Applied Mathematics and Nonlinear Sciences 4(1) (2019) 139–150

Applied Mathematics and Nonlinear Sciences

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BSDEs driven by two mutually independent fractional Brownian motions with stochastic Lipschitz coefficients

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Submission Info

Communicated by Juan Luis García Guirao
Received March 9th 2019
Accepted April 25th 2019
Available online June 25th 2019

Abstract

This paper deals with a class of backward stochastic differential equation driven by two mutually independent fractional Brownian motions. We essentially establish existence and uniqueness of a solution in the case of stochastic Lipschitz coefficients. The stochastic integral used throughout the paper is the divergence-type integral.

Keywords: backward stochastic differential equation, stochastic Lipschitz coefficients, Malliavin derivative and fractional Itô's formula.
MSC codes: 60H05, 60H07, 60G22.

1 Introduction

Backward stochastic differential equations (BSDEs in short) were first introduced by Pardoux and Peng [7]. They proved the celebrated existence and uniqueness result under Lipschitz assumption. This pioneer work was extensively used in many fields like stochastic interpretation of solutions of PDEs and financial mathematics.

Few years later, several authors investigated BSDEs with respect to fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$ with Hurst parameter H. Since B^H is not a semimartingale when $H \neq \frac{1}{2}$, we cannot use the beautiful classical theory of stochastic calculus to define the fractional stochastic integral. It is a significant and challenging problem to extend the results in the classical stochastic calculus to this fractional Brownian motion. Essentially, two different types of integrals with respect to a fractional Brownian motion have been defined and studied. The first one is the pathwise Riemann-Stieltjes integral (see Young [10]). This integral has a proprieties of Stratonovich integral, which leads to difficulties in applications. The second one, introduced in Decreusefond and Ustunel [3] is the divergence operator (or Skorohod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. Since this stochastic integral satisfies the zero mean property and it can be expressed as the limit of Riemann sums defined using Wick products, it was later developed by many authors.

ISSN 2444-8656 doi:10.2478/AMNS.2019.1.00014

Recently, new classes of BSDEs driven by both standard and fractional Brownian motions were introduced by Fei et al [4]. They established the existence and uniqueness of solutions.

In this paper, our aim is to generalize the result established in [2] to the following equation called fractional BSDE under stochastic conditions on the generator:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s} ds - \int_t^T Z_{1,s} dB_{1,t}^{H_1} - \int_t^T Z_{2,s} dB_{2,t}^{H_2}, \quad t \in [0,T], \tag{1.1}$$

where $\left(B_{1,t}^{H_1}\right)_{t\geq 0}$ and $\left(B_{2,t}^{H_2}\right)_{t\geq 0}$ are two mutually independent fractional Brownian motions. The novelty in these types of stochastic equations lies in the fact of coupling two mutually independent fractional Brownian motions. In this work, the authors established some properties of solutions of a fractional BSDE with Lipschitz coefficients. By the help of the fixed point principle, we establish existence and uniqueness of solutions.

The paper is organized as follows: In Section 2, we introduce some preliminaries, before studying the solvability of our equation under Lipschitz conditions on the generator in Section 3. Using this result, we prove existence and uniqueness of the solution with a coefficient satisfying rather weaker conditions.

2 Fractional Stochastic calculus

Let us assume given two mutually independent fractional Brownian motions $B^H \in \left\{B_1^{H_1}, B_2^{H_2}\right\}$ with Hurst parameter $H \geq \frac{1}{2}$ is given.
Let Ω be a non-empty set, $\mathscr{F}$ a $\sigma-$algebra of sets Ω, $\mathbf{P}$ a probability measure defined on $\mathscr{F}$ and $\{\mathscr{F}_t,\, t \in [0,T]\}$ a $\sigma-$algebra generated by both fractional Brownian motions.
The triplet $(\Omega, \mathscr{F}, \mathbf{P})$ defines a probability space and $\mathbf{E}$ the mathematical expectation with respect to the probability measure $\mathbf{P}$.

The fractional Brownian motion B^H is a zero mean Gaussian process with the covariance function

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \geq 0.$$

Denote $\phi(t,s) = H(2H-1)|t-s|^{2H-2}$, $(t,s) \in \mathbf{R}^2$.
Let ξ and η be measurable functions on $[0,T]$. Define

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u,v)\xi(u)\eta(v)dudv \quad \text{and} \quad \|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

Note that, for any $t \in [0,T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let $\mathscr{H}$ be the completion of the set of continuous functions under this Hilbert norm $\|\cdot\|_t$ and $(\xi_n)_n$ be a sequence in $\mathscr{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$.

Let $\mathscr{P}_T^H$ be the set of all polynomials of fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$. Namely, $\mathscr{P}_T^H$ contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \ldots, \int_0^T \xi_n(t)dB_t^H\right)$$

where f is a polynomial function of n variables.

