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Dual skew Heyting almost distributive lattices

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Abstract

In this paper, we introduce the concept of dual skew Heyting almost distributive lattices (dual skew HADLs) and characterise it in terms of dual HADL. We define an equivalence relation θ on a dual skew HADL L and prove that θ is a congruence relation on the equivalence class $[x]\theta$ so that each congruence class is a maximal rectangular subalgebra and the quotient $[y]\theta/\theta$ is a maximal lattice image of $[x]\theta$ for any $y \in [x]\theta$. Moreover, we show that if the set $PI(L)$ of all the principal ideals of an ADL L with 0 is a dual skew Heyting algebra then L becomes a dual skew HADL. Further we present different conditions on which an ADL with 0 becomes a dual skew HADL.

Keywords: almost distributive lattice (ADL), maximal element, Heyting almost distributive lattice(HADL), Heyting algebra, skew lattice, skew Heyting algebra and skew Heyting almost distributive lattice(skew HADL).
MSC codes: 03G12, 06D15

1 Introduction

The foundation for the modern theory of skew lattice was laid by Jonathan Leech in 1989 [5]. Leech [6, 7] showed that each right handed skew Boolean algebra can be embedded in to a generic skew Boolean algebra of partial functions from a given set in the co-domain $\{0,1\}$. Heyting algebra is a relatively pseudo-complemented distributive lattice that arises from non-classical logic, and it is named after a Dutch mathematician Arend Heyting, it was introduced by G. Birkhoff [1] and was developed by H. B. Curry around the year 1963. While Boolean

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algebras provide algebraic models of classical logic, Heyting algebras provide algebraic models of intuitionistic logic. The notion of skew Heyting algebra was introduced by Karin Cvetko-vah [8]. That paper it is proved that a skew Heyting algebras form a variety and that the maximal lattice image of a skew Heyting algebra is a generalised Heyting algebra.
The concept of an almost distributive lattice (ADL) was introduced by U.M.Swamy and G.C.Rao [9] as a common abstraction for most of the existing ring theoretic generalisations of Boolean algebra and distributive lattices. G.C.Rao, Berhanu Assaye and M.V.Ratnamani in [3] introduced Heyting ADLs (HADLs) as a generalisation of Heyting algebra in the class of ADLs, and they characterise an HADL in terms of the set of all its principal ideals. Unlike in lattices, the dual of an ADL is not an ADL in general. For this reason Rao, G. C. and Naveen Kumar K. introduced the concept of a dual Heyting ADL (dual HADL) [2]. They derived a number of important laws and results satisfied by a dual HADL and dual L-ADL. They also characterised a dual HADL in terms of the lattice of all its principal ideals.

This paper consists of three sections. The first section describes the preliminary concepts that can be used in proving lemmas, theorems and corollaries in the sequel. In the second section of this paper, we introduce the concept of dual skew HADLs. We define an equivalence relation θ on a dual skew HADL and show that: θ is a congruence relation on each equivalence class, and each congruence class is a maximal rectangular subalgebra of the equivalence class and the quotient lattice $[a]\theta$, where $a \in [x]\theta$ such that $[x]\theta$ is the equivalence class of L. Moreover, given that a skew ADL L with 0 on which the set of all the principal ideals of L is a dual skew Heyting algebra, then we prove that L is a dual skew HADL.

2 Preliminaries

In this section, we give the necessary definitions and results on ADLs, HADLs, dual HADLs and skew Heyting algebras which will be used in the subsequent sections.

Definition 2.1. [9] An algebra $(H, \vee, \wedge, 0)$ of type $(2,2,0)$ is called almost distributive lattice (ADL) with 0 if it satisfies the following axioms: for all $x, y, z \in H$,

(1) $x \vee 0 = x$

(2) $0 \wedge x = 0$

(3) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(6) $(x \vee y) \wedge y = y$

Theorem 2.2. [9] *Let H be an ADL. Then for any $w, x, y, z \in H$, we have the following.*

(1) $\wedge$ *is associative*

(2) $x \wedge y \wedge z = y \wedge x \wedge z$

(3) $(x \vee y) \wedge z = (y \vee x) \wedge z$

(4) $\{w \vee (x \vee y)\} \wedge z = \{(w \vee x) \vee y\} \wedge z$

(5) If $x \leq z$ and $y \leq z$, then $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.

Definition 2.3. [3] An algebra $(H, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2,2,2,0,0)$ is called a Heyting algebra if it satisfies the following conditions.

(1) $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice

(2) $x \to x = 1$

(3) $y \leq x \to y$

(4) $x \wedge (x \to y) = x \wedge y$

(5) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

for all $x, y, z \in H$.

Lemma 2.4. [3] *Let H be a Heyting algebra, then an equivalence relation θ on H is a congruence relation if and only if for any $(a, b) \in \theta, d \in H$,*

(1) $(a \wedge d, b \wedge d) \in \theta$

(2) $(a \vee d, b \vee d) \in \theta$

(3) $(a \to d, b \to d) \in \theta$

(4) $(d \to a, d \to b) \in \theta$.

