1)}}\,\mathrm{nc}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right), \tag{64}$$

where $c_0 = \frac{c_2^2 m^2(m^2-1)}{c_4(2m^2-1)^2}$, $c_2 > 0$ and $\delta > 0$. As $m \to 0$, solution (64) degenerates to solution (42).

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(1-m^2)(k_1^2+k_2^2+k_3^2)}{\delta(2-m^2)}}\,\mathrm{nd}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right), \tag{65}$$

where $c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}$, $c_2 > 0$ and $\delta > 0$.

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta(2-m^2)}}\,\frac{1+\varepsilon\,\mathrm{dn}\left(\sqrt{-\frac{2c_2}{2-m^2}}\xi\right)}{\mathrm{sn}\left(\sqrt{-\frac{2c_2}{2-m^2}}\xi\right)}, \tag{66}$$

where $c_0 = \frac{c_2^2 m^4}{4c_4(2-m^2)^2}$, $c_2 < 0$ and $\delta < 0$. As $m \to 0$, solution (66) gives rise to solution (43) and as $m \to 1$, solution (66) changes to

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\frac{1+\varepsilon\,\mathrm{sech}\left(\sqrt{-2c_2}\xi\right)}{\tanh\left(\sqrt{-2c_2}\xi\right)}. \tag{67}$$

$$u(x,y,z,t) = \varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta(1-2m^2)}}\,\frac{1+\varepsilon\,\mathrm{cn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)}{\mathrm{sn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)}, \tag{68}$$

where $c_0 = \frac{c_2^2}{4c_4(1-2m^2)^2}$, $c_2 > 0$ and $\delta < 0$. As $m \to 0$, solution (68) transforms to

$$u(x,y,z,t) = \varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\frac{1+\varepsilon\cos\left(\sqrt{2c_2}\xi\right)}{\sin\left(\sqrt{2c_2}\xi\right)}, \tag{69}$$

and as $m \to 1$, solution (68) degenerates to solution (67). In solutions (62)–(69), $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} - c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set III. $a_0 = -d\sqrt{-\frac{2c_4B}{A}}$, $a_1 = 0$, $a_{-1} = \frac{(c_2+2c_4d^2)}{2c_4}\sqrt{-\frac{2c_4B}{A}}$, $\nu = -c_2B$, $c_0 = \frac{c_2^2}{4c_4}$.

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\frac{\sqrt{-c_2c_4}+\sqrt{2}c_4d\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)}{2c_4d+\sqrt{-2c_2c_4}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)}, \tag{70}$$

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\frac{\sqrt{-c_2c_4}+\sqrt{2}c_4d\coth\left(\sqrt{-\frac{c_2}{2}}\xi\right)}{2c_4d+\sqrt{-2c_2c_4}\coth\left(\sqrt{-\frac{c_2}{2}}\xi\right)}, \tag{71}$$

where $c_2<0,\ c_4>0$ and $\delta<0$. Note that when $d=0$ solution (70) converts to solution (63) whereas solution (71) turns to solution (55).

$$u(x,y,z,t)=\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{\sqrt{2}c_4d\tan\left(\sqrt{\frac{c_2}{2}}\xi\right)-\sqrt{c_2c_4}}{\sqrt{2c_2c_4}\tan\left(\sqrt{\frac{c_2}{2}}\xi\right)+2c_4d},\tag{72}$$

$$u(x,y,z,t)=\varepsilon\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{\sqrt{2}c_4d\cot\left(\sqrt{\frac{c_2}{2}}\xi\right)-\sqrt{c_2c_4}}{\sqrt{2c_2c_4}\cot\left(\sqrt{\frac{c_2}{2}}\xi\right)+2c_4d},\tag{73}$$

where $c_2>0,\ c_4>0$ and $\delta<0$. In solutions (70)–(73), $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}-c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set IV. $a_0=-\frac{1}{2}\sqrt{-\frac{2B}{A}(c_2+2c_4d^2)},\ a_1=0,\ a_{-1}=d\sqrt{-\frac{2B}{A}(c_2+2c_4d^2)},\ \nu=\frac{B}{2}(c_2-6c_4d^2),\ c_0=c_4d^4$