The Malliavin derivative D_t^H of F is given by

$$D_s^H F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \ldots, \int_0^T \xi_n(t)dB_t^H\right)\xi_j(s), \quad 0 \leq s \leq T.$$

Now we introduce the Malliavin ϕ-derivative $\mathbb{D}_t^H$ of F by

$$\mathbb{D}_t^H F = \int_0^T \phi(t,s) D_s^H F ds.$$

We have the following (see[[5], Proposition 6.25]):

Theorem 2.1. *Let $F : (\Omega, \mathscr{F}, \mathbf{P}) \longrightarrow \mathscr{H}$ be a stochastic processes such that*

$$\mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 dsdt \right) < +\infty.$$

Then, the Itô-Skorohod-type stochastic integral denoted by $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathscr{F}, \mathbf{P})$ and satisfies

$$\mathbf{E}\left(\int_0^T F_s dB_s^H\right) = 0 \quad and \quad \mathbf{E}\left(\int_0^T F_s dB_s^H\right)^2 = \mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s dsdt \right).$$

Let us recall the fractional Itô formula (see[[4], Theorem 3.1]).

Theorem 2.2. *Let $\sigma_1, \sigma_2 \in \mathscr{H}$ be deterministic continuous functions. Denote*

$$X_t = X_0 + \int_0^t \alpha(s)ds + \int_0^t \sigma_1(s) dB_{1,s}^{H_1} + \int_0^t \sigma_2(s) dB_{2,s}^{H_2},$$

where X_0 is a constant and $\alpha(t)$ is a deterministic function with $\int_0^t |\alpha(s)|ds < +\infty$.
Let $F(t,x)$ be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then

$$\begin{aligned} F(t,X_t) &= F(0,X_0) + \int_0^t \frac{\partial F}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial F}{\partial x}(s,X_s)dX_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s,X_s)\left[\frac{d}{ds}\|\sigma_1\|_s^2 + \frac{d}{ds}\|\sigma_2\|_s^2\right] ds, \quad 0 \le t \le T. \end{aligned}$$

Let us finish this section by giving a fractional Itô chain rule (see[[4], Theorem 3.2]).

Theorem 2.3. *Assume that for $j = 1,2$, the processes μ_j, α_j and ϑ_j, satisfy*

$$\mathbf{E}\left[\int_0^T \mu_j^2(s)ds + \int_0^T \alpha_j^2(s)ds + \int_0^T \vartheta_j^2(s)ds\right] < +\infty.$$

Suppose that $\mathbb{D}_t^{H_1}\alpha_j(s)$ and $\mathbb{D}_t^{H_2}\vartheta_j(s)$ are continuously differentiable with respect to $(s,t) \in [0,T]^2$ for almost all $\omega \in \Omega$. Let X_t and Y_t be two processes satisfying

$$\begin{aligned} X_t &= X_0 + \int_0^t \mu_1(s)ds + \int_0^t \alpha_1(s)dB_{1,s}^{H_1} + \int_0^t \vartheta_1(s)dB_{2,s}^{H_2}, \qquad 0 \le t \le T, \\ Y_t &= Y_0 + \int_0^t \mu_2(s)ds + \int_0^t \alpha_2(s)dB_{1,s}^{H_1} + \int_0^t \vartheta_2(s)dB_{2,s}^{H_2}, \qquad 0 \le t \le T. \end{aligned}$$

If the following conditions hold:

$$\mathbf{E}\left[\int_0^T |\mathbb{D}_t^{H_1}\alpha_i(s)|^2 dsdt\right] < +\infty \quad and \quad \mathbf{E}\left[\int_0^T |\mathbb{D}_t^{H_2}\vartheta_i(s)|^2 dsdt\right] < +\infty$$

then

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s$$
$$+ \int_0^t \left[\alpha_1(s)\mathbb{D}_s^{H_1} Y_s + \alpha_2(s)\mathbb{D}_s^{H_1} X_s + \vartheta_1(s)\mathbb{D}_s^{H_2} Y_s + \vartheta_2(s)\mathbb{D}_s^{H_2} X_s\right] ds,$$

which may be written formally as

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + \left[\alpha_1(t)\mathbb{D}_t^{H_1} Y_t + \alpha_2(t)\mathbb{D}_t^{H_1} X_t + \vartheta_1(t)\mathbb{D}_t^{H_2} Y_t + \vartheta_2(t)\mathbb{D}_t^{H_2} X_t\right] dt.$$

3 Fractional BSDEs

3.1 Definitions and notations

Let $T > 0$ be fixed throughout this paper. Let $\left\{B_{1,t}^{H_1}\right\}_{t\in[0,T]}$ and $\left\{B_{2,t}^{H_2}\right\}_{t\in[0,T]}$ be two mutually independent fractional Brownian motions processes, with respectively $H_1 \geq \frac{1}{2}$ and $H_2 \geq \frac{1}{2}$, defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Let N denote the class of P-null sets of $\mathscr{F}$.
we define

$$\mathscr{F}_s = \mathscr{F}_{0,s}^{B_1^{H_1}} \vee \mathscr{F}_{s,T}^{B_2^{H_2}}, \quad s \in [0,T],$$

where for any process $\{\psi_t\}_{t\geq 0}$, $\mathscr{F}_{s,t}^{\psi} = \sigma\{\psi_r - \psi_s,\ s \leq r \leq t\} \vee \mathscr{N}$.