Definition 2.5. [3] Let $(H, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose $\to$ is a binary operation on H satisfying the following conditions.

(1) $x \to x = m$

(2) $(x \to y) \wedge y = y$

(3) $x \wedge (x \to y) = x \wedge y \wedge m$

(4) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(5) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

for all $x, y, z \in H$. Then $(H, \vee, \wedge, \to, 0, m)$ is called a Heyting ADL (HADL).

Let H be an HADL and $\to$ be a binary operation on H such that $x \to y \in H$ for any $x, y \in H$. Then $y \leq x \to y$ implies that $(x \to y) \wedge y = y$, but the converse is not always true. The converse becomes true whenever H is a lattice, and therefore an HADL becomes a Heyting algebra.

Theorem 2.6. [3] *Let H be an ADL with 0 and a maximal element m, then the following are equivalent.*

(1) H is an HADL

(2) $[0, a]$ *is a Heyting algebra for all $a \in H$*

(3) $[0, m]$ *is a Heyting algebra.*

Definition 2.7. [4] Let P be a nonempty set and $\leq, \leq'$ be two partial orders on P. Then we say that $\leq$ and $\leq'$ are dual to each other whenever for any $a, b \in P$, $a \leq b$ if and only if $b \leq' a$.

Note that the dual of any partial order $\leq$ on a nonempty set P is again a partial order on P and is unique. The poset $(P, \leq')$ is the dual of the poset $(P, \leq)$.
Since the dual H^d of a distributive lattice $(H, \vee, \wedge)$ is again a distributive lattice, we give the following definition. A distributive lattice $(H, \vee, \wedge)$ is called a dual Heyting algebra if its dual H^d of H is a Heyting algebra.

Theorem 2.8. [2] *A distributive lattice* $(H, \vee, \wedge, 0, 1)$ *is a dual Heyting algebra if and only if there exists a binary operation* $\leftarrow$ *satisfying the following:*

(1) $x \leftarrow x = 0$

(2) $x \vee (x \leftarrow y) = x \vee y$

(3) $(x \leftarrow y) \vee y = y$

(4) $x \leftarrow (y \vee z) = (x \leftarrow y) \vee (x \leftarrow z)$

(5) $(x \wedge y) \leftarrow z = (x \leftarrow z) \vee (y \leftarrow z)$ *for all* $x, y, z \in H$.

Definition 2.9. [2] An ADL $(H, \vee, \wedge, 0)$ with a maximal element m is called a dual Heyting almost distributive lattice (dual HADL), if for each $a \in H$ the distributive lattice $([0,a], \vee, \wedge, 0, a)$ is a dual Heyting algebra with respect to the binary operation denoted by $\leftarrow_a$.

Theorem 2.10. [2] *Let H be an ADL with a maximal element m. Then the following are equivalent:*

(1) H is a dual HADL

(2) $[0,m]$ *is a dual Heyting algebra*

(3) There exists a binary operation $\leftarrow$ *on H satisfying the following conditions:*

(i) $x \leftarrow x = 0$

(ii) $((x \leftarrow y) \vee y) \wedge m = y \wedge m$

(iii) $(x \vee (x \leftarrow y)) \wedge m = (x \vee y) \wedge m$

(iv) $z \leftarrow (x \vee y) = (z \leftarrow x) \vee (z \leftarrow y)$

(v) $(x \wedge y) \leftarrow z = (x \leftarrow z) \vee (y \leftarrow z)$ *for all* $x, y, z \in H$.

Definition 2.11. [5] A skew lattice is an algebra $\mathbf{S} = (S, \wedge, \vee)$ of type $(2,2)$ such that $\wedge$ and $\vee$ are both idempotent and associative, and they satisfy the following absorption laws:
$x \wedge (x \vee y) = x = x \vee (x \wedge y)$ and $(x \wedge y) \vee y = y = (x \vee y) \wedge y$ for all $x, y \in S$.

A skew lattice is called strongly distributive if for all $x, y, z \in S$ it satisfies the following identities: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$; and it is called co-strongly distributive if it satisfies the identities: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

Definition 2.12. [8] An algebra $\mathbf{S} = (S, \vee, \wedge, \rightarrow, 1)$ of type $(2,2,2,0)$ is said to be a skew Heyting algebra whenever the following conditions are satisfied:

(1) $(S; \vee, \wedge, 1)$ is a co-strongly distributive skew lattice with top 1.

(2) For any $a \in S$, an operation $\rightarrow_a$ can be defined on $a\uparrow = \{a \vee x \vee a | x \in S\} = \{x \in S | a \leq x\}$ such that $(u\uparrow, \vee, \wedge, \rightarrow_a, 1, a)$ is a Heyting algebra with top 1 and bottom a.

(3) An induced binary operation $\rightarrow$ from $\rightarrow_a$ is defined on S by $x \rightarrow y = (y \vee x \vee y) \rightarrow_y y$.