$$u(x,y,z,t)=\varepsilon\frac{m+1}{2}\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(m^2+1)}}\frac{1-\sqrt{-m}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)}{1+\sqrt{-m}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)},\tag{74}$$

where $c_0=\frac{c_2^2m^2}{c_4(m^2+1)^2},\ c_2<0,\ \delta>0$ and $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}+\frac{(m-1)^2-4m}{2(m^2+1)}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m\to1$, solution (74) becomes

$$u(x,y,z,t)=\varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{1-i\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)}{1+i\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)}.\tag{75}$$

where $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}-c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. Using the relation $\tanh(\theta)=-i\tan(i\theta)$ solution (75) converts to

$$u(x,y,z,t)=\varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\frac{1+\tan\left(\sqrt{\frac{c_2}{2}}\xi\right)}{1-\tan\left(\sqrt{\frac{c_2}{2}}\xi\right)},\tag{76}$$

where $c_2>0$ and $\delta<0$.

$$u(x,y,z,t)=\varepsilon\frac{\sqrt{m^2-1}+m}{2}\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(2m^2-1)}}\frac{1-\sqrt{-\frac{m}{\sqrt{m^2-1}}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{1+\sqrt{-\frac{m}{\sqrt{m^2-1}}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)},\tag{77}$$

where $c_0=\frac{c_2^2m^2(m^2-1)}{c_4(2m^2-1)^2},\ c_2>0,\ \delta<0$ and $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}+\frac{2m^2-1-6m\sqrt{m^2-1}}{2(2m^2-1)}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

$$u(x,y,z,t)=\varepsilon\frac{\sqrt{1-m^2}+1}{2}\sqrt{-\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(2-m^2)}}\frac{1-\sqrt{-\frac{1}{\sqrt{1-m^2}}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{1+\sqrt{-\frac{1}{\sqrt{1-m^2}}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)},\tag{78}$$

where $c_0=\frac{c_2^2(1-m^2)}{c_4(2-m^2)^2},\ c_2>0,\ \delta<0$ and $\xi=k_1\frac{x^\alpha}{\alpha}+k_2\frac{y^\alpha}{\alpha}+k_3\frac{z^\alpha}{\alpha}+\frac{2-m^2-6\sqrt{1-m^2}}{2(2-m^2)}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

$$u(x,y,z,t)=\varepsilon\sqrt{-\frac{3c_2(1-m^2)(k_1^2+k_2^2+k_3^2)}{\delta(1-2m^2)}}\frac{1+\varepsilon\,\mathrm{cn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)-\mathrm{sn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)}{1+\varepsilon\,\mathrm{cn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)+\mathrm{sn}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right)},\tag{79}$$

where $c_0 = \frac{c_2^2}{4c_4(1-2m^2)^2}$, $c_2 > 0$, $\delta < 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} - (1+m^2)c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m \to 0$, solution (68) transforms to

$$u(x,y,z,t) = \varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}\frac{1+\varepsilon\cos\left(\sqrt{2c_2}\xi\right)-\sin\left(\sqrt{2c_2}\xi\right)}{1+\varepsilon\cos\left(\sqrt{2c_2}\xi\right)+\sin\left(\sqrt{2c_2}\xi\right)}}, \tag{80}$$

where $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} - c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

$$u(x,y,z,t) = \varepsilon d\sqrt{-\frac{3c_4(k_1^2+k_2^2+k_3^2)}{\delta}\frac{1-\sqrt{-2}\,\mathrm{ds}\left(d\sqrt{-2c_4}\xi, \frac{\sqrt{2}}{2}\right)}{1+\sqrt{-2}\,\mathrm{ds}\left(d\sqrt{-2c_4}\xi, \frac{\sqrt{2}}{2}\right)}}, \tag{81}$$

where $c_2 = 0$, $c_4 < 0$ and $\delta > 0$.