For every $\mathscr{F}$-adapted random process $\alpha = (\alpha(t))_{t\geq 0}$ with positive values, we define an increasing process $(A(t))_{t\geq 0}$ by setting $A(t) = \int_0^t \alpha^2(s)ds$.

For a fixed $\beta > 0$, we will use the following sets:

- $\mathscr{C}_{\text{pol}}^{1,2}([0,T]\times\mathbf{R})$ is the space of all $\mathscr{C}^{1,2}$-functions over $[0,T]\times\mathbf{R}$, which together with their derivative is of polynomial growth.
- $\mathscr{L}^2(\beta, \mathscr{F}_t, \mathbf{R}) = \left\{\xi : \Omega \to \mathbf{R} \mid \xi \text{ is } \mathscr{F}_t - \text{measurable}, \mathbf{E}[e^{\beta A(T)}|\xi|^2] < +\infty\right\}$,
- $\mathscr{V}_{[0,T]} = \left\{Y = \psi(\cdot,\eta) : \ \psi \in \mathscr{C}^{1,2}([0,T]\times\mathbf{R}), \frac{\partial\psi}{\partial t} \text{ is bounded}, \ t \in [0,T]\right\}$,
- $\widetilde{\mathscr{V}}_{[0,T]}^{\beta}$ and $\widetilde{\mathscr{V}}_{[0,T]}^{a,\beta}$ are the completion of $\mathscr{V}_{[0,T]}$ under the following norm

$$\|Y\|_{\alpha,\beta} = \left(\mathbf{E}\int_0^T \alpha^2(t)e^{\beta A(t)}|Y_t|^2 dt\right)^{1/2}, \quad \|Z\|_{\beta} = \left(\mathbf{E}\int_0^T e^{\beta A(t)}|Z_t|^2 dt\right)^{1/2}$$

- $\mathscr{B}^2([0,T],\mathbf{R}) = \widetilde{\mathscr{V}}_{[0,T]}^{\alpha,\beta} \times \widetilde{\mathscr{V}}_{[0,T]}^{\beta} \times \widetilde{\mathscr{V}}_{[0,T]}^{\beta}$ is a Banach space with the norm

$$\|(Y,Z_1,Z_2)\|_{\mathscr{B}}^2 = \|Y\|_{\alpha,\beta}^2 + \|Z_1\|_{\beta}^2 + \|Z_2\|_{\beta}^2.$$

Let us consider

$$\eta_t = \eta_0 + \int_0^t b(s)ds + \int_0^t \sigma_1(s)dB_{1,s}^{H_1} + \int_0^t \sigma_2(s)dB_{2,s}^{H_2}, \quad 0 \leq t \leq T$$

where the coefficients η_0, b, σ_1 and σ_2 satisfy:

- η_0 is a given constant and $b : [0,T] \to \mathbf{R}$ is a deterministic continuous function,
- $\sigma_1, \sigma_2 : [0,T] \to \mathbf{R}$ are deterministic differentiable continuous functions, and $\sigma_1(t) \neq 0$, and $\sigma_2(t) \neq 0$ such that

$$|\sigma|_t^2 = \|\sigma_1\|_t^2 + \|\sigma_2\|_t^2, \quad 0 \leq t \leq T, \tag{3.1}$$

where $$\|\sigma_i\|_t^2 = H_i(2H_i - 1) \int_0^t \int_0^t |u-v|^{2H_i-2} \sigma_i(u)\sigma_i(v) du dv, \quad i = 1, 2.$$

The next Remark will be useful in the sequel.

Remark 3.1. *There exists a constant* $C_0 \in (0,1)$ *such that* $\inf_{0 \leq t \leq T} \frac{\hat{\sigma}_i(t)}{\sigma_i(t)} \geq C_0$ *where* $\hat{\sigma}_i(t) = \int_0^t \phi(t,v)\sigma_i(v) dv \quad i = 1, 2.$

We are interested in the following one-dimensional fractional BSDE:

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_{1,s}, Z_{2,s}) ds - \int_t^T Z_{1,s} dB_{1,s}^{H_1} - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0,T]. \tag{3.2}$$

Definition 3.2. *A triplet of processes* (Y, Z_1, Z_2) *is called a solution to fractional BSDE* (3.2)*, if* $(Y, Z_1, Z_2) \in \mathscr{B}^2([0,T], \mathbf{R})$ *and satisfies eq.*(3.2)*.*

The next proposition will be useful in the sequel.

