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Dimensionless characterization of the non-linear soil consolidation problem of Davis and Raymond. Extended models and universal curves

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Abstract

The dimensionless groups that govern the Davis and Raymond non-linear consolidation model, and its extended versions resulting from eliminating several restrictive hypotheses, were deduced. By means of the governing equations nondimensionalization technique and introducing the characteristic time concept, both in terms of settlement and pressures, was obtained (for the most general model) that the average degree of settlement only depends on the dimensionless time while the average degree of pressure dissipation does it, additionally, on the loading ratio. These results allowed the construction of universal curves expressing the solutions of the unknowns of interest in a direct and simple way.

Keywords: non-linear consolidation, dimensionless groups, characteristic time, average degree of settlement

1 Introduction

The soil consolidation problem under linear behaviour is governed by the diffusion equation of excess pore pressure, with well-known analytical solutions for the usual boundary conditions. However, under the hypotheses of non-linear dependencies of hydraulic conductivity and void ratio with the effective stress, as well as under the consideration of variable volume elements in thickness, numerical solutions are used. Among the most common non-linear models used in literature are those of Davis and Raymond [1, 2], Juárez-Badillo [3–5] and Cornetti and Battaglio [6, 7]. The interest of the non-linear models against the linear ones is consequence of the important deviations that emerge from their solutions, above 100% [2] both in relation to the time characteristic of the process and the evolution of the average degree of consolidation.

In this paper, we address, on the one hand, the deduction and verification of the dimensionless groups that govern the solutions of the Davis and Raymond model and the extensions derived from eliminating one or more of its restrictive hypotheses (constant 1+e, c_v and dz) and, on the other hand, based on these groups, construction

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of universal curves that allow the engineer to obtain the solutions of the most important unknowns of the problem in a direct way. In order to verify the results obtained, a set of significant cases has been simulated, widely covering the range of values of the physical and geometric parameters and the loading ratio (final and initial effective stress) that can take place in real problems. In each case, the value of one or more of the parameters involved has been significantly altered, but the numerical value of the dimensionless groups has been retained so that it is verified that the same pattern of solutions is obtained for the whole set of cases; or the value of the groups has been modified appropriately to confirm that the form of the solutions changes.
As a deduction technique for these groups, the nondimensionalization of governing equations [8–10] has been used, a form of application of the pi theorem [11], thus allowing the dimensionless parameters involved in the problem (void ratio and compression index), and that they would form independent dimensionless groups, to be included in the inferred groups. In the nondimensionalization process, which has been carried out in terms of both excess pore pressure dissipation and settlement, the characteristic time of consolidation is introduced as a reference, an unknown that is incorporated into one of the resulting monomials and whose dependency with the rest of the groups is established by the pi theorem. Despite suppressing the restrictive original hypotheses of the authors [1], the resulting number of dimensionless groups is small enough to be able to represent their dependencies by means of universal curves obtained through numerical simulations using the network simulation method [12, 13].

For the most general and precise case (non-constant 1+e, c_v and dz), the characteristic time in terms of settlement is defined by a single group, so it has a direct relationship with the parameters of the problem and a single test is sufficient to obtain the proportionality factor. However, in terms of pressure, the characteristic time is a function of the final and initial effective stress ratio. On the other hand, the average degree of settlement is a direct function of the dimensionless characteristic time, while the average degree of pressure dissipation depends, additionally, on the loading ratio, so its universal representation needs to be done by means of an abacus.

2 Davis and Raymond consolidation model

2.1 Davis and Raymond proposed model

The general consolidation equation is obtained by matching, in a soil element, the temporal change in water volume with the temporal change in void volume [14]. This balance, together with the constitutive equation that associates the variables flow and pressure gradient (Darcy's law) and the empirical expressions that relate the parameters of the problem with the dependent variable (excess pore pressure, u, or effective pressure, σ'), allows to deduce the consolidation equation for the general case (linear or non-linear processes) expressed in terms of the independent variables position and time. In linear cases, this equation has a semi-analytical solution for all geometries, while for non-linear cases, in general, the use of numerical calculation is required. In short, equalling the expressions $\dot{q}_w = \frac{dV_w}{dt} = -A\frac{\partial v_z}{\partial z}dz$ and $\dot{q}_v = \frac{dV_v}{dt} = A\,(dz)\,\frac{\partial}{\partial t}\left(\frac{e}{1+e}\right)$, we have

$$\frac{\partial v_z}{\partial z} = -\frac{\partial}{\partial t}\left(\frac{e}{1+e}\right) \tag{1}$$

Using Darcy's Law, $v_z = -\frac{k}{\gamma_w}\frac{\partial u}{\partial z}$,
expression (1) is finally written in the form

$$\frac{\partial}{\partial z}\left(\frac{k}{\gamma_w}\frac{\partial u}{\partial z}\right) = \frac{1}{(1+e)^2}\left(\frac{\partial e}{\partial t}\right) \tag{2}$$

where k (hydraulic conductivity) and e (void ratio) are, in general, parameters that depend on the effective stress (σ') through the so-called constitutive equations of the ground. In turn, the excess pore pressure (u) and the effective stress are related by the Terzaghi's hypothesis under oedometric conditions, $\sigma = \sigma' + \mathrm{u}$, from which it follows that $\frac{\partial \sigma'}{\partial t} = -\frac{\partial u}{\partial t}$ and $\frac{\partial \sigma'}{\partial z} = -\frac{\partial u}{\partial z}$.

In their model [1], Davis and Raymond assumed the following classical hypotheses: i) secondary consolidation is ignored, ii) soil particles and pore water are incompressible, iii) soil is saturated, iv) soil weight is negligible and v) the thickness change is considered negligible compared to the initial thickness, this is, 1+e constant. In addition, they assumed a constant consolidation coefficient based on the fact that, in a real mass of soil, compressibility and hydraulic conductivity vary during the consolidation process, both decreasing while increasing the effective pressure but in such a way that the changes in both are compensated so that the consolidation coefficient $c_v = \frac{k}{m_v \gamma_w}$ remains more or less constant. On the other hand, they adopted an e$\sim \sigma'$ dependency governed by the following empirical law [15]:

$$\mathrm{e} = \mathrm{e_o} - \mathrm{I_c} \log_{10}\left(\frac{\sigma'}{\sigma'_o}\right) \tag{3}$$

where I_c is the compression index, a constant parameter.

From the definition of the volumetric compressibility coefficient, $m_v = -\frac{\partial e}{\partial \sigma'}\left(\frac{1}{1+e}\right)$, and the expression $\frac{\partial e}{\partial \sigma'} = -\frac{I_c}{\ln(10)\sigma'}$ obtained from (3), we can write

$$m_v = \frac{I_c}{\ln(10)(1+e)\,\sigma'} \tag{4}$$

which gives us the volumetric compressibility coefficient as a function of the effective stress.

Finally, the assumption of a constant value for both the consolidation coefficient and the factor 1+e allows writing

$$c_v = \frac{k\sigma'(1+e)\ln(10)}{I_c \gamma_w} \propto (k\sigma') \tag{5}$$

an equation equivalent to assuming that k and σ' change inversely during the consolidation process.