$$u(x,y,z,t) = \varepsilon d\sqrt{-\frac{3c_4(k_1^2+k_2^2+k_3^2)}{\delta}\frac{1-\mathrm{ns}\left(2d\sqrt{c_4}\xi, \frac{\sqrt{2}}{2}\right)-\mathrm{cs}\left(2d\sqrt{c_4}\xi, \frac{\sqrt{2}}{2}\right)}{1+\mathrm{ns}\left(2d\sqrt{c_4}\xi, \frac{\sqrt{2}}{2}\right)+\mathrm{cs}\left(2d\sqrt{c_4}\xi, \frac{\sqrt{2}}{2}\right)}}, \tag{82}$$

where $c_2 = 0$, $c_4 > 0$ and $\delta < 0$. In solutions (81) and (82), $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} - 3c_4d^2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

Set V. $a_0 = 0$, $a_1 = \sqrt{-\frac{2c_4B}{A}}$, $a_{-1} = \sqrt{-\frac{2c_0B}{A}}$, $\nu = B(6\sqrt{c_0c_4} - c_2)$, $d = 0$

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(m^2+1)}\frac{1-m\,\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)}{\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)}}, \tag{83}$$

where $c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}$, $c_2 < 0$, $\delta < 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + \frac{6m-m^2-1}{m^2+1}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m \to 0$, solution (83) converts to solution (43) while as $m \to 1$, solution (83) degenerates to

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}\frac{\mathrm{sech}^2\left(\sqrt{-\frac{c_2}{2}}\xi\right)}{\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)}}, \tag{84}$$

where $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + 2c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(2m^2-1)}\frac{\sqrt{m^2-1}-m\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}{\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)}}, \tag{85}$$

where $c_0 = \frac{c_2^2m^2(m^2-1)}{c_4(2m^2-1)^2}$, $c_2 > 0$, $\delta > 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + \frac{6m\sqrt{m^2-1}-2m^2+1}{2m^2-1}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m \to 0$, solution (85) converts to solution (42) while as $m \to 1$, solution (85) degenerates to solution (40).

$$u(x,y,z,t) = \varepsilon\sqrt{\frac{6c_2(k_1^2+k_2^2+k_3^2)}{\delta(2-m^2)}\frac{\sqrt{1-m^2}-\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}{\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)}}, \tag{86}$$

where $c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}$, $c_2 > 0$, $\delta > 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + \frac{6\sqrt{1-m^2}+m^2-2}{2-m^2}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m \to 1$, solution (86) reduces to solution (40).

$$u(x,y,z,t) = 2\varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta(1-2m^2)}}\,\mathrm{ns}\left(\sqrt{\frac{2c_2}{1-2m^2}}\xi\right), \tag{87}$$

where $c_0 = \frac{c_2^2}{4c_4(1-2m^2)^2}$, $c_2 > 0$, $\delta < 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + \frac{2(1+m^2)}{1-2m^2}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m \to 0$, solution (87) reduces to

$$u(x,y,z,t) = 2\varepsilon\sqrt{-\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\csc\left(\sqrt{2c_2}\xi\right), \tag{88}$$

where $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + 2c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m \to 1$, solution (87) becomes

$$u(x,y,z,t) = 2\varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\coth\left(\sqrt{-2c_2}\xi\right), \tag{89}$$

where $c_2 < 0$, $\delta < 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} - 4c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

$$u(x,y,z,t) = 2\varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta(2-m^2)}}\,\mathrm{ds}\left(\sqrt{-\frac{2c_2}{2-m^2}}\xi\right), \tag{90}$$

where $c_0 = \frac{c_2^2m^4}{4c_4(2-m^2)^2}$, $c_2 < 0$, $\delta > 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + \frac{4m^2-2}{2-m^2}c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$. As $m \to 0$, solution (90) turns to solution (43) while as $m \to 1$, solution (90) reduces to

$$u(x,y,z,t) = 2\varepsilon\sqrt{\frac{3c_2(k_1^2+k_2^2+k_3^2)}{\delta}}\,\mathrm{csch}\left(\sqrt{-2c_2}\xi\right), \tag{91}$$

where $c_2 < 0$, $\delta < 0$ and $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + 2c_2k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