Proposition 3.3. *Let* (Y, Z_1, Z_2) *be a solution of the fractional BSDE* (3.2)*. Then for almost* $t \in [0,T]$*, we have*

$$\mathbb{D}_t^{H_1} Y_t = \frac{\hat{\sigma}_1(t)}{\sigma_1(t)} Z_{1,t} \quad \textit{and} \quad \mathbb{D}_t^{H_2} Y_t = \frac{\hat{\sigma}_2(t)}{\sigma_2(t)} Z_{2,t}.$$

3.2 The case of stochastic Lipschitz coefficient

3.2.1 Aassumptions

In the following, we assume that f satisfies assumptions **(H1)**:

(H1.1): There exist three non-negative processes $\{\mu(t)\}_{0 \leq t \leq T}$, $\{\nu(t)\}_{0 \leq t \leq T}$ and $\{\vartheta(t)\}_{0 \leq t \leq T}$ such that:

- **i)** for any $t \in [0,T]$, $\mu(t)$, $\nu(t)$, $\vartheta(t)$ are $\mathscr{F}_t$-measurable,
- **ii)** for any $t \in [0,T]$, $x, y, y', z_1, z_1', z_2, z_2' \in \mathbf{R}$, we have

$$\left| f(t,x,y,z_1,z_2) - f\left(t,x,y',z_1',z_2'\right) \right| \leq \mu(t)|y-y'| + \nu(t)|z_1 - z_1'| + \vartheta(t)|z_2 - z_2'|.$$

(H1.2): for any $t \in [0,T]$, $\alpha^2(t) = \mu(t) + \nu^2(t) + \vartheta^2(t) > 0$.

3.2.2 Existence and uniqueness of the solution

The main result of this section is the following theorem:

Theorem 3.4. *Let the assumptions* **(H1)** *be satisfied. Then the fractional BSDE* (3.2) *has a unique solution* (Y, Z_1, Z_2) *in the space* $\mathscr{B}^2([0,T], \mathbf{R})$*.*

Proof. Let us consider the mapping $\Gamma : \mathscr{B}^2([0,T], \mathbf{R}) \to \mathscr{B}^2([0,T], \mathbf{R})$ driven by $(U, V_1, V_2) \longmapsto \Gamma(U, V_1, V_2) = (Y, Z_1, Z_2)$.

We will show that the mapping Γ is a contraction, where (Y, Z_1, Z_2) is a solution of the following fractional BSDE:

$$Y_t = \int_t^T f(s, \eta_s, U_s, V_{1,s}, V_{2,s}) ds - \int_t^T Z_{1,s} dB_{1,s}^{H_1} - \int_t^T Z_{2,s} dB_{2,s}^{H_2}, \quad t \in [0,T]. \tag{3.3}$$

Let us define for a process $\delta \in \{Y, Z_1, Z_2, U, V_1, V_2\}$, $\overline{\delta} = \delta - \delta'$ where $\delta' \in \widetilde{\mathscr{V}}^{\beta}_{[0,T]}$ and the function

$$\Delta f(t) = f(t, \eta_t, U_t, V_{1,t}, V_{2,t}) - f(t, \eta_t, U'_t, V'_{1,t}, V'_{2,t}).$$

Then, the triplet $(\overline{Y}, \overline{Z}_1, \overline{Z}_2)$ solves the fractional BSDE

$$\overline{Y}_t = \int_t^T \Delta f(s) ds - \int_t^T \overline{Z}_{1,s} dB^{H_1}_{1,s} - \int_t^T \overline{Z}_{2,s} dB^{H_2}_{2,s}, \quad t \in [0,T]. \tag{3.4}$$

By the fractional Itô chain rule, we have

$$\begin{aligned} |\overline{Y}_t|^2 &= 2\int_t^T \overline{Y}_s \Delta f(s) ds - 2\int_t^T \overline{Z}_{1,s} \mathbb{D}^{H_1}_s \overline{Y}_s ds - 2\int_t^T \overline{Z}_{2,s} \mathbb{D}^{H_2}_s \overline{Y}_s ds \\ &\quad - 2\int_t^T \overline{Y}_s \overline{Z}_{1,s} dB^{H_1}_{1,s} - 2\int_t^T \overline{Y}_s \overline{Z}_{2,s} dB^{H_2}_{2,s}. \end{aligned}$$

Applying the Itô formula to $e^{\beta A(t)}|\overline{Y}_t|^2$, we obtain that

$$\begin{aligned} e^{\beta A(t)}|\overline{Y}_t|^2 &= 2\int_t^T e^{\beta A(s)} \overline{Y}_s \Delta f(s) ds - 2\int_t^T e^{\beta A(s)} \overline{Z}_{1,s} \mathbb{D}^{H_1}_s \overline{Y}_s ds - 2\int_t^T e^{\beta A(s)} \overline{Z}_{2,s} \mathbb{D}^{H_2}_s \overline{Y}_s ds \\ &\quad - 2\int_t^T e^{\beta A(s)} \overline{Y}_s \overline{Z}_{1,s} dB^{H_1}_{1,s} - 2\int_t^T e^{\beta A(s)} \overline{Y}_s \overline{Z}_{2,s} dB^{H_2}_{2,s} - \beta \int_t^T e^{\beta A(s)} \alpha^2(s) |\overline{Y}_s|^2 ds. \end{aligned}$$