The Davis and Raymond consolidation equation [1] is derived from expression (1) and considering constant the factor 1+e, resulting in

$$\frac{\partial}{\partial z}\left(\frac{k}{\gamma_w}\frac{\partial u}{\partial z}\right) = \frac{1}{(1+e)}\left(\frac{\partial e}{\partial t}\right) \tag{6}$$

an expression very similar to that of equation (2), with the exception that the factor 1+e does not appear squared (note that, in this case, $e = e_o$). Since k and σ' vary inversely ($k\sigma = k_o \sigma_o$) and $\frac{\partial \sigma'}{\partial z} = -\frac{\partial u}{\partial z}$, equation (6) can be written as

$$-\frac{k_o {\sigma'}_o}{\gamma_w}\frac{\partial}{\partial z}\left(\frac{1}{\sigma'}\frac{\partial \sigma'}{\partial z}\right) = \frac{1}{(1+e_o)}\left(\frac{\partial e}{\partial \sigma'}\frac{\partial \sigma'}{\partial t}\right) \tag{7}$$

which, after mathematical manipulation, results in

$$\frac{k_o {\sigma'}_o (1+e_o)\ln(10)}{I_c \gamma_w}\left\{\frac{\partial^2 \sigma'}{\partial z^2} - \frac{1}{\sigma'}\left(\frac{\partial \sigma'}{\partial z}\right)^2\right\} = \frac{\partial \sigma'}{\partial t} \tag{8}$$

and, from expression (5), since c_v is constant we have $c_v = \frac{k_o {\sigma'}_o (1+e_o)\ln(10)}{I_c \gamma_w}$, finally obtaining:

$$c_v\left\{\frac{\partial^2 \sigma'}{\partial z^2} - \frac{1}{\sigma'}\left(\frac{\partial \sigma'}{\partial z}\right)^2\right\} = \frac{\partial \sigma'}{\partial t} \tag{9a}$$

Expression (9a), a clearly non-linear equation, is the Davis and Raymond consolidation equation [1] in terms of the effective stress (σ'). Its form in terms of the excess pore pressure (u) is

$$c_v\left\{\frac{\partial^2 u}{\partial z^2} + \frac{1}{\sigma - u}\left(\frac{\partial u}{\partial z}\right)^2\right\} = \frac{\partial u}{\partial t} \tag{9b}$$

The authors, for the usual boundary conditions [14], obtained the analytical solution for the variable (u), which besides depending on position (z) and time (t) also depends on the ratio σ'_f/σ'_o [1]. In this way, the average degree of pressure dissipation throughout the domain ($\bar{U}_{\sigma'}$) will depend only on time and the ratio between the final and initial effective pressures.

On the other hand, looking for a simplification (by means of a change in variable) that reduces the consolidation equation (9) to a linear one, the authors proposed the introduction of a new variable (w), to which they did not assign any explicit physical meaning, whose relation with the effective stress is in the form

$$w = \log_{10}\left(\frac{\sigma'}{\sigma'_f}\right) = \log_{10}\left(\frac{\sigma'_f - u}{\sigma'_f}\right) \tag{10}$$

The introduction of this new variable, through its spatial and temporal derivatives, into equation (9) leads to a pure diffusion equation

$$c_v \frac{\partial^2 w}{\partial z^2} = \frac{\partial w}{\partial t} \tag{11}$$

What is diffused in this linear equation, then, is a new local magnitude (w) that, although the authors did not mention, is directly related to the settlements (changes in the void ratio) of the problem. Thus, from equation (3), it is easy to obtain that

$$\frac{e_f - e}{I_c} = \log_{10}\left(\frac{\sigma'}{\sigma'_f}\right) \tag{12}$$

It follows from equation (11) that the solution for the variable w depends on position and time, and therefore, as the authors conclude [1], the average degree of settlement in the domain ($\bar{U}_s$) is only function of time.

2.2 Extended models

Within this section, we will consider two models that eliminate some restrictive hypotheses considered by Davis and Raymond, giving rise to more general and, therefore, more precise models: i) the first one in which the value of 1+e is assumed to be non-constant but keeping the consolidation coefficient c_v constant and ii) a more general second model in which both 1+e and c_v are considered non-constant.

2.2.1 Model with non-constant 1+e and constant c_v

In this case, we start from expression (2), which represents the general consolidation equation under the hypothesis of 1+e non-constant. For the changes in the void ratio with the effective stress, we maintain the constitutive relation (3). On the other hand, if we consider that c_v is constant throughout the process, on this criterion it is true that

$$c_v = \frac{k_o \sigma'_o (1+e_o) \ln(10)}{I_c \gamma_w} = \frac{k\sigma' (1+e) \ln(10)}{I_c \gamma_w} \tag{13}$$

so that the relation between the hydraulic conductivity and the effective pressure, necessarily, becomes now

$$k\sigma' = k_o \sigma'_o \frac{(1+e_o)}{(1+e)} \tag{14}$$

that is, the changes in k are inversely proportional to those in σ', but, in addition, they also depend on the changes in the factor 1+e. With all this, equation (2) can be written as

$$-\frac{k_o \sigma'_o (1+e_o)}{\gamma_w} \frac{\partial}{\partial z}\left(\frac{1}{\sigma'(1+e)} \frac{\partial \sigma'}{\partial z}\right) = \frac{1}{(1+e)^2}\left(\frac{\partial e}{\partial \sigma'} \frac{\partial \sigma'}{\partial t}\right) \tag{15}$$

After mathematical operation, it is deduced that

$$\frac{k_o \sigma'_o (1+e_o)(1+e) \ln(10)}{I_c \gamma_w} \left\{ \frac{\partial^2 \sigma'}{\partial z^2} - \frac{1}{\sigma'}\left(\frac{\partial \sigma'}{\partial z}\right)^2 + \frac{I_c}{\sigma'(1+e)\ln(10)}\left(\frac{\partial \sigma'}{\partial z}\right)^2 \right\} = \frac{\partial \sigma'}{\partial t} \tag{16}$$

which, from the deduction of expression (13), finally leads to:

$$c_v \left\{ (1+e)\frac{\partial^2\sigma'}{\partial z^2} + \left(\frac{I_c}{\sigma' \ln(10)} - \frac{1+e}{\sigma'}\right)\left(\frac{\partial\sigma'}{\partial z}\right)^2 \right\} = \frac{\partial\sigma'}{\partial t} \tag{17}$$

2.2.2 Model with both non-constant 1+e and c_v

For this extended model, we again begin from expressions (2) and (3). Maintaining the Davis and Raymond philosophy, we return to the relation $k\sigma' = k_o\sigma'_o$ to represent the variation in the hydraulic conductivity versus the effective stress. In this way, now we have

$$c_v = c_{vo}\frac{1+e}{1+e_o} \tag{18}$$

With all this, equation (2) can be written as

$$\frac{k_o\sigma'_o(1+e)^2\ln(10)}{I_c\gamma_w}\left\{\frac{\partial^2\sigma'}{\partial z^2} - \frac{1}{\sigma'}\left(\frac{\partial\sigma'}{\partial z}\right)^2\right\} = \frac{\partial\sigma'}{\partial t} \tag{19}$$

which expressed in terms of the consolidation coefficient takes the form

$$c_v(1+e)\left\{\frac{\partial^2\sigma'}{\partial z^2} - \frac{1}{\sigma'}\left(\frac{\partial\sigma'}{\partial z}\right)^2\right\} = \frac{\partial\sigma'}{\partial t} \tag{20}$$

or, in terms of the initial consolidation coefficient (a constant parameter), results in

$$c_{vo}\frac{(1+e)^2}{1+e_o}\left\{\frac{\partial^2\sigma'}{\partial z^2} - \frac{1}{\sigma'}\left(\frac{\partial\sigma'}{\partial z}\right)^2\right\} = \frac{\partial\sigma'}{\partial t} \tag{21}$$

3 Dimensionless characterization of the Davis and Raymond model and its extended variants

In this section, we proceed to obtain the dimensionless groups that govern the solution of the non-linear consolidation scenarios presented in the previous section to subsequently address a universal representation of the main variables of interest: average degree of consolidation and characteristic time of the duration of the process. By defining the variables of the problem (effective stress, medium depth and characteristic time) in a dimensionless form, the non-linear equation is nondimensionalized. Given the different hypotheses assumed in the different models, the nondimensionalization process deduces different dependencies in each case and, at the same time, different solutions.