3 Dual Skew HADLs

In this section, we introduce the concept of dual skew HADLs and characterise it in terms of dual skew Heyting algebras. We define a congruence relations on a dual skew HADL and show that each congruence class is a maximal rectangular subalgebra. More over we investigate some of its algebraic properties.

First we define the concept of dual skew Heyting algebra.

Definition 3.1. An algebra $(L, \vee, \wedge, \leftarrow, 0)$ of type $(2,2,2,0)$ is said to be a dual skew Heyting algebra whenever the following conditions are satisfied:

(1) $(L, \vee, \wedge, 0)$ is a strongly distributive skew lattice with bottom 0.

(2) For any $a \in L$, an operation $\leftarrow_a$ can be defined on $a\downarrow = \{x \in L | x \leq a\}$ such that $(a\downarrow, \vee, \wedge, \leftarrow_a, 0, a)$ is a dual Heyting algebra with top a and bottom 0.

(3) An induced binary operation $\leftarrow$ from $\leftarrow_a$ is defined on L by

$$x \leftarrow y = (y \wedge x \wedge y) \leftarrow_y y.$$

Theorem 3.2. *A strongly distributive skew lattice $(H, \vee, \wedge)$ with bottom 0 is a dual skew Heyting algebra if and only if the following conditions are satisfied:*

(1) For any $a \in H^d$, where H^d is the dual of $(H, \vee, \wedge)$ an operation ${}_a\!\leftarrow$ can be defined on $a\uparrow$ such that $(a\uparrow, \vee, \wedge, 0, a)$ is a dual Heyting algebra.

(2) An induced binary operation $\leftarrow$ can be defined on H^d by $x \leftarrow y = (y \wedge x \wedge y)_y\!\leftarrow y$

Through out this chapter for a non empty set L and for any $a, b, c, z \in L$ we use the following notations:

(1) $L^z = \{z \vee c | c \in L\}$

(2) For any $a \in L$, $\leftarrow_a$ is the binary operation defined on $[0, a]$

(3) For any $b, c \in L$, ${}_b\!\leftarrow$ is the binary operation defined on L^b.

For an ADL L and $z \in L$, the set $L_z = \{x \wedge z | x \in L\}$ is a distributive lattice. But the set L^z is not necessarily a distributive lattice. This motivates us to study an algebra defined in terms of the set L^z.

Definition 3.3. Let L be an ADL with out a maximal element m but with 0. Then L is said to be a dual skew HADL if, for each $z \in L$ the algebra $(L^z, \vee, \wedge, {}_z\!\leftarrow, z)$ is a dual skew Heyting algebra.

Example 3.4. Let L be an ADL with 0. For any $z \in L$ we define a binary operation $\leftarrow_z$ on L_z by

$$x \leftarrow_z y = \begin{cases} 0 & \text{if } x \geq y \\ y & \text{otherwise.} \end{cases}$$

It is easy to show that $(L_z, \vee, \wedge, \leftarrow_z, 0, z)$ is a dual Heyting algebra.

The following theorem characterises dual skew HADLs.

Theorem 3.5. *Let L be an ADL with 0. Then the following conditions are equivalent.*

(1) L is a dual skew HADL

(2) (i) for any $z \in L$, L^z is a skew lattice

(ii) for any $b \in L$, $[z,b]$ is a dual HADL

(iii) there exists a binary operation $\leftarrow$ on L defined by $x \leftarrow y = (x \wedge y) \leftarrow_y y$.

Proof. Assume L to be a dual skew HADL. This implies that for any $z \in L$, L^z is a dual skew Heyting algebra and hence it is a skew lattice. In particular $L^0 = L$ is a dual skew Heyting algebra. Consequently, from the definition of dual skew Heyting algebra we have seen that for any $b \in L$ there exists $z \in L^z$ and $([z,b], \vee, \wedge, \leftarrow_b, z, b)$ is a dual Heyting algebra. Hence $[z,b]$ is a dual HADL. Since L^z is a dual skew Heyting algebra, the induced operation ${}_z\!\leftarrow$ on L^z from $\leftarrow_a$ on $[z,a]$, is given by $x_z\!\leftarrow\! y = (y \wedge x \wedge y) \leftarrow_y y$. Thus it is possible to define a binary operation $\leftarrow$ on L by $x \leftarrow y = (y \wedge x \wedge y) \leftarrow_y y = (x \wedge y) \leftarrow_y y$.