It is known that, by Proposition 3.3, $\mathbb{D}^{H_1}_s \overline{Y}_s = \frac{\hat{\sigma}_1(s)}{\sigma_1(s)} \overline{Z}_{1,s}$ and $\mathbb{D}^{H_2}_s \overline{Y}_s = \frac{\hat{\sigma}_2(s)}{\sigma_2(s)} \overline{Z}_{2,s}$.
Then, we have

$$\begin{aligned} &\mathbf{E}\left[e^{\beta A(t)}|\overline{Y}_t|^2\right] + \beta \mathbf{E}\int_t^T e^{\beta A(s)} \alpha^2(s) |\overline{Y}_s|^2 ds + 2\mathbf{E}\int_t^T e^{\beta A(s)} \left[\frac{\hat{\sigma}_1(s)}{\sigma_1(s)} |\overline{Z}_{1,s}|^2 + \frac{\hat{\sigma}_2(s)}{\sigma_2(s)} |\overline{Z}_{2,s}|^2\right] ds \\ &\quad = 2\mathbf{E}\int_t^T e^{\beta A(s)} \overline{Y}_s \Delta f(s) ds. \end{aligned} \tag{3.5}$$

Using standard estimates and assumption **(H1.1)**, we obtain that

$$\begin{aligned} 2\mathbf{E}\int_t^T e^{\beta A(s)} \overline{Y}_s \Delta f(s) ds &\le 2\mathbf{E}\int_t^T e^{\beta A(s)} |\overline{Y}_s| \left[\mu(s)|\overline{U}_s| + \nu(s)|\overline{V}_{1,s}| + \vartheta(s)|\overline{V}_{2,s}|\right] ds \\ &\le \frac{1}{C_0} \mathbf{E}\int_t^T e^{\beta A(s)} \left[\mu(s) + \nu^2(s) + \vartheta^2(s)\right] |\overline{Y}_s|^2 ds \\ &\quad + C_0 \mathbf{E}\int_t^T e^{\beta A(s)} \mu(s) |\overline{U}_s|^2 ds + C_0 \mathbf{E}\int_t^T e^{\beta A(s)} \left[|\overline{V}_{1,s}|^2 + |\overline{V}_{2,s}|^2\right] ds, \end{aligned}$$

and using in addition asumption **(H1.2)**,

$$\begin{aligned} 2\mathbf{E}\int_t^T e^{\beta A(s)} \overline{Y}_s \Delta f(s) ds &\le C_0 \mathbf{E}\int_t^T e^{\beta A(s)} \left[\alpha^2(s)|\overline{U}_s|^2 + |\overline{V}_{1,s}|^2 + |\overline{V}_{2,s}|^2\right] ds \\ &\quad + \frac{1}{C_0} \mathbf{E}\int_t^T e^{\beta A(s)} \alpha^2(s) |\overline{Y}_s|^2 ds \end{aligned} \tag{3.6}$$

Using the abovementioned inequality, from (3.5) we deduce that

$$\begin{aligned} &\mathbf{E}\left[e^{\beta A(t)}|\overline{Y}_t|^2\right] + \beta \mathbf{E}\int_t^T e^{\beta A(s)} \alpha^2(s) |\overline{Y}_s|^2 ds + 2\mathbf{E}\int_t^T e^{\beta A(s)} \left[\frac{\hat{\sigma}_1(s)}{\sigma_1(s)} |\overline{Z}_{1,s}|^2 + \frac{\hat{\sigma}_2(s)}{\sigma_2(s)} |\overline{Z}_{2,s}|^2\right] ds \\ &\quad \le C_0 \mathbf{E}\int_t^T e^{\beta A(s)} \left[\alpha^2(s)|\overline{U}_s|^2 + |\overline{V}_{1,s}|^2 + |\overline{V}_{2,s}|^2\right] ds \\ &\quad + \frac{1}{C_0} \mathbf{E}\int_t^T e^{\beta A(s)} \alpha^2(s) |\overline{Y}_s|^2 ds \end{aligned} \tag{3.7}$$

By Remark 3.1, we obtain

$$\mathbf{E}\left[e^{\beta A(t)}|\overline{Y}_t|^2\right]+\beta\mathbf{E}\int_t^T e^{\beta A(s)}\alpha^2(s)|\overline{Y}_s|^2ds+2C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[|\overline{Z}_{1,s}|^2+|\overline{Z}_{2,s}|^2\right]ds$$
$$\leq C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[\alpha^2(s)|\overline{U}_s|^2+|\overline{V}_{1,s}|^2+|\overline{V}_{2,s}|^2\right]ds$$
$$+\frac{1}{C_0}\mathbf{E}\int_t^T e^{\beta A(s)}\alpha^2(s)|\overline{Y}_s|^2ds \tag{3.8}$$