It will also be analyzed, for the models in which 1+e is non-constant, the influence of the elimination of another restrictive hypothesis, consequence of this, which is the consideration of the variation in the volume element size dz. This last question, which implies addressing a moving boundary problem (and even closer to the real consolidation problem), involves changes in the governing equations, which will result in different solutions for the variables of interest, as we will see in the following.

The consolidation problem describes an interesting phenomenon whereby as the excess pore pressure is relaxed (allowing water to escape from the system), the ground settlement takes place. Two processes closely linked but, due to the non-linearity of the problem, develop in a different way, with dependencies that are not always equal. Therefore, the characterization of the problem will be carried out, for all scenarios, both in terms of pressure and settlement.

3.1 Characterization of the model proposed by Davis and Raymond

3.1.1 Nondimensionalization of the model proposed by Davis and Raymond in terms of pressure

For the nondimensionalization of this model (9a), we will adopt the following dimensionless variables:

$$\left(\sigma'\right)' = \frac{\sigma' - \sigma'_{\mathrm{o}}}{\sigma'_{\mathrm{f}} - \sigma'_{\mathrm{o}}} \tag{22}$$

normalized to the interval [0,1]; $\tau_{o,\sigma'}$ is the characteristic time that takes the excess pore pressure to relax to approximately the value of zero (throughout the domain). Substituting (22) in equation (9a), through mathematical manipulations, we get

$$\mathrm{c_v}\left\{\left(\frac{\sigma'_{\mathrm{f}} - \sigma'_{\mathrm{o}}}{\mathrm{H_o^2}}\right)\frac{\partial^2(\sigma')'}{\partial \mathrm{z}'^2} - \left(\frac{(\sigma'_{\mathrm{f}} - \sigma'_{\mathrm{o}})^2}{\mathrm{H_o^2}}\right)\left(\frac{1}{\sigma'}\right)\left(\frac{\partial(\sigma')'}{\partial \mathrm{z}'}\right)^2\right\} = \left(\frac{\sigma'_{\mathrm{f}} - \sigma'_{\mathrm{o}}}{\tau_{\mathrm{o},\sigma'}}\right)\frac{\partial\,(\sigma')'}{\partial \mathrm{t}'} \tag{23}$$

As they are normalized variables, all derivative terms of $(\sigma')'$, z' and t' are averaged to the unit, while the value of σ' is averaged to an arbitrary (characteristic) value σ'_{m}. In this way, the coefficients of this equation (once simplified), of the same order of magnitude, are three:

$$\frac{\mathrm{c_v}}{\mathrm{H_o^2}} \quad \frac{c_v}{H_o^2}\left(\frac{\sigma'_f - \sigma'_o}{\sigma'_m}\right) \quad \frac{1}{\tau_{o,\sigma'}} \tag{24}$$

from which, dividing by the first, monomials result:

$$\pi_{\mathrm{I}} = \frac{\mathrm{H_o^2}}{\mathrm{c_v}\tau_{\mathrm{o},\sigma'}} \text{ or, alternatively, } \pi_{\mathrm{I}} = \frac{\tau_{\mathrm{o},\sigma'}\mathrm{c_v}}{\mathrm{H_o^2}} \tag{25}$$

$$\pi_{\mathrm{II}} = \frac{\sigma'_{\mathrm{f}} - \sigma'_{\mathrm{o}}}{\sigma'_{\mathrm{m}}}$$

that, by means of the pi theorem [11], provide the solution for the characteristic time ($\pi_{\mathrm{I}} = \Psi\left[\pi_{\mathrm{II}}\right]$)

$$\tau_{\mathrm{o},\sigma'} = \frac{\mathrm{H_o^2}}{\mathrm{c_v}}\Psi\left[\frac{\sigma'_{\mathrm{f}} - \sigma'_{\mathrm{o}}}{\sigma'_{\mathrm{m}}}\right] \tag{26}$$

where Ψ is an arbitrary and unknown function of its argument. Adopting for σ'_m the value, for example, σ'_o (it can also be the value σ'_f or the average between them), the above equation is simplified to

$$\tau_{\mathrm{o},\sigma'} = \frac{\mathrm{H_o^2}}{\mathrm{c_v}}\Psi\left[\frac{\sigma'_{\mathrm{f}}}{\sigma'_{\mathrm{o}}}\right] \tag{27}$$

which means that the characteristic time taken by the excess pore pressure to relax to approximately the value of zero is a function of the ratio between the final and initial effective stresses, in addition to being directly proportional to $\frac{\mathrm{H_o^2}}{\mathrm{c_v}}$. In this way, and based on the result obtained, the evolution of the average degree of pressure dissipation, $\bar{\mathrm{U}}_{\sigma'}$, can be expressed as

$$\bar{\mathrm{U}}_{\sigma'} = \Psi\left[\frac{\mathrm{t}}{\tau_{\mathrm{o},\sigma'}}, \frac{\sigma'_{\mathrm{f}}}{\sigma'_{\mathrm{o}}}\right] \tag{28}$$

In short, the monomials that govern the solution of the characteristic time associated with the dissipation of interstitial pressure throughout the soil are two:

$$\pi_1 = \frac{\tau_{\mathrm{o},\sigma'}\mathrm{c_v}}{\mathrm{H_o^2}} \quad \pi_2 = \sigma'_{\mathrm{f}}/\sigma'_{\mathrm{o}} \tag{29}$$

and, therefore, the average degree of pressure dissipation ($\bar{\mathrm{U}}_{\sigma'}$) will also be a function dependent on π_1 and π_2.

At the local level, the characteristic time will depend, additionally, on the depth z, which when expressed in dimensionless form is $z' = \frac{z}{H_o}$. The solution for $(\sigma')'$ or σ' is governed by the expression

$$\left(\sigma'\right)' = \frac{\sigma' - \sigma'_o}{\sigma'_f - \sigma'_o} = \Psi\left(\frac{t}{\tau_{o,\sigma'}}, \frac{\sigma'_f}{\sigma'_o}, \frac{z}{H_o}\right) \tag{30}$$

where Ψ, an arbitrary function of its arguments, can also be written in the form

$$\left(\sigma'\right)' = \frac{\sigma' - \sigma'_o}{\sigma'_f - \sigma'_o} = \Psi\left(\frac{tc_v}{H_o^2}, \frac{\sigma'_f}{\sigma'_o}, \frac{z}{H_o}\right) \tag{31}$$

a result that coincides with the analytical solution (Davis and Raymond [1]) in terms of argument dependencies.