Conversely, suppose that condition (2) hold and let $z \in L$. Since the meet operation in an ADL is distributive over the join operation, and by (i), L^z is a strongly distributive skew lattice with bottom z. By (ii) for any $b \in L^z, [z,b]$ is a dual HADL. Since $[z,b]$ is a lattice $([z,b], \vee, \wedge, \leftarrow_b, z, b)$ is a dual Heyting algebra. Now using (ii) it is possible to define ${}_z\!\leftarrow$ on L^z by $x\ {}_z\!\leftarrow y = (x \wedge y) \leftarrow_y y$. But $x \wedge y = y \wedge x \wedge y$. Hence $x\ {}_z\!\leftarrow y = (y \wedge x \wedge y) \leftarrow_y y$. Therefore $(L^z, \vee, \wedge, {}_z\!\leftarrow, z)$ is a dual skew Heyting algebra and hence L is a dual skew HADL. □

Corollary 3.6. *Let L be a dual skew HADL. Then for any $a \in L, L_a$ is a dual Heyting algebra.*

Proof. Clear by Theorem 3.5(2). □

Lemma 3.7. *Let L be a dual skew HADL. Then for any $z \in L, L^z$ is a dual skew HADL.*

Proof. Suppose that L is a dual skew HADL. Then for any $z \in L$, L^z is a dual skew Heyting algebra. Take any $y \in L^z$ as $y \in L$, L^y is also a dual skew Heyting algebra. Therefore L^z is a dual skew HADL. □

Here is an important result that we use in the sequel.

Corollary 3.8. *Let L be a dual skew HADL. If $x,y \in L$ such that $x \leq y$ and $a,b \in L_x$, then $a\ {}_x\!\leftarrow b = a\ {}_y\!\leftarrow b$.*

Proof. Let L be a dual skew HADL and $x,y \in L$ such that $x \leq y$. Then $L_x \subseteq L_y$. If $a,b \in L_x$, then $a \leftarrow_x b \in L_x$ and hence $a \leftarrow_x b \in L_y$. Since $a,b \in L_y$, $a \leftarrow_y b$ also belongs to L_y. The minimal element characterisation of $a \leftarrow_x b$ and $a \leftarrow_y b$ on the dual Heyting algebra L_y forces the two elements to be equal. □

The next theorem characterises a dual skew HADL interms of a congruence relation θ defined on it. First observe the following lemma.

Lemma 3.9. *Let H be a dual skew HADL and $x,y,z \in H$ such that $x \wedge m = y \wedge m$. Then the following statements hold:*

(1) $x \leftarrow y = 0$

(2) $x \leftarrow z = y \leftarrow z.$

Proof. Let L be a dual skew HADL and $x,y,z \in L$. Clearly, $x \wedge y = y$ and $y \wedge x = x$. Then (1) $x \leftarrow y = x \leftarrow (x \wedge y) = (x \wedge (x \wedge y)) \leftarrow_{(x \wedge y)} (x \wedge y) = (x \wedge y) \leftarrow_{(x \wedge y)} (x \wedge y) = 0$.
To prove (2), taking $z \in L$, we obtain that

$$\begin{aligned} x \leftarrow z &= (x \wedge z) \leftarrow_z z \\ &= ((y \wedge x) \wedge z) \leftarrow_z z \\ &= ((x \wedge y) \wedge z) \leftarrow_z z \\ &= (y \wedge z) \leftarrow_z z \\ &= y \leftarrow z. \end{aligned}$$

□

Lemma 3.10. *Let L be a relatively complemented ADL with 0. If L is a dual skew HADL and θ is defined by*

$$\theta = \{(x,y) \in L \times L | x \vee y = x \text{ and } y \vee x = y\},$$

then θ is an equivalence relation on L.

Proof. Suppose L is a dual skew HADL. Let $x,y,z \in L$. It is easly seen that θ is reflexive and symmetric. Assume that $(x,y) \in \theta$ and $(y,z) \in \theta$. Then $x \vee y = x$, $y \vee x = y$, $y \vee z = y$ and $z \vee y = z$. Hence $z \vee x = (z \vee y) \vee x = z \vee (y \vee x) = z \vee y = z$ and $x \vee z = (x \vee y) \vee z = x \vee (y \vee z) = (x \vee y) = x$, it follows that $(x,z) \in \theta$. Consequently θ is transitive and therefore it is an equivalence relation.

□

Theorem 3.11. *Let a relatively complemented ADL L with 0 be a dual skew HADL. For each $b \in L$, consider the equivalence class $[b]\theta$ where θ is a relation in the above lemma. Then the following conditions hold:*

(1) θ is a congruence relation on $[b]\theta$

(2) For each $x \in [b]\theta$, the congruence class $[x]\theta$ is the maximal rectangular subalgebra of $[b]\theta$

(3) $[x]\theta/\theta$ is the maximal lattice image of $[x]\theta$.