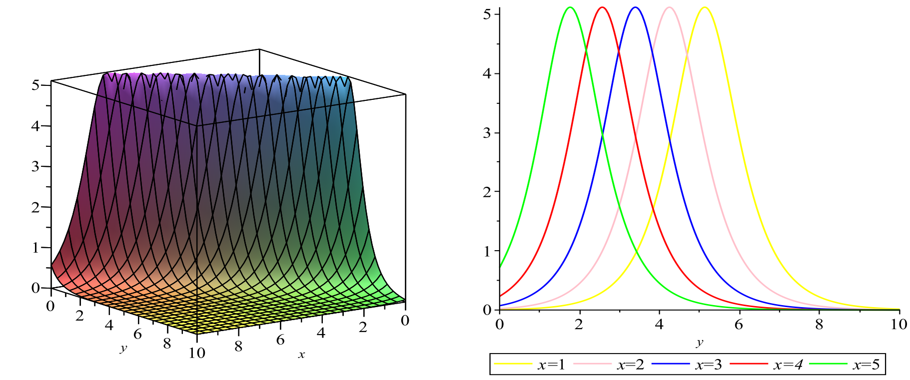
$$u(x,y,z,t) = \varepsilon(-c_0c_4)^{1/4}\sqrt{\frac{3(k_1^2+k_2^2+k_3^2)}{\delta}}\,\frac{1+2i\,\mathrm{ds}^2\left((-4c_0c_4)^{1/4}\xi,\frac{\sqrt{2}}{2}\right)}{\mathrm{dn}\left((-4c_0c_4)^{1/4}\xi,\frac{\sqrt{2}}{2}\right)}, \tag{92}$$

where $c_2 = 0$, $c_0c_4 < 0$ and $\delta > 0$.

$$u(x,y,z,t) = \varepsilon(c_0c_4)^{1/4}\sqrt{-\frac{6(k_1^2+k_2^2+k_3^2)}{\delta}}\,\frac{1+\left(\mathrm{ns}\left(2(c_0c_4)^{1/4}\xi,\frac{\sqrt{2}}{2}\right)+\mathrm{cs}\left(2(c_0c_4)^{1/4}\xi,\frac{\sqrt{2}}{2}\right)\right)^2}{\mathrm{ns}\left(2(c_0c_4)^{1/4}\xi,\frac{\sqrt{2}}{2}\right)+\mathrm{cs}\left(2(c_0c_4)^{1/4}\xi,\frac{\sqrt{2}}{2}\right)}, \tag{93}$$

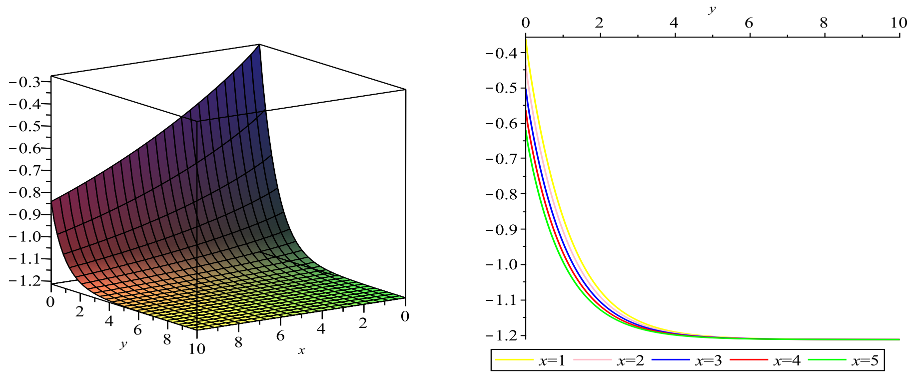
where $c_2 = 0$, $c_0c_4 > 0$ and $\delta < 0$. In solutions (92) and (93), $\xi = k_1\frac{x^\alpha}{\alpha} + k_2\frac{y^\alpha}{\alpha} + k_3\frac{z^\alpha}{\alpha} + 6\sqrt{c_0c_4}k_1(k_1^2+k_2^2+k_3^2)\frac{t^\alpha}{\alpha}$.