3.1.2 Nondimensionalization of the model proposed by Davis and Raymond in terms of settlement

Davis and Raymond did not seem to notice that the new local magnitude (w), equation (10), is proportional to the differential void ratio '$e_f - e$', equation (12) and, under this dependency, also the local settlement. It would have seemed more illustrative and didactic, although in essence it is the same approach, starting from the empirical relation $e = e(\sigma')$, equation (3), written in the form

$$e = e_f - I_c \log_{10}\left(\frac{\sigma'}{\sigma'_f}\right) \tag{32}$$

where e_f denotes the final void ratio (corresponding to the final effective stress σ'_f), and define the new variable $\zeta = e - e_f$, with a clear physical meaning (differential local void ratio, a kind of local degree of settlement, difference between the current void ratio and the final void ratio, in each position; a positive number). It is evident that the variable ζ thus defined must lead to a pure linear diffusion equation, with universal solutions. Indeed, we can rewrite (32) as

$$\zeta = -I_c \log_{10}\left(\frac{\sigma'}{\sigma'_f}\right) = -\frac{I_c}{\ln(10)} \ln\left(\frac{\sigma'}{\sigma'_f}\right) \tag{33}$$

while its partial derivatives with respect to position and time are

$$\frac{\partial \zeta}{\partial z} = -\frac{I_c}{\ln(10)} \frac{1}{\sigma'} \frac{\partial \sigma'}{\partial z} \tag{34}$$

$$\frac{\partial \zeta}{\partial t} = -\frac{I_c}{\ln(10)} \frac{1}{\sigma'} \frac{\partial \sigma'}{\partial t} \tag{35}$$

By calculating the partial derivatives of σ' with respect to position and time, we obtain

$$\frac{\partial \sigma'}{\partial z} = -\frac{\ln(10)}{I_c} \sigma' \frac{\partial \zeta}{\partial z} \tag{36}$$

$$\frac{\partial \sigma'}{\partial t} = -\frac{\ln(10)}{I_c} \sigma' \frac{\partial \zeta}{\partial t} \tag{37}$$

expressions from which it is deduced that

$$\left(\frac{\partial \sigma'}{\partial z}\right)^2 = \left(\frac{\ln(10)}{I_c}\right)^2 \sigma'^2 \left(\frac{\partial \zeta}{\partial z}\right)^2 \tag{38}$$

$$\frac{\partial^2 \sigma'}{\partial z^2} = \left(\frac{\ln(10)}{I_c}\right)^2 \sigma' \left(\frac{\partial \zeta}{\partial z}\right)^2 - \frac{\ln(10)}{I_c} \sigma' \frac{\partial^2 \zeta}{\partial z^2} \tag{39}$$

Now, starting from (9a), it is immediate to reach

$$c_v \frac{\partial^2 \zeta}{\partial z^2} = \frac{\partial \zeta}{\partial t} \tag{40}$$

A pure diffusion equation of the local variable differential void ratio directly related to the local degree of settlement. For its nondimensionalization, the dimensionless variables are defined as

$$(\zeta)' = \frac{\zeta - \zeta_o}{\zeta_f - \zeta_o} = \frac{\zeta_o - \zeta}{\zeta_o} = \frac{e - e_o}{e_f - e_o} \quad z' = \frac{z}{H_o} \quad t' = \frac{t}{\tau_{o,s}} \tag{41}$$

normalized to the interval [0,1], with $\tau_{o,s}$ as the characteristic time taken by the variable ζ to cover approximately its entire range of values. Substituting in equation (40), through mathematical manipulations, we reach

$$c_v \frac{\zeta_f - \zeta_o}{H_o^2} \frac{\partial^2 \zeta'}{\partial z'^2} = \frac{\zeta_f - \zeta_o}{\tau_{o,s}} \frac{\partial \zeta'}{\partial t'} \tag{42}$$

The coefficients of this equation, once simplified, result in

$$\frac{c_v}{H_o^2} \quad \frac{1}{\tau_{o,s}} \tag{43}$$

that, by division, provides a single monomial solution

$$\pi_1 = \frac{\tau_{o,s} c_v}{H_o^2} \tag{44}$$

and a value for the characteristic time of settlement ($\tau_{o,s}$) of the same order of magnitude as that obtained by Davis and Raymond [1], independent of the loading ratio $\frac{\sigma'_f}{\sigma'_o}$.

$$\tau_{o,s} \approx \frac{H_o^2}{c_v} \tag{45}$$

In this way, and based on the result obtained, the average degree of settlement ($\bar{U}_s$) is expressed as

$$\bar{U}_s = \Psi\left[\frac{t}{\tau_{o,s}}\right] \tag{46}$$

a result also coinciding with that obtained by Davis and Raymond in terms of argument dependencies.

3.2 Characterization of the model with non-constant 1+e and constant c_v

3.2.1 Nondimensionalization of the model with non-constant 1+e and constant c_v in terms of pressure

In this case, starting from equation (17), which we repeat here for convenience

$$c_v \left\{ (1+e) \frac{\partial^2 \sigma'}{\partial z^2} + \left(\frac{I_c}{\sigma' \ln(10)} - \frac{1+e}{\sigma'} \right) \left(\frac{\partial \sigma'}{\partial z} \right)^2 \right\} = \frac{\partial \sigma'}{\partial t} \tag{17}$$

and with the dimensionless variables of the expression (22), we reach

$$c_v \left\{ (1+e) \frac{\sigma'_f - \sigma'_o}{H_o^2} \frac{\partial^2 (\sigma')'}{\partial z'^2} + \left(\frac{I_c}{\sigma' \ln(10)} - \frac{1+e}{\sigma'} \right) \left(\frac{\sigma'_f - \sigma'_o}{H_o} \right)^2 \left(\frac{\partial (\sigma')'}{\partial z'} \right)^2 \right\} = \frac{\sigma'_f - \sigma'_o}{\tau_{o,\sigma'}} \frac{\partial (\sigma')'}{\partial t'} \tag{47}$$

from which the following coefficients result in:

$$\frac{c_v I_c\left(\sigma'_f - \sigma'_o\right)}{\ln(10)\, H_o^2 \sigma'_m} \quad \frac{c_v(1+e_m)(\sigma'_f - \sigma'_o)}{H_o^2 \sigma'_m} \frac{c_v(1+e_m)}{H_o^2} \quad \frac{1}{\tau_{o,\sigma'}} \tag{48}$$

where the values of σ' and e have been averaged to arbitrary characteristic values (σ'_m and e_m). Dividing the coefficients by the third coefficient, we obtain the monomials

$$\pi_I = \frac{\tau_{o,\sigma'} c_v\,(1+e_m)}{H_o^2} \quad \pi_{II} = \frac{\sigma'_f - \sigma'_o}{\sigma'_m} \quad \pi_{III} = \frac{I_c(\sigma'_f - \sigma'_o)}{\ln(10)(1+e_m)\sigma'_m} \tag{49}$$

that provide the solution for the characteristic time ($\pi_I = \Psi[\pi_{II},\ \pi_{III}]$).

Adopting for σ'_m and e_m, for instance, the values σ'_o and e_o, the previous monomials can be simplified to

$$\pi_1 = \frac{\tau_{o,\sigma'} c_v\,(1+e_o)}{H_o^2} \quad \pi_2 = \frac{\sigma'_f}{\sigma'_o} \quad \pi_3 = \frac{I_c}{1+e_o} \tag{50}$$

allowing to write now more comfortably ($\pi_1 = \Psi[\pi_2,\ \pi_3]$)

$$\tau_{o,\sigma'} = \frac{H_o^2}{c_v\,(1+e_o)} \Psi\left[\frac{\sigma'_f}{\sigma'_o}, \frac{I_c}{1+e_o}\right] \tag{51}$$

On this occasion, the evolution of the average degree of pressure dissipation ($\bar{U}_{\sigma'}$) can be expressed as

$$\bar{U}_{\sigma'} = \left[\frac{t}{\tau_{o,\sigma'}}, \frac{\sigma'_f}{\sigma'_o}, \frac{I_c}{1+e_o}\right] \tag{52}$$