Taking β such that $\beta = 2C_0+\frac{1}{C_0}$, we get

$$\mathbf{E}\int_0^T e^{\beta A(s)}\left[\alpha^2(s)|\overline{Y}_s|^2+|\overline{Z}_{1,s}|^2+|\overline{Z}_{2,s}|^2\right]ds\leq\frac{1}{2}\mathbf{E}\int_0^T e^{\beta A(s)}\left[\alpha^2(s)|\overline{U}_s|^2+|\overline{V}_{1,s}|^2+|\overline{V}_{2,s}|^2\right]ds. \tag{3.9}$$

Thus, the mapping $(U,V_1,V_2)\longmapsto\Gamma(U,V_1,V_2)=(Y,Z_1,Z_2)$ determined by the fractional BSDE (3.2) is a strict contraction on $\mathscr{B}^2([0,T],\mathbf{R})$. Using the fixed point principle, we deduce the solution to the fractional BSDE (3.2) that exists uniquely. This completes the proof. □

3.3 The case of weak stochastic Lipschitz coefficient

3.3.1 Aassumptions

In the following, we assume that f satisfies assumptions **(H2)**:

(H2.1): There exist three non-negative processes $\{\mu(t)\}_{0\leq t\leq T}$, $\{\nu(t)\}_{0\leq t\leq T}$ and $\{\vartheta(t)\}_{0\leq t\leq T}$ such that:

i) for any $t\in[0,T]$, $\mu(t)$, $\nu(t)$, $\vartheta(t)$ are $\mathscr{F}_t$-measurable,

ii) for any $t\in[0,T]$, $x,y,y',z_1,z_1',z_2,z_2'\in\mathbf{R}$, we have

$$\left|f(t,x,y,z_1,z_2)-f\left(t,x,y',z_1',z_2'\right)\right|\leq\mu^{\frac{1}{2}}(t)\rho^{\frac{1}{2}}\left(t,|y-y'|^2\right)$$
$$+\nu(t)|z_1-z_1'|+\vartheta(t)|z_2-z_2'|,$$

where $\rho(t,\nu):[0,T]\times\mathbf{R}_+\to\mathbf{R}_+$ satisfies:

- For fixed $t\in[0,T]$, $\rho(t,\cdot)$ is a continuous, concave and nondecreasing such that

$$\rho(t,0)=0,\qquad\text{and}\quad\forall\alpha>0\quad\alpha\rho(t,\nu)=\rho(t,\alpha\nu).$$

- The ordinary differential equation (ODE)

$$\nu'(t)=-\rho(t,\nu(t)),\quad v(T)=0, \tag{3.10}$$

has a unique solution $\nu(t)=0,\quad 0\leq t\leq T$.

- There exist two continuous and non-negative functions a and b such that

$$\rho(t,\nu)\leq a(t)+b(t)\nu\quad\text{and}\quad\int_0^T[a(t)+b(t)]dt<\infty.$$

(H2.2): for any $t\in[0,T]$, $\alpha^2(t)=\mu(t)+\nu^2(t)+\vartheta^2(t)>0$.

(H2.3): The integrability condition holds:

$$\mathbf{E}\int_0^T e^{\beta A(t)}\frac{|f(t,\eta_t,0,0)|^2}{\alpha^2(t)}dt<+\infty,\quad t\in[0,T].$$

3.3.2 Existence and uniqueness of the solution

For $n \geq 1$, we can construct the Picard approximate sequence of eq.(3.2) as follows

$$\begin{cases} Y_t^0 = 0, \quad Z_{t,1}^0 = 0, \quad Z_{t,2}^0 = 0 \quad t \in [0,T] \\ Y_t^n = \xi + \int_t^T f(s,\eta_s,Y_s^{n-1},Z_{1,s}^n,Z_{2,s}^n)ds - \int_t^T Z_{1,s}^n dB_{1,s}^{H_1} - \int_t^T Z_{2,s}^n dB_{2,s}^{H_2}, \quad t \in [0,T]. \end{cases} \tag{3.11}$$

Thanks to Theorem 3.4, this sequence is well defined.

Lemma 3.5. *Assume that assumptions* **(H2)** *are true. Then for all $n,m \geq 1$ and $t \in [0,T]$, we have*

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^{n+m} - Y_t^n|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta A(s)}|Y_s^{n+m-1} - Y_s^{n-1}|^2\right]\right)ds.$$

Proof. Let us define for a process $\delta \in \{Y,Z_1,Z_2\}$, $n,m \geq 1$, $\overline{\delta}^{n,m} = \delta^{n+m} - \delta^n$ and the function $\Delta f^{(n,m)}(s) = f(s,\eta_s,Y_s^{n+m-1},Z_{1,s}^{n+m},Z_{2,s}^{n+m}) - f(s,\eta_s,Y_s^{n-1},Z_{1,s}^n,Z_{2,s}^n)$.