In what follows, we depict the dynamics of solitons and periodic waves in the model of the (3+1)-dimensional space-time fractional modified KdV–Zakharov–Kuznetsov equation to show a clear understanding of the physical properties of obtained results. The numerical results of some of the derived solutions are exhibited by selecting different values for the constants $\varepsilon, \delta, \alpha, c_2$ and k_l $(l = 1,2,3)$. For example, the 3D and 2D plots of the bell-shaped solitary wave solution (40) are displayed in Figure 1 with $\varepsilon = 1, k_1 = 1, k_2 = 1.2, k_3 = 2.2, c_2 = 1.5, \delta = 2.5$ when $\alpha = 0.95$. Figure 2 shows the 3D and 2D plots of the kink-shaped solitary wave solution (44) for $\varepsilon = -1, k_1 = 0.2, k_2 = 1.5, k_3 = 0.25, c_2 = 0.5, \delta = -1.2$ when $\alpha = 0.9$. In Figure 3, the 3D and 2D plots of the singular soliton solution (49) are depicted for $\varepsilon = 1, k_1 = 0.5, k_2 = 1.5, k_3 = 0.3, c_2 = 1.3, \delta = -1$



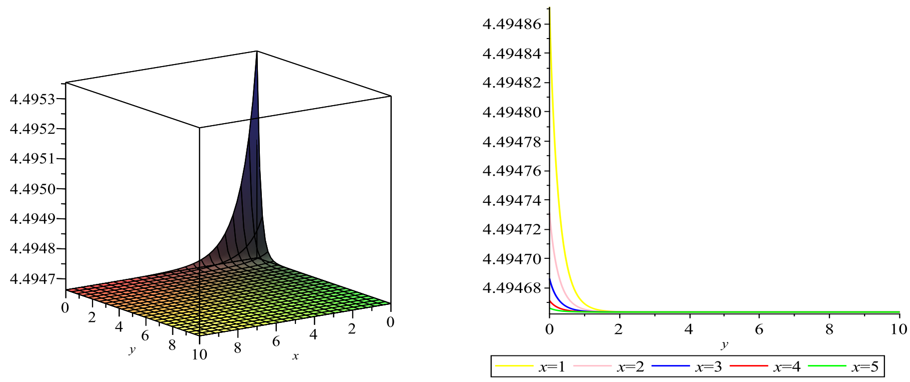
(a) 3D (b) 2D

Fig. 1 The solitary wave solution (40) for $k_1 = 1, k_2 = 1.2, k_3 = 2.2, c_2 = 1.5, \delta = 2.5, \alpha = 0.95$.



(a) 3D (b) 2D

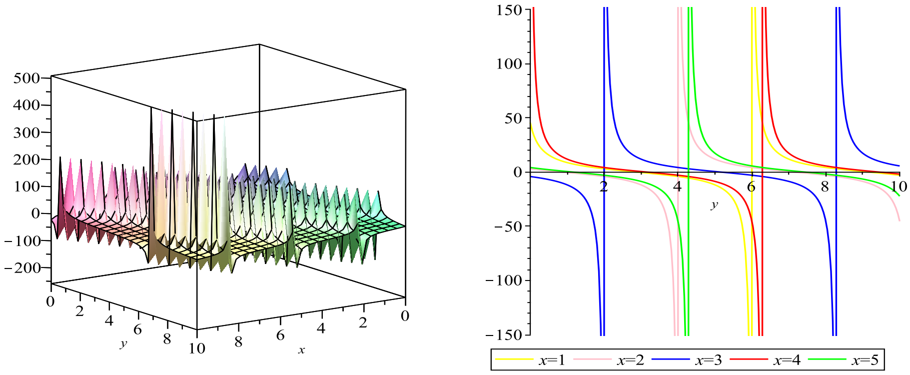
Fig. 2 The kink wave solution (44) for $k_1 = 0.2, k_2 = 1.5, k_3 = 0.25, c_2 = 0.5, \delta = -1.2, \alpha = 0.9$.