If, in addition, we take into account the variation of the volume element size dz throughout the consolidation process, considering the relation $\frac{H}{H_o} = \frac{\Delta z}{\Delta z_o} = \frac{1+e}{1+e_o}$, we just have to add to equation (47) the dependency

$$\frac{1}{\Delta z^2} = \frac{1}{\Delta {z_o}^2} \frac{(1+e_o)^2}{(1+e)^2} \tag{53}$$

resulting in the coefficients

$$\frac{c_v I_c\left(\sigma'_f - \sigma'_o\right)}{\ln(10)\, H_o^2 \sigma'_m} \frac{(1+e_o)^2}{(1+e_m)^2} \quad \frac{c_v(\sigma'_f - \sigma'_o)}{H_o^2 \sigma'_m} \frac{(1+e_o)^2}{(1+e_m)} \quad \frac{c_v}{H_o^2} \frac{(1+e_o)^2}{(1+e_m)} \quad \frac{1}{\tau_{o,\sigma'}} \tag{54}$$

and, dividing by the third coefficient, the monomials

$$\pi_I = \frac{\tau_{o,\sigma'} c_v (1+e_o)^2}{H_o^2\,(1+e_m)} \quad \pi_{II} = \frac{\sigma'_f - \sigma'_o}{\sigma'_m} \quad \pi_{III} = \frac{I_c(\sigma'_f - \sigma'_o)}{\ln(10)(1+e_m)\sigma'_m} \tag{55}$$

Once we adopt the values σ'_o and e_o for σ'_m and em, the resulting monomials are on this occasion the same (50) as for the case of constant dz. Thus, the fact of adopting the more general (and more precise) hypothesis, dz variable, does not increase the number of argument dependencies of the problem, which is governed by three monomials. Therefore, the expressions for the characteristic time ($\tau_{o,\sigma'}$) and the average degree of pressure dissipation ($\bar{U}_{\sigma}{}'$) are those already expressed in equations (51) and (52).

3.2.2 Nondimensionalization of the model with non-constant 1+e and constant c_v in terms of settlement

In this case, starting from (17) and through the expressions (37–39), it is immediate to reach

$$c_v(1+e)\frac{\partial^2\zeta}{\partial z^2}-c_v\left(\frac{\partial\zeta}{\partial z}\right)^2=\frac{\partial\zeta}{\partial t} \tag{56}$$

and with the dimensionless variables of the expression (41), we obtain

$$c_v(1+e)\frac{\zeta_f-\zeta_o}{H_o^2}\frac{\partial^2\zeta'}{\partial z'^2}-c_v\left(\frac{\zeta_f-\zeta_o}{H_o}\right)^2\left(\frac{\partial\zeta'}{\partial z'}\right)^2=\frac{\zeta_f-\zeta_o}{\tau_{o,s}}\frac{\partial\zeta'}{\partial t'} \tag{57}$$

The coefficients of this equation, once simplified, result in

$$\frac{c_v(1+e_m)}{H_o^2}\quad\frac{c_v(\zeta_f-\zeta_o)}{H_o^2}\quad\frac{1}{\tau_{o,s}} \tag{58}$$

that, dividing by the second, provide the monomials

$$\pi_I=\frac{\tau_{o,s}c_v\zeta_o}{H_o^2}\quad\pi_{II}\frac{\zeta_o}{1+e_m} \tag{59}$$

of which it is easy to reach

$$\pi_I=\frac{\tau_{o,s}c_v(1+e_o)}{H_o^2}\frac{I_c\log_{10}\left(\frac{\sigma_f'}{\sigma_o'}\right)}{1+e_o}=\frac{\tau_{o,s}c_v(1+e_o)}{H_o^2}\frac{H_o-H_f}{H_o}\quad\pi_{II}=\frac{I_c\log_{10}\left(\frac{\sigma_f'}{\sigma_o'}\right)}{1+e_m}=\frac{H_o-H_f}{H_m} \tag{60}$$

what allows, finally, to write

$$\pi_1=\frac{\tau_{o,s}c_v(1+e_o)}{H_o^2}\quad\pi_2=\frac{H_f}{H_o} \tag{61}$$

For the case of variable *dz*, proceeding as in the previous section, the coefficients obtained are

$$\frac{c_v}{H_o^2}\frac{(1+e_o)^2}{(1+e_m)}\quad\frac{c_v(\zeta_f-\zeta_o)}{H_o^2}\frac{(1+e_o)^2}{(1+e_m)}\quad\frac{1}{\tau_{o,s}} \tag{62}$$

providing the monomials

$$\pi_I=\frac{\tau_{o,s}c_v\zeta_o}{H_o^2}\frac{(1+e_o)^2}{(1+e_m)^2}\quad\pi_{II}=\frac{\zeta_o}{1+e_m} \tag{63}$$

from which it is easy to reach the same monomials (61) as for the case of constant dz, allowing to write now for the characteristic time of settlement ($\tau_{o,s}$), from $\pi_1=\Psi[\pi_2]$

$$\tau_{o,s}=\frac{H_o^2}{c_v(1+e_o)}\Psi\left[\frac{H_f}{H_o}\right] \tag{64}$$

whereas the evolution of the average degree of settlement ($\bar{U}_s$) can be expressed a

$$\bar{U}_s=\Psi\left[\frac{t}{\tau_{o,s}},\frac{H_f}{H_o}\right] \tag{65}$$

3.3 Characterization of the model with both non-constant 1+e and c_v

3.3.1 Nondimensionalization of the model with both non-constant 1+e and c_v in terms of pressure

In this case, starting from equation (21), which we repeat here for convenience

$$c_{vo}\frac{(1+e)^2}{1+e_o}\left\{\frac{\partial^2\sigma'}{\partial z^2}-\frac{1}{\sigma'}\left(\frac{\partial\sigma'}{\partial z}\right)^2\right\}=\frac{\partial\sigma'}{\partial t} \tag{21}$$

and with the dimensionless variables of expression (22), we have

$$c_{vo}\frac{(1+e)^2}{1+e_o}\left\{\frac{\sigma'_f-\sigma'_o}{H_o^2}\frac{\partial^2(\sigma')'}{\partial z'^2}-\frac{1}{\sigma'}\left(\frac{\sigma'_f-\sigma'_o}{H_o}\right)^2\left(\frac{\partial(\sigma')'}{\partial z'}\right)^2\right\}=\frac{\sigma'_f-\sigma'_o}{\tau_{o,\sigma'}}\frac{\partial(\sigma')'}{\partial t'} \tag{66}$$

from which the following coefficients result:

$$\frac{c_{vo}(1+e_m)^2}{H_o^2(1+e_o)}\quad\frac{c_{vo}(1+e_m)^2(\sigma'_f-\sigma'_o)}{H_o^2(1+e_o)\sigma'_m}\quad\frac{1}{\tau_{o,\sigma'}} \tag{67}$$

Dividing these coefficients by the first, we get the groups

$$\pi_I=\frac{\tau_{o,\sigma'}c_{vo}(1+e_m)^2}{H_o^2(1+e_o)}\quad\pi_{II}=\frac{\sigma'_f-\sigma'_o}{\sigma'_m} \tag{68}$$

Adopting for σ'_m and e_m, again, the values σ'_o and e_o, the previous monomials can be simplified to (after the necessary partition of the group π_I)

$$\pi_1=\frac{\tau_{o,\sigma'}c_{vo}(1+e_o)}{H_o^2}\quad\pi_2=\frac{\sigma'_f}{\sigma'_o}\quad\pi_3=\frac{H_f}{H_o} \tag{69a}$$

or since $\frac{H_f}{H_o}=1-\frac{I_c\log_{10}\left(\frac{\sigma'_f}{\sigma'_o}\right)}{(1+e_o)}$

$$\pi_1=\frac{\tau_{o,\sigma'}c_{vo}(1+e_o)}{H_o^2}\quad\pi_2=\frac{\sigma'_f}{\sigma'_o}\quad\pi_3=\frac{I_c}{1+e_o} \tag{69b}$$