Proof. To show that θ is a congruence relation on $[b]\theta$, for any $x,y,a \in [b]\theta$ we need to check that $((x \leftarrow a),(y \leftarrow a)) \in \theta$, $((a \leftarrow x),(a \leftarrow y)) \in \theta$ and the substitution properties are satisfied. Clearly for any $x,y,a \in [b]\theta$, $(x \wedge a) \vee (y \wedge a) = (x \vee y) \wedge a = x \wedge a$ and $(y \wedge a) \vee (x \wedge a) = (y \vee x) \wedge a = y \wedge a$. Thus $((x \wedge a),(y \wedge a)) \in \theta$. Moreover,

$$\begin{aligned}
(x \vee a) \wedge (y \vee a) &= ((x \vee a) \wedge y) \vee ((x \vee a) \wedge a) \\
&= ((x \vee a) \wedge y) \vee a \\
&= ((a \vee x) \wedge y) \vee a \\
&= ((a \wedge y) \vee (x \wedge y)) \vee a \\
&= ((a \wedge y) \vee y) \vee a \\
&= y \vee a.
\end{aligned}$$

Using the same procedure we have $(y \vee a) \wedge (x \vee a) = x \vee a$. From these results we obtain that $(x \vee a) \vee (y \vee a) = (x \vee a) \vee ((x \vee a) \wedge (y \vee a)) = x \vee a$ and $(y \vee a) \vee (x \vee a) = (y \vee a) \vee ((y \vee a) \wedge (x \vee a)) = y \vee a$. Thus $((x \vee a),(y \vee a)) \in \theta$ and $((y \vee a),(x \vee a)) \in \theta$. This shows that θ satisfies the substitution property.

One can simply observe that $x \wedge y = (x \vee y) \wedge y = y$ and $y \wedge x = (y \vee x) \wedge x = x$. Indeed, point (2) of the above lemma assures that $((x \leftarrow a),(y \leftarrow a)) \in \theta$. Furthermore, using Lemma 3.9 (1) we obtain that $(a \leftarrow x) \vee (a \leftarrow y) = (a \leftarrow x) \vee 0 = a \leftarrow x$ and $(a \leftarrow y) \vee (a \leftarrow x) = (a \leftarrow y) \vee 0 = a \leftarrow y$. Thus $((a \leftarrow x),(a \leftarrow y)) \in \theta$. Hence θ is a congruence relation on each equivalence class.

Suppose $y,z \in [b]\theta$. It is obvious that $(y,z) \in \theta$. Following this we obtain that $z \wedge y = (z \vee y) \wedge (y \vee z) = (y \vee z) \wedge (y \vee z) = y \vee z$ so that each congruence class is rectangular. Now take an arbitrary element $x \in [b]\theta$ and consider the congruence class $[x]\theta$. Let T be a rectangular subalgebra of $[b]\theta$ such that $[x]\theta \subseteq T$. Let $r \in T$. Since $x \in T$ and T is a rectangular subalgebra of $[x]\theta$ we have $r \vee x = x \wedge r$ and $x \vee r = r \wedge x$. Thus $x \wedge r = (r \wedge x) \wedge r = (x \vee r) \wedge r = r$ which implies that $r \vee x = x \wedge r = r$ and $x \vee r = x \vee (x \wedge r) = x$. Hence $r \in [x]\theta$. Therefore $T \subseteq [x]\theta$ and we conclude that $[x]\theta = T$. Hence $[x]\theta$ is maximal, i.e., each congruence class $[x]\theta$ is a maximal rectangular subalgebra of $[b]\theta$. Next, to show that $[x]\theta/\theta$ is a lattice we need to have the following results. Let $x, y \in [x]\theta$. Then

$$\begin{aligned} x \vee y &= ((y \vee x) \wedge x) \vee y \\ &= ((y \vee x) \wedge x) \vee ((x \vee y) \wedge y) \\ &= ((y \vee x) \wedge x) \vee ((y \vee x) \wedge y) \\ &= (y \vee x) \wedge (x \vee y). \end{aligned}$$

Similarly,

$$\begin{aligned} y \vee x &= ((x \vee y) \wedge y) \vee x \\ &= ((x \vee y) \wedge y) \vee ((y \vee x) \wedge x) \\ &= ((x \vee y) \wedge y) \vee ((x \vee y) \wedge x) \\ &= (x \vee y) \wedge (y \vee x). \end{aligned}$$

Hence $(x \vee y) \vee (y \vee x) = (x \vee y) \vee ((x \vee y) \wedge (y \vee x)) = x \vee y$ and $(y \vee x) \vee (x \vee y) = (y \vee x) \vee ((y \vee x) \wedge (x \vee y)) = y \vee x$. Therefore $((x \vee y), (y \vee x)) \in \theta$.
Similarly, we have

$$\begin{aligned} (x \wedge y) \vee (y \wedge x) &= (x \wedge y) \vee (x \wedge y \wedge x) \\ &= (x \wedge y) \vee ((x \wedge y) \wedge x) \\ &= x \wedge y \end{aligned}$$

and

$$\begin{aligned} (y \wedge x) \vee (x \wedge y) &= (y \wedge x) \vee (y \wedge x \wedge y) \\ &= (y \wedge x) \vee ((y \wedge x) \wedge y) \\ &= y \wedge x. \end{aligned}$$

Hence, $((x \wedge y), (y \wedge x)) \in \theta$.