Then, it is obvious that $(\overline{Y}^{n,m},\overline{Z}_1^{n,m},\overline{Z}_2^{n,m})$ solves the fractional BSDE

$$\overline{Y}_t^{n,m} = \int_t^T \Delta f^{(n,m)}(s)ds - \int_t^T \overline{Z}_{1,s}^{n,m} dB_{1,s}^{H_1} - \int_t^T \overline{Z}_{2,s}^{n,m} dB_{2,s}^{H_2}, \quad t \in [0,T]. \tag{3.12}$$

By the fractional Itô chain rule, we have

$$\begin{aligned} |\overline{Y}_t^{n,m}|^2 &= 2\int_t^T \overline{Y}_s^{n,m}\Delta f^{(n,m)}(s)ds - 2\int_t^T \overline{Z}_{1,s}^{n,m}\mathbb{D}_s^{H_1}\overline{Y}_s^{n,m}ds - 2\int_t^T \overline{Z}_{2,s}^{n,m}\mathbb{D}_s^{H_2}\overline{Y}_s^{n,m}ds \\ &\quad - 2\int_t^T \overline{Y}_s^{n,m}\overline{Z}_{1,s}^{n,m}dB_{1,s}^{H_1} - 2\int_t^T \overline{Y}_s^{n,m}\overline{Z}_{2,s}^{n,m}dB_{2,s}^{H_2}. \end{aligned}$$

Applying Itô formula to $e^{\beta A(t)}|\overline{Y}_t^{n,m}|^2$, we obtain that

$$\begin{aligned} e^{\beta A(t)}|\overline{Y}_t^{n,m}|^2 &= 2\int_t^T e^{\beta A(s)}\overline{Y}_s^{n,m}\Delta f^{(n,m)}(s)ds - 2\int_t^T e^{\beta A(s)}\overline{Z}_{1,s}^{n,m}\mathbb{D}_s^{H_1}\overline{Y}_s^{n,m}ds \\ &\quad - 2\int_t^T e^{\beta A(s)}\overline{Z}_{2,s}^{n,m}\mathbb{D}_s^{H_2}\overline{Y}_s^{n,m}ds - 2\int_t^T e^{\beta A(s)}\overline{Y}_s^{n,m}\overline{Z}_{1,s}^{n,m}dB_{1,s}^{H_1} \\ &\quad - 2\int_t^T e^{\beta A(s)}\overline{Y}_s^{n,m}\overline{Z}_{2,s}^{n,m}dB_{2,s}^{H_2} - \beta\int_t^T e^{\beta A(s)}\alpha^2(s)|\overline{Y}_s^{n,m}|^2ds. \end{aligned}$$

By Proposition 3.3, we have

$$\begin{aligned} &\mathbf{E}\left[e^{\beta A(t)}|\overline{Y}_t^{n,m}|^2\right] + \mathbf{E}\int_t^T e^{\beta A(s)}\left[\beta\alpha^2(s)|\overline{Y}_s^{n,m}|^2 + 2\frac{\hat{\sigma}_1(s)}{\sigma_1(s)}|\overline{Z}_{1,s}^{n,m}|^2 + 2\frac{\hat{\sigma}_2(s)}{\sigma_2(s)}|\overline{Z}_{2,s}^{n,m}|^2\right]ds \\ &= 2\mathbf{E}\int_t^T e^{\beta A(s)}\overline{Y}_s^{n,m}\Delta f^{(n,m)}(s)ds. \end{aligned} \tag{3.13}$$

Applying standard estimates and asumption **(H2.1)**, we obtain that

$$\begin{aligned} 2\mathbf{E}\int_t^T e^{\beta A(s)}\overline{Y}_s^{n,m}\Delta f^{(n,m)}(s)ds &\leq 2\mathbf{E}\int_t^T e^{\beta A(s)}|\overline{Y}_s^{n,m}|\mu^{\frac{1}{2}}(s)\rho^{\frac{1}{2}}(s,|Y_s^{n+m-1} - Y_s^{n-1}|^2)ds \\ &\quad + 2\mathbf{E}\int_t^T e^{\beta A(s)}|\overline{Y}_s^{n,m}|\left[\nu(s)|\overline{Z}_{1,s}^{n,m}| + \vartheta(s)|\overline{Z}_{2,s}^{n,m}|\right]ds \\ &\leq \frac{1}{C_0}\mathbf{E}\int_t^T e^{\beta A(s)}\left[\mu(s) + \nu^2(s) + \vartheta^2(s)\right]|\overline{Y}_s^{n,m}|ds \\ &\quad + C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[\rho(s,|Y_s^{n+m-1} - Y_s^{n-1}|^2) + |\overline{Z}_{1,s}^{n,m}|^2 + |\overline{Z}_{2,s}^{n,m}|^2\right]ds \end{aligned}$$