Now we can write for the characteristic time in terms of pressure ($\tau_{o,\sigma'}$), from $\pi_1=\Psi[\pi_2,\ \pi_3]$

$$\tau_{o,\sigma'}=\frac{H_o^2}{c_{vo}(1+e_o)}\Psi\left[\frac{\sigma'_f}{\sigma'_o},\frac{I_c}{1+e_o}\right] \tag{70}$$

and for the evolution of the average degree of pressure dissipation, $\bar{U}_{\sigma'}$

$$\bar{U}_{\sigma'}=\Psi\left[\frac{t}{\tau_{o,\sigma'}},\frac{\sigma'_f}{\sigma'_o},\frac{I_c}{1+e_o}\right] \tag{71}$$

For the variant dz, proceeding as in the previous sections, we have the coefficients

$$\frac{c_{vo}(1+e_o)}{H_o^2}\quad\frac{c_{vo}(1+e_o)(\sigma'_f-\sigma'_o)}{H_o^2\sigma'_m}\quad\frac{1}{\tau_{o,\sigma'}} \tag{72}$$

which, dividing by the first, provide the monomials

$$\pi_I=\frac{\tau_{o,\sigma'}c_{vo}(1+e_o)}{H_o^2}\quad\pi_{II}=\frac{\sigma'_f-\sigma'_o}{\sigma'_m} \tag{73}$$

that can be simplified to

$$\pi_1 = \frac{\tau_{o,\sigma'} c_{vo}\left(1+e_o\right)}{H_o^2} \quad \pi_2 = \frac{\sigma'_f}{\sigma'_o} \tag{74}$$

being able to write for the characteristic time in terms of pressure ($\tau_{o,\sigma'}$), from $\pi_1 = \Psi\left[\pi_2\right]$ more comfortably

$$\tau_{o,\sigma'} = \frac{H_o^2}{c_{vo}\left(1+e_o\right)}\left[\frac{\sigma'_f}{\sigma'_o}\right] \tag{75}$$

and for the evolution of the average degree of pressure dissipation, $\bar{U}_{\sigma'}$

$$\bar{U}_{\sigma'} = \Psi\left[\frac{t}{\tau_{o,\sigma'}}, \frac{\sigma'_f}{\sigma'_o}\right] \tag{76}$$

Thus, in view of the expressions (69–71) and (74–76), the fact of adopting the more general (and more precise) hypothesis, dz variable, decreases in this case the number of argument dependencies of the problem, which is reduced to only two monomials.

3.3.2 Nondimensionalization of the model with both non-constant 1+e and c_v in terms of settlement

In this case, starting from (21) and through expressions (37–39), it is immediate to reach

$$c_{vo}\frac{(1+e)^2}{1+e_o}\frac{\partial^2\zeta}{\partial z^2} = \frac{\partial\zeta}{\partial t} \tag{77}$$

and with the dimensionless variables of expression (41), we have

$$c_{vo}\frac{(1+e)^2}{1+e_o}\frac{\zeta_f-\zeta_o}{H_o^2}\frac{\partial^2\zeta'}{\partial z'^2} = \frac{\zeta_f-\zeta_o}{\tau_{o,s}}\frac{\partial\zeta'}{\partial t'} \tag{78}$$

The coefficients of this equation, once simplified, result in

$$\frac{c_{vo}}{H_o^2}\frac{\left(1+e_m\right)^2}{1+e_o} \quad \frac{1}{\tau_{o,s}} \tag{79}$$

that, by dividing, provide the monomial

$$\pi_I = \frac{\tau_{o,s} c_{vo}\left(1+e_m\right)^2}{H_o^2\left(1+e_o\right)} \tag{80}$$

and, proceeding as in the previous sections, we reach

$$\pi_1 = \frac{\tau_{o,s} c_{vo}\left(1+e_o\right)}{H_o^2} \quad \pi_2 = \frac{H_f}{H_o} \tag{81}$$

So the expressions for the characteristic time ($\tau_{o,s}$) and the average degree of settlement ($\bar{U}_s$) remain

$$\tau_{o,s} = \frac{H_o^2}{c_{vo}\left(1+e_o\right)}\Psi\left[\frac{H_f}{H_o}\right] \tag{82}$$

$$\bar{U}_s = \Psi\left[\frac{t}{\tau_{o,s}}, \frac{H_f}{H_o}\right] \tag{83}$$

Finally, adding the condition of variable *dz* in expression (78), the following coefficients are obtained:

$$\frac{c_{vo}\left(1+e_o\right)}{H_o^2} \quad \frac{1}{\tau_{o,s}} \tag{84}$$

These coefficients provide the monomial

$$\pi_1 = \frac{\tau_{o,s} c_{vo} (1 + e_o)}{H_o^2} \tag{85}$$

and the expressions for the characteristic time ($\tau_{o,s}$) and the average degree of settlement ($\bar{U}_s$) are

$$\tau_{o,s} \approx \frac{H_o^2}{c_{vo} (1 + e_o)} \tag{86}$$

$$\bar{U}_s = \Psi \left[\frac{t}{\tau_{o,s}} \right] \tag{87}$$

Thus, in view of the expressions (81–83) and (85–87), the fact of adopting the more general (and more precise) hypothesis, dz variable, decreases in this case the number of argument dependencies of the problem, which is reduced to a single monomial.

Table 1 summarizes the expressions that govern each of the models addressed in this section, whereas Table 2 shows the monomials that rule their solution patterns.

	Pressure	Settlement
Davis and Raymond	$c_v \left\{ \frac{\partial^2 \sigma'}{\partial z^2} - \frac{1}{\sigma'} \left(\frac{\partial \sigma'}{\partial z} \right)^2 \right\} = \frac{\partial \sigma'}{\partial t}$	$c_v \frac{\partial^2 \zeta}{\partial z^2} = \frac{\partial \zeta}{\partial t}$
Davis and Raymond 1+e $\neq$ constant* c_v constant	$c_v \left\{ (1+e) \frac{\partial^2 \sigma'}{\partial z^2} + \left(\frac{I_c}{\sigma' \ln(10)} - \frac{1+e}{\sigma'} \right) \left(\frac{\partial \sigma'}{\partial z} \right)^2 \right\} = \frac{\partial \sigma'}{\partial t}$	$c_v (1+e) \frac{\partial^2 \zeta}{\partial z^2} - c_v \left(\frac{\partial \zeta}{\partial z} \right)^2 = \frac{\partial \zeta}{\partial t}$
Davis and Raymond 1+e $\neq$ constant* $c_v \neq$ constant	$c_{vo} \frac{(1+e)^2}{1+e_o} \left\{ \frac{\partial^2 \sigma'}{\partial z^2} - \frac{1}{\sigma'} \left(\frac{\partial \sigma'}{\partial z} \right)^2 \right\} = \frac{\partial \sigma'}{\partial t}$	$c_{vo} \frac{(1+e)^2}{1+e_o} \frac{\partial^2 \zeta}{\partial z^2} = \frac{\partial \zeta}{\partial t}$

*the assumption of variable dz adds $dz = dz_o \frac{1+e}{1+e_o}$

Table 1 Governing equations for the different variants of the Davis and Raymond model

4 Verification of results and universal curves

The objective of this section is the verification of the deduced dimensionless groups that govern the consolidation problem of Davis and Raymond and, once tested, the obtention of universal curves based on these groups for the variables of greatest interest to geotechnical engineers: characteristic time and average degree of consolidation.