Now, assume that $[y]\theta, [z]\theta \in [x]\theta/\theta$. Define $[y]\theta \vee [z]\theta = [y \vee z]\theta$ and $[y]\theta \wedge [z]\theta = [y \wedge z]\theta$. Then $t \in [y \vee z]\theta \Leftrightarrow (t, (y \vee z)) \in \theta \Leftrightarrow (t, (z \vee y)) \in \theta \Leftrightarrow t \in [z \vee y]\theta$. Thus $[y]\theta \vee [z]\theta = [z]\theta \vee [y]\theta$. Similarly, $[y]\theta \wedge [z]\theta = [z]\theta \wedge [y]\theta$. Therefore $[x]\theta/\theta$ is a lattice. Let β be a congruence relation on $[x]\theta$ such that $[x]\theta/\beta$ is a lattice. Suppose $(y, z) \in \theta$. Clearly

$$\begin{aligned} y \vee z = y \text{ and } z \vee y = z &\Rightarrow [y \vee z]\beta = [y]\beta \text{ and } [z \vee y]\beta = [z]\beta \\ &\Rightarrow [y]\beta \vee [z]\beta = [y]\beta \text{ and } [z]\beta \vee [y]\beta = [z]\beta \\ &\Rightarrow [y]\beta = [z]\beta, \text{ since } [y]\beta \vee [z]\beta = [z]\beta \vee [y]\beta \\ &\Rightarrow (y, z) \in \beta. \end{aligned}$$

Therefore, $\theta \subseteq \beta$. Suppose H is a lattice image of L. Then there exist an epimorphism $f : [x]\theta \longrightarrow H$. Define the kernel K of f by

$$K = \{(y, z) \in [b]\theta^2 | f(y) = f(z)\}.$$

K is reflexive, symmetric and transitive. Let $(x_1, y_1), (x_2, y_2) \in K$. Since f is a homomorphism, we have $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = f(y_1) \vee f(y_2) = f(y_1 \vee y_2)$ so that $((x_1 \vee x_2), (y_1 \vee y_2)) \in K$. Using a similar procedure for $\wedge$ and $\leftarrow$ we obtain that $((x_1 \wedge x_2), (y_1 \wedge y_2)) \in K$ and $((x_1 \leftarrow x_2), (y_1 \leftarrow y_2)) \in K$. This shows that K is a congruence relation on $[x]\theta$. Hence we obtain that $[x]\theta/Kerf \cong f([x]\theta) = H$. Therefore any lattice image of $[x]\theta$ is of the form $[x]\theta/\phi$ for some congruence relation ϕ on $[x]\theta$.

Consider that the set $\mathfrak{L} = \{[x]\theta/\phi | \phi$ is a congruence on $[x]\theta$ and $\theta \subseteq \phi$ $\}$ of lattices. Now, take the congruence relations θ and β discussed earlier and assume that $[x]\theta/\theta \subseteq [x]\theta/\beta$. Let $a, c \in [x]\theta$ such that $(a,c) \in \beta$. Since $\theta \subseteq \beta$ and $[x]\theta/\theta \subseteq [x]\theta/\beta$ there exist $a', c' \in [x]\theta$ such that $a \in [a]\theta = [a']\beta$ and $c \in [x]\theta = [c']\beta$. Thus $(a,a'), (c,c') \in \theta$ and $(a,a'), (c,c') \in \beta$. Hence $(a,c), (c,c') \in \beta \Rightarrow (a,c') \in \beta \Rightarrow a \in [b']\beta = [x]\theta \Rightarrow (a,c) \in \theta$. Therefore $\beta \subseteq \theta$ and hence $\theta = \beta$. Consequently, $[x]\theta/\theta = [x]\theta/\beta$ so that $[x]\theta/\theta$ is the maximal lattice image of $[x]\theta$. □

Lemma 3.12. *Let L be an ADL with 0 and $z \in L$. If L is a dual skew HADL, then for any $w, x, y \in L_z$ the following conditions hold:*

(1) $x \geq w \leftarrow_z x$

(2) $(x \leftarrow_z y) \vee w = w$ *if and only if* $y \vee w \vee x = w \vee x$.

Proof. Suppose that L is a dual skew HADL and $z \in L$ such that $w, x, y \in L_z$. Then by definition L^z is a dual skew Heyting algebra. This implies that $L^0 = L$ is a dual skew Heyting algebra so that L_z is a dual Heyting algebra. Thus (1) follows. Now, assume that $(x \leftarrow_z y) \vee w = w$. Then

$$\begin{aligned} w \vee x &= ((x \leftarrow_z y) \vee w) \vee x \\ &= ((x \leftarrow_z y) \vee x) \vee w \\ &= x \vee y \vee w \\ &= y \vee w \vee x. \end{aligned}$$