(a) 3D (b) 2D

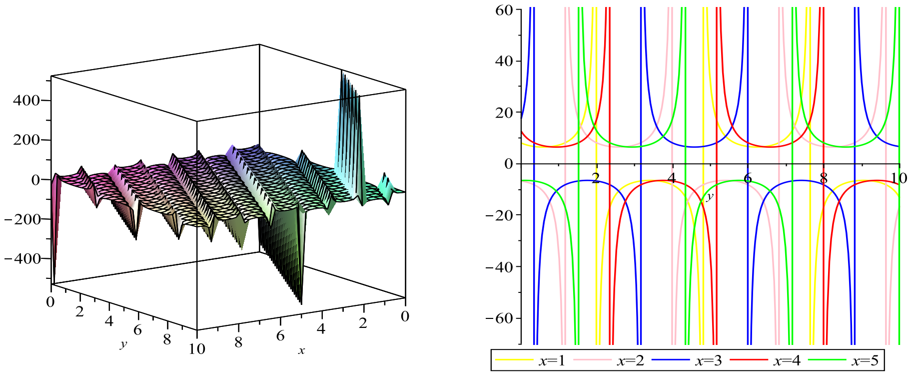
Fig. 3 The singular soliton solution (49) for $k_1 = 0.5, k_2 = 1.5, k_3 = 0.3, c_2 = 1.3, \delta = -1, \alpha = 0.95$.

when $\alpha = 0.95$. Further to this, the 3D and 2D plots of the singular periodic solution (69) are described in Figure 4 for $\varepsilon = 1, k_1 = 1, k_2 = 0.5, k_3 = 1.5, c_2 = 2, \delta = -0.5$ when $\alpha = 1$. Figure 5 shows the 3D and 2D plots of the singular periodic solution (88) for $\varepsilon = 1, k_1 = -1.5, k_2 = 0.75, k_3 = 1, c_2 = 1.1, \delta = -1.2$ when $\alpha = 1$. Figure 6 presents the 3D and 2D plots of the singular soliton solution (91) with $\varepsilon = -1, k_1 = 0.5, k_2 = -0.5, k_3 = 0.5, c_2 = -1.2, \delta = -1.5$ when $\alpha = 1$.



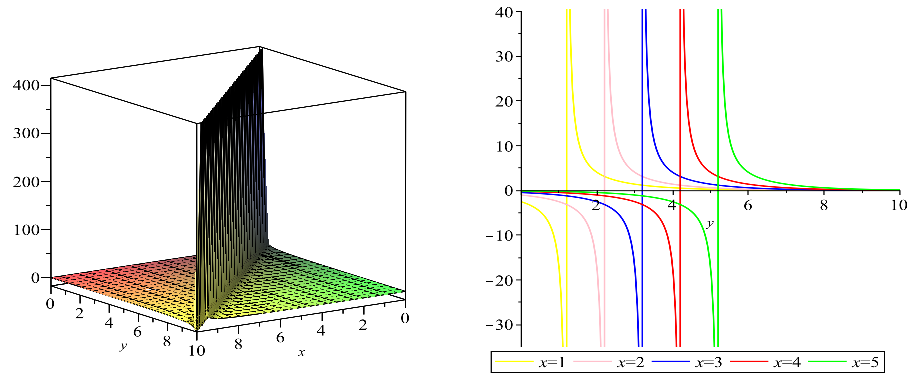
(a) 3D (b) 2D

Fig. 4 The singular periodic solution (69) for $k_1 = 1, k_2 = 0.5, k_3 = 1.5, c_2 = 2, \delta = -0.5, \alpha = 1$.



(a) 3D (b) 2D

Fig. 5 The singular periodic solution (88) for $k_1 = -1.5, k_2 = 0.75, k_3 = 1, c_2 = 1.1, \delta = -1.2, \alpha = 1$.



(a) 3D (b) 2D

Fig. 6 The singular soliton solution (91) for $k_1 = 0.5, k_2 = -0.5, k_3 = 0.5, c_2 = -1.2, \delta = -1.5, \alpha = 1$.

Remark 2. All of the results are calculated by using Maple, when $t = 1$ and $z = 2$ with the interval $0 < x, y \leq 10$.

5 Conclusions

In this study, we have investigated the exact analytic solutions to the (3+1)-dimensional space-time fractional mKdV–ZK equation. By means of conformable fractional derivative and wave transformation, the fractional mKdV–ZK equation is changed to an ODE. Then, the resulting ODE is solved by applying the variable separated ODE method. All obtained solutions are verified by utilising symbolic computation. To shed light on the behaviour of extracted results, some of derived solutions are displayed graphically. Moreover, it is found that the implemented method is a powerful mathematical tool for solving ODE and provides more exact solutions such as solitary, kink and periodic waves. Consequently, it can be applied to different physical models to generate various types of solutions.

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