In order not to make this section too extensive, we exclusively stick to the most general and precise Davis and Raymond model, that is, the one that considers both 1+e and c_v non-constant, with the added hypothesis of variable dz. For the rest of the models, of less interest, universal solutions can be found (based on the monomials obtained) in the works of Davis and Raymond [1] (for the original model of the authors with constant cv and 1+e) and García-Ros [16] (for the extended models with constant c_v and non-constant 1+e).

4.1 Verification of results and universal curves for the model with both non-constant c_v and 1+e and variable *dz*

Table 3 shows a series of nine simulations for the most general Davis and Raymond model (non-constant 1+e and c_v, with variable dz) in which several parameters or initial values are modified in order to check and

	Pressure	Settlement
Davis and Raymond	$\pi_1 = \frac{\tau_{o,\sigma'} c_v}{H_o^2}$ $\pi_2 = \frac{\sigma'_f}{\sigma'_o}$	$\pi_1 = \frac{\tau_{o,s} c_v}{H_o^2}$
Davis and Raymond 1+e $\neq$ constant c_v constant *dz* constant and *dz* $\neq$ constant	$\pi_1 = \frac{\tau_{o,\sigma'} c_v (1+e_o)}{H_o^2}$ $\pi_2 = \frac{\sigma'_f}{\sigma'_o}$ $\pi_3 = \frac{I_c}{1+e_o}$	$\pi_1 = \frac{\tau_{o,s} c_v (1+e_o)}{H_o^2}$ $\pi_2 = \frac{H_f}{H_o}$
Davis and Raymond 1+e $\neq$ constant $c_v \neq$ constant *dz* constant	$\pi_1 = \frac{\tau_{o,\sigma'} c_{vo} (1+e_o)}{H_o^2}$ $\pi_2 = \frac{\sigma'_f}{\sigma'_o}$ $\pi_3 = \frac{I_c}{1+e_o}$	$\pi_1 = \frac{\tau_{o,s} c_{vo} (1+e_o)}{H_o^2}$ $\pi_2 = \frac{H_f}{H_o}$
Davis and Raymond 1+e $\neq$ constant $c_v \neq$ constant *dz* $\neq$ constant	$\pi_1 = \frac{\tau_{o,\sigma'} c_{vo} (1+e_o)}{H_o^2}$ $\pi_2 = \frac{\sigma'_f}{\sigma'_o}$	$\pi_1 = \frac{\tau_{o,s} c_{vo} (1+e_o)}{H_o^2}$

Table 2 Dimensionless groups that characterize the solutions for the different variants of the Davis and Raymond model

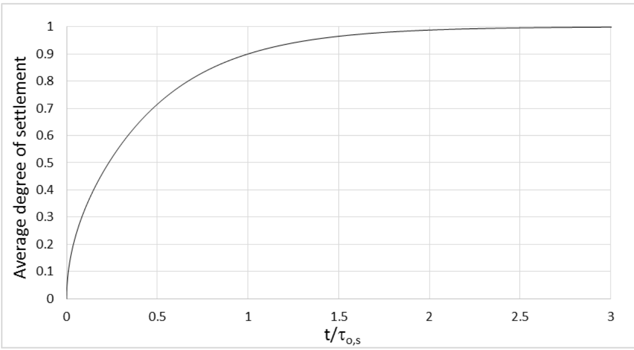

Fig. 1 Average degree of settlement evolution for the extended Davis and Raymond model with both non-constant c_v and 1+e and variable dz.

verify the dependencies of the different solutions of the models with respect to the monomials of Table 2: π_1 and π_2 in terms of pressure (hereinafter $\pi_{1\sigma'}$ and $\pi_{2\sigma'}$) and π_1 in terms of settlement (henceforth π_{1s}). The objective is to show (and verify) that, independently of the values that the particular parameters of the problem take, the monomials $\pi_{1\sigma'}$, $\pi_{2\sigma'}$ and π_{1s} govern the solution pattern of the problem.

For this purpose, a first reference case is established, on the basis of which we will modify the different parameters or initial values to define the other cases. For all models, the potentially variable physical and geometric characteristics are k_o (m/yr), e_o, I_c, σ'_o (N/m^2), H_o (m) and σ'_f (N/m^2). The values of c_{vo} (m^2/yr) and $\pi_{2\sigma'}$ are deduced from them, whereas $\pi_{1\sigma'}$ and π_{1s} are obtained once we know the values of $\tau_{o,\sigma'}$ and $\tau_{o,s}$ (yr), characteristic times corresponding to pressure or settlement, respectively; these times are recorded once the simulation has been performed. As a criterion for the choice of the value of the characteristic time, it has been considered to take this time value for which 90% of the definitive settlement ($\tau_{o,s}$), or 90% of an average pressure dissipation ($\tau_{o,\sigma'}$), has been reached.

In Table 3, cases 01–05 represent different consolidation scenarios, but the monomial remains $\pi_{2\sigma'} = \frac{\sigma'_f}{\sigma'_o}$ constant. In view of the results, it is observed how each scenario can present different values for both the initial consolidation coefficient (c_{vo}) and the characteristic time ($\tau_{o,\sigma'}$); however while the value of the monomial $\pi_{2\sigma'}$ ($\frac{\sigma'_f}{\sigma'_o}$ equals to 2) remains unchanged, the dimensionless expression of the characteristic time $\pi_{1\sigma'}$ will also not vary. Scenarios 06 and 07 show how by varying the loading ratio ($\frac{\sigma'_f}{\sigma'_o}$ now equal to 4), the value of the monomial

Case	K_o (m/yr)	e_o	I_c	σ'_o (N/m^2)	H_o (m)	σ'_f (N/m^2)	c_{vo} (m^2/yr)	$\tau_{o,\sigma'}$ (yr)	$\tau_{o,s}$ (yr)	$\pi_{1\sigma'}$	$\pi_{2\sigma'}$	π_{1s}
01	0.02	1.5	0.45	30000	1	60000	0.783	0.4941	0.4328	0.967	2.0	0.847
02	0.04	1.5	0.45	15000	1	30000	0.783	0.4941	0.4328	0.967	2.0	0.847
03	0.02	0.25	0.1125	30000	1	60000	1.566	0.4941	0.4328	0.967	2.0	0.847
04	0.04	1.5	0.45	60000	2	120000	3.133	0.4941	0.4328	0.967	2.0	0.847
05	0.03	1	0.3	25000	1.5	50000	1.175	0.926	0.811	0.967	2.0	0.847
06	0.02	1.5	0.45	30000	1	120000	0.783	0.5501	0.4328	1.077	4.0	0.847
07	0.02	1.5	0.45	30000	2	120000	0.783	2.2004	1.7312	1.077	4.0	0.847
08	0.02	1.5	0.45	30000	1	240000	0.783	0.6001	0.4328	1.175	8.0	0.847
09	0.02	1.5	0.45	30000	1	480000	0.783	0.6444	0.4328	262	16.0	0.847

Table 3 Verification of the dimensionless groups for the extended Davis and Raymond model with both non-constant c_v and 1+e and variable dz.

$\pi_{1\sigma'}$ changes with respect to the first 5 cases, having the same value for cases 06 and 07 despite having very different characteristic times ($\tau_{o,\sigma'}$). Cases 08 and 09 complete the verification, showing how the value of $\pi_{1\sigma'}$ changes (growing) as $\frac{\sigma'_f}{\sigma'_o}$ increases.