On the other hand given that $y \vee w \vee x = w \vee x$, we obtain $x \leftarrow_z (w \vee x) = x \leftarrow_z (y \vee w \vee x)$. Hence $(x \leftarrow_z w) = (x \leftarrow_z y) \vee (x \leftarrow_z w)$. Thus

$$\begin{aligned} (x \leftarrow_z y) \vee w &= (x \leftarrow_z y) \vee ((x \leftarrow_z w) \vee w) \\ &= ((x \leftarrow_z y) \vee (x \leftarrow_z w)) \vee w \\ &= (x \leftarrow_z w) \vee w \\ &= w. \end{aligned}$$

Therefore (2) is satisfied. □

Theorem 3.13. *Let L be an ADL with 0 and for any $z \in L$, L^z is a skew lattice. Then L is a dual skew HADL if for any $b \in L^z$ and $w, x, y \in [z, b]$ the following conditions are satisfied:*

(1) $x \geq w \leftarrow_b x$

(2) $(x \leftarrow_b y) \vee w = w$ *if and only if* $y \vee w \vee x = w \vee x$.

Proof. Assume that conditions (1) and (2) hold. For any $b \in L^z$ and $w, x, y \in [z, b]$ we have by (1), $y \geq x \leftarrow_b y$ such that $x \leftarrow_b y \in [z, b]$ and then by (2) we get

$$\begin{aligned} w \geq x \leftarrow_b y &\Leftrightarrow w = (x \leftarrow_b y) \vee w \\ &\Leftrightarrow w \vee x = y \vee w \vee x \\ &\Leftrightarrow w \vee x \geq y. \end{aligned}$$

Hence, $[z, b]$ is a dual Heyting algebra. Defining a binary operation ${}_z\!\leftarrow$ on L^z by $x\,{}_z\!\leftarrow y = (y \wedge x \wedge y) \leftarrow_y y$ makes L^z is a dual skew Heyting algebra and therefore L is a dual skew HADL. □

Corollary 3.14. *On a dual skew HADL L, $(x \wedge y) \leftarrow_y y = 0$ if and only if $x \vee y = x$.*

Proof. Suppose that L is a dual skew HADL and $(x\wedge y)\leftarrow_y y=0$. Since L_y is a dual Heyting algebra (Corollary 3.6), we obtain that

$$\begin{aligned}x &= x\vee(x\wedge y)\\ &= x\vee((x\wedge y)\vee 0)\\ &= x\vee((x\wedge y)\vee((x\wedge y)\leftarrow_y y))\\ &= x\vee((x\wedge y)\vee y)\\ &= x\vee y.\end{aligned}$$

Hence $x\vee y=x$. The converse is straight forward. □

Theorem 3.15. *Let L is a skew ADL with 0. If the set $PI(L)$ of all principal ideals of L is a dual skew Heyting algebra, then L is a dual skew HADL.*

Proof. Suppose $PI(L)$ be a dual skew Heyting algebra. For each $x\in L$ we show that L^x is a dual skew Heyting algebra. Define ${}_x\!\leftarrow$ on L^x by $a\ {}_x\!\leftarrow b=c\wedge x$, for some $c\in L$ such that $(a]\leftarrow(b]=(c]$. Let $(u]=(v]$ for some $u,v\in L$. Since $(u]\subseteq(v]$ and $(v]\subseteq(u]$ implies that $u\wedge v=v$ and $v\wedge u=u$ respectively, we have $u\wedge x=v\wedge u\wedge x=u\wedge v\wedge x=v\wedge x$. Now take $a,b,c,d\in L^x$ such that $a=c,b=d$. Then we obtain that $(a]=(c]$ and $(b]=(d]$. Consequently $(a]\leftarrow(b]=(c]\leftarrow(d]$. Let $(a]\leftarrow(b]=(e],(c]\leftarrow(d]=(f]$ for some $e,f\in L$, we have $a\ {}_x\!\leftarrow b=e\wedge x$ and $c\ {}_x\!\leftarrow d=f\wedge x$. Hence $e\wedge x=f\wedge x$ implies that $a\ {}_x\!\leftarrow b=c\ {}_x\!\leftarrow d$. Therefore the binary operation ${}_x\!\leftarrow$ is well defined on L^x. It is clear that L^x is a strongly distributive skew lattice with bottom x. For any $y\in L^x$, we claim that $[x,y]$ is a dual Heyting algebra. Let $a,b,c\in[x,y]$ and define $\leftarrow_y$ on $[x,y]$ by $a\leftarrow_y b=a\ {}_x\!\leftarrow b$. Clearly $(a],(b],(c]\in(y]\downarrow$ and $(y]\downarrow$ is a dual Heyting algebra. Then

(i) $(a]\leftarrow(a]=(y]$. Then we have $a\leftarrow_y a=a\leftarrow_x a=y\wedge x=x$.