Therefore, we can write

$$2\mathbf{E}\int_t^T e^{\beta A(s)}\overline{Y}_s^{n,m}\Delta f^{(n,m)}(s)ds \le \frac{1}{C_0}\mathbf{E}\int_t^T e^{\beta A(s)}\alpha^2(s)|\overline{Y}_s^{n,m}|ds + C_0\mathbf{E}\int_t^T e^{\beta A(s)}|\overline{Z}_{1,s}^{n,m}|^2 ds$$
$$+C_0\mathbf{E}\int_t^T e^{\beta A(s)}|\overline{Z}_{2,s}^{n,m}|^2 ds + C_0\mathbf{E}\int_t^T e^{\beta A(s)}\rho(s,|Y_s^{n+m-1}-Y_s^{n-1}|^2)ds.$$

Using the abovementioned inequality and Remark 3.1, from (3.13), we deduce that

$$\mathbf{E}\left[e^{\beta A(t)}|\overline{Y}_t^{n,m}|^2\right] + \beta\mathbf{E}\int_t^T e^{\beta A(s)}\alpha^2(s)|\overline{Y}_s^{n,m}|^2 ds + 2C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[|\overline{Z}_{1,s}^{n,m}|^2 + |\overline{Z}_{2,s}^{n,m}|^2\right]ds$$
$$\le \frac{1}{C_0}\mathbf{E}\int_t^T e^{\beta A(s)}\alpha^2(s)|\overline{Y}_s^{n,m}|ds + C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[|\overline{Z}_{1,s}^{n,m}|^2 + |\overline{Z}_{2,s}^{n,m}|^2\right]ds$$
$$+C_0\mathbf{E}\int_t^T e^{\beta A(s)}\rho(s,|Y_s^{n+m-1}-Y_s^{n-1}|^2)ds.$$

Choosing β such that $\beta > \frac{1}{C_0}$, we have

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^{n+m}-Y_t^n|^2\right] \le C_0\int_t^T \rho\left(s,\mathbf{E}\left[e^{\beta A(s)}|Y_s^{n+m-1}-Y_s^{n-1}|^2\right]\right)ds.$$

Finally, by Remark 3.1 we obtain

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^{n+m}-Y_t^n|^2\right] \le \int_t^T \rho\left(s,\mathbf{E}\left[e^{\beta A(s)}|Y_s^{n+m-1}-Y_s^{n-1}|^2\right]\right)ds.$$

□

Lemma 3.6. *Let the assumption* **(H2)** *be satisfied. Then there exists a constant $M \ge 0$ and $T_1 \in [0,T]$ such that*

$$\forall n \ge 1, \quad \mathbf{E}\left[e^{\beta A(t)}|Y_t^n|^2\right] \le M, \qquad t \in [T_1,T].$$

Proof. Using the same method as in the proof of Lemma 3.5, we obtain that

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^n|^2\right] + \beta\mathbf{E}\int_t^T e^{\beta A(s)}|Y_s^n|^2 ds + 2C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[|Z_{1,s}^n|^2 + |Z_{2,s}^n|^2\right]ds$$
$$\le \mathbf{E}\left[e^{\beta A(T)}|\xi|^2\right] + 2\mathbf{E}\int_t^T e^{\beta A(s)}Y_s^n f(s,\eta_s,Y_s^{n-1},Z_{1,s}^n,Z_{2,s}^n)ds. \tag{3.14}$$

Applying standard estimates and asumption **(H2.1)**, we obtain that

$$2\mathbf{E}\int_t^T e^{\beta A(s)}Y_s^n f(s,\eta_s,Y_s^{n-1},Z_{1,s}^n,Z_{2,s}^n)ds \le C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[\rho(s,|Y_s^{n-1}|^2) + |\overline{Z}_{1,s}^{n,m}|^2 + |Z_{2,s}^n|^2\right]ds$$
$$+\mathbf{E}\int_t^T e^{\beta A(s)}\frac{|f(s,\eta_s,0,0,0)|^2}{\alpha^2(s)}ds + \left(1+\frac{1}{C_0}\right)\mathbf{E}\int_t^T e^{\beta A(s)}\alpha^2(s)|Y_s^n|ds$$

Using the abovementioned inequality, from (3.14) we deduce that

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^n|^2\right] + [\beta-(1+\frac{1}{C_0})]\mathbf{E}\int_t^T e^{\beta A(s)}|Y_s^n|^2 ds + C_0\mathbf{E}\int_t^T e^{\beta A(s)}\left[|Z_{1,s}^n|^2 + |Z_{2,s}^n|^2\right]ds$$
$$\le \Lambda_t + C_0\int_t^T \rho(s,\mathbf{E}\left[e^{\beta A(s)}|Y_s^{n-1}|^2\right])ds, \tag{3.15}$$

where $\Lambda_t = \mathbf{E}\left(e^{\beta A(T)}|\xi|^2 + \int_t^T e^{\beta A(s)}\frac{|f(s,\eta_s,0,0,0)|^2}{\alpha^2(s)}ds\right)$.

Choosing β such that $\beta - \frac{1}{C_0} > 1$ we have

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^n|^2\right] \leq \Lambda_