Regarding the dimensionless form π_{1s} of the characteristic time of settlement ($\tau_{o,s}$), it is verified, as it has been deduced from the nondimensionalization process, that its value does not depend on any group of parameters of the problem, remaining constant at all times (π_{1s} equals to 0.847). In this way, from expression (85), where π_1 is of the order of magnitude of the unit, once the problem has been numerically solved and the veracity of the proposed dimensionless groups verified, the universal solution for the characteristic time in terms of settlement is reached:

$$\tau_{o,s} = \frac{0.847 H_o^2}{c_{vo}\left(1+e_o\right)}. \tag{88}$$

Once the expression for the characteristic time in terms of settlement ($\tau_{o,s}$) has been obtained, we can represent the universal curve for the average degree of settlement ($\bar{U}_s$) as a function of the dimensionless time $t/\tau_{o,s}$ (Figure 1). Considering the reader that, to obtain this curve, only a single simulation (of any consolidation scenario) has been necessary.

Regarding the problem in terms of pressure, as can be deduced from the expressions (74–76), the dimensionless form of the characteristic time is a function of $\frac{\sigma'_f}{\sigma'_o}$. On the other hand, in view of expressions (75) and (86), the characteristic time in terms of both pressure and settlement is proportional to the factor $\frac{H_o^2}{c_{vo}(1+e_o)}$. In addition, in practice, although what interests the geotechnical engineer about the consolidation process is the evolution of settlements, in many cases, it is easier to track in situ the evolution of interstitial pressure. For this reason, it seems interesting to know the relation between the two characteristic times for the different loading ratios $\frac{\sigma'_f}{\sigma'_o}$ (Figure 2).

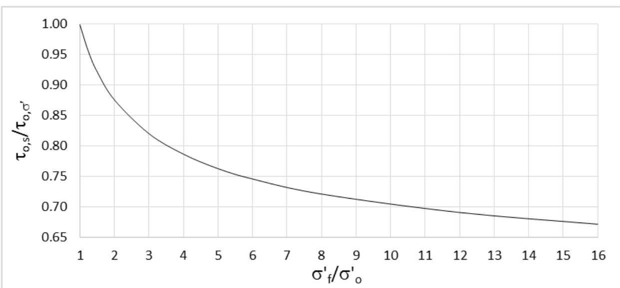

Fig. 2 Ratio $\tau_{o,s}/\tau_{o,\sigma'}$ as a function of σ'_f/σ'_o for the extended model of Davis and Raymond with both non-constant c_v and 1+e and variable dz.

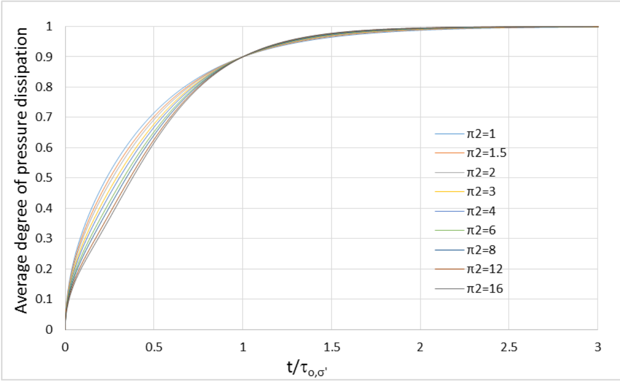

Fig. 3 Evolution of average degree of pressure dissipation for the extended Davis and Raymond model with both non-constant c_v and 1+e and variable *dz*.

Finally, once the value for the characteristic time in terms of pressure ($\tau_{o,\sigma'}$) is known, we can represent the universal curve for the average degree of pressure dissipation ($\bar{U}_{\sigma\prime}$) as a function of the dimensionless time $t/\tau_{o,\sigma'}$ and the monomial $\frac{\sigma'_f}{\sigma'_o}$ (Figure 3).

5 Final comments and conclusions

The search for the dimensionless groups that govern the non-linear consolidation problem based on the Davis and Raymond original and extended models, by means of the nondimensionalization technique for governing equations, has led to simple solutions despite the enormous set of physical and geometrical parameters, in addition to those referred to the boundary conditions, involved in the problem.

By introducing as reference different characteristic times of consolidation (parameters of great interest in ground engineering) in order to nondimensionalize the real time, the groups have been deduced by means of the dimensional coefficients derived from the mathematical treatment of the governing equations. In this way, these same characteristic times can be expressed as a function of the emerging dimensionless groups.

The comparison between the most complex model (non-constant 1+e and c_v and variable d*z*) and the original (whose groups can be deduced from the analytical expressions reported by Davis and Raymond) has given rise, curiously, to the same number of monomials, one for settlement and two for pressure (despite having introduced two new parameters in the extended model: initial void ratio and initial consolidation coefficient), which can be considered a contribution of great interest given the higher precision of the extended model.

It is worth mentioning that in the less general extended models, case i) non-constant 1+e and constant c_v and *dz*, and case ii) non-constant 1+e and c_v and constant *dz,* one more group has emerged in both settlement and pressure, which is undoubtedly due to the inconsistencies resulting in, the first case, assuming 1+e non-constant but keeping constant cv and, the second case, assuming both 1+e and cv non-constant but *dz* constant.

The application of the pi theorem has allowed to represent the results as a function of the dimensionless characteristic time, for both the average degree of settlement and the average degree of pressure dissipation, in the second case by means of an abacus that used the loading ratio as a parameter. It has been observed that the characteristic time of settlement is always lower than the characteristic time in terms of pressure and that this decreases depending on the loading ratio. This is, undoubtedly, due to the non-linear nature of the e$\sim \sigma'$ constitutive relation, according to which as the effective soil stress is more, an increase of this leads to a diminution of the void ratio every time smaller.

Nomenclature

A cross section area (m^2)

c_v consolidation coefficient (m^2/s or m^2/yr)
c_{vo} initial consolidation coefficient (m^2/s or m^2/yr)
dz volume element size (m)
dz_o initial volume element size (m)
e void ratio (dimensionless)
e_f final void ratio (dimensionless)
e_m mean void ratio (dimensionless)
e_o initial void ratio (dimensionless)
H soil thickness (m)
H_f final soil thickness (m)
H_m mean soil thickness (m)
H_o initial soil thickness (m)
I_c compression index (dimensionless)
k hydraulic conductivity (m/s)
k_o initial hydraulic conductivity (m/s)
m_v volumetric compressibility coefficient (m^2/N)
$\dot{q}_v$ void volume temporal change (m^3/s)
$\dot{q}_w$ water volume temporal change (m^3/s)
t time (s or yr)
t' dimensionless time (dimensionless)
u excess pore or interstitial pressure (N/m^2)
$\bar{U}$ average degree of consolidation (dimensionless)
$\bar{U}_s$ average degree of settlement (dimensionless)
$\bar{U}_{\sigma'}$ average degree of pressure dissipation (dimensionless)
V_v void volume (m^3)
V_w water volume (m^3)
v_z water velocity in the vertical spatial direction z (m/s)
w auxiliary variable of Davis and Raymond (dimensionless)
z spatial direction z (m)
z' dimensionless spatial direction z (dimensionless)
Ψ unknown arbitrary function
Δz volume element thickness (m)
Δz_o initial volume element thickness (m)
γ_w water-specific weight (N/m^3)
σ total pressure or stress (N/m^2)
σ' effective pressure or stress (N/m^2)
$(\sigma')'$ dimensionless effective pressure or stress (dimensionless)
σ'_f final effective pressure or stress (N/m^2)
σ'_m mean effective pressure or stress (N/m^2)
σ'_o initial effective pressure or stress (N/m^2)
τ_o characteristic time (s or yr)
$\tau_{o,s}$ characteristic time of settlement (s or yr)
$\tau_{o,\sigma'}$ characteristic time of pressure dissipation (s or yr)
ζ differential local void ratio (dimensionless)
ζ_f final differential local void ratio (dimensionless)
ζ_o initial differential local void ratio (dimensionless)

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