(ii) Let $(a]\leftarrow(b]=(t]$ for some $t\in L$. Then $a\ {}_x\!\leftarrow b=t\wedge x$. Therefore

$$\begin{aligned}(a\leftarrow_y b)\vee b &= (a\ {}_x\!\leftarrow b)\vee b\\ &= (t\wedge x)\vee b\\ &= (b\wedge t\wedge x)\vee b\\ &= (t\wedge b\wedge x)\vee b\\ &= (t\wedge x\wedge b)\vee b\\ &= b.\end{aligned}$$

This is because of $(a]\leftarrow(b]\subseteq(b]\Rightarrow(t]\subseteq(b]\Rightarrow b\wedge t=t$.

(iii) Let $(a]\leftarrow(b]=(t]$ for some $t\in L$. Then $a\ {}_x\!\leftarrow b=t\wedge x$. Now $(a\vee t]=(a]\vee(t]=(a]\vee((a]\leftarrow(b])=(a]\vee(b]=(a\vee b]$. Hence $(a\vee t)\wedge x=(a\vee b)\wedge x$. Therefore

$$\begin{aligned}a\vee(a\leftarrow_y b) &= a\vee(a\ {}_x\!\leftarrow b)\\ &= a\vee(t\wedge x)\\ &= (a\vee t)\wedge(a\vee x)\\ &= (a\vee t)\wedge x\\ &= (a\vee b)\wedge x\\ &= a\vee b.\end{aligned}$$

(iv) Let $(a] \leftarrow (c] = (u]$ and $(b] \leftarrow (c] = (v]$ for some $u, v \in L$. Then we obtain that $a\ {}_x\!\!\leftarrow c = u \wedge x$ and $b\ {}_x\!\!\leftarrow c = v \wedge x$. Further

$$\begin{aligned}(a \wedge b] \leftarrow (c] &= ((a] \wedge (b]) \leftarrow (c] \\ &= ((a] \leftarrow (c]) \vee ((b] \leftarrow (c]) \\ &= (u] \vee (v] \\ &= (u \vee v].\end{aligned}$$

Hence we obtain that

$$\begin{aligned}(a \wedge b) \leftarrow_y c &= (a \wedge b)\ {}_x\!\!\leftarrow c \\ &= (u \vee v) \wedge x \\ &= (u \wedge x) \vee (v \wedge x) \\ &= (a\ {}_x\!\!\leftarrow c) \vee (b\ {}_x\!\!\leftarrow c) \\ &= (a \leftarrow_y c) \vee (b \leftarrow_y c).\end{aligned}$$

(v) Let $(a] \leftarrow (b] = (u]$ and $(a] \leftarrow (c] = (v]$ for some $u, v \in L$. Then $a\ {}_x\!\!\leftarrow b = u \wedge x$ and $a\ {}_x\!\!\leftarrow c = v \wedge x$. Since

$$\begin{aligned}(a] \leftarrow (b \vee c] &= (a] \leftarrow ((b] \vee (c]) \\ &= ((a] \leftarrow (b]) \vee ((a] \leftarrow (c]) \\ &= (u] \vee (v] \\ &= (u \vee v],\end{aligned}$$

we get

$$\begin{aligned}a \leftarrow_y (b \vee c) &= a\ {}_x\!\!\leftarrow (b \vee c) \\ &= (u \vee v) \wedge x \\ &= (u \wedge x) \vee (v \wedge x) \\ &= (a\ {}_x\!\!\leftarrow b) \vee (a\ {}_x\!\!\leftarrow c) \\ &= (a \leftarrow_y b) \vee (a \leftarrow_y c).\end{aligned}$$

Hence $[x, y]$ is a dual Heyting algebra.

On $PI(L)$ we have $(a] \leftarrow_{(b]} (b] = (a \leftarrow_b b]$. Then for $a, b \in L^x$ we have the following

$$\begin{aligned}(a] \leftarrow (b] &= ((b] \wedge (a] \wedge (b]) \leftarrow_{(b]} (b] \\ &= (b \wedge a \wedge b] \leftarrow_{(b]} (b] \\ &= ((b \wedge a \wedge b) \leftarrow_b b].\end{aligned}$$

This implies that $a\ {}_x\!\!\leftarrow b = ((b \wedge a \wedge b) \leftarrow_b b) \wedge x$. Thus $a \leftarrow_x b = (b \wedge a \wedge b) \leftarrow_b b$. Hence L^x is a dual skew Heyting algebra. Therefore L is a dual skew HADL. □

Corollary 3.16. *On a dual skew HADL H, $(x \wedge y)_y\!\!\leftarrow y = 0$ if and only if $x \vee y = x$.*

Proof. Suppose that H is a dual skew HADL and $(x \wedge y)_y\!\!\leftarrow y = 0$. Then

$$\begin{aligned}x &= x \vee 0 \\ &= x \vee ((x \wedge y)_y\!\!\leftarrow y) \\ &= x \vee ((x_y\!\!\leftarrow y) \vee (y_y\!\!\leftarrow y)) \\ &= x \vee (x_y\!\!\leftarrow y) \vee 0 \\ &= x \vee (x_y\!\!\leftarrow y) \\ &= x \vee y.\end{aligned}$$

Hence $x \vee y = x$. The converse is straight forward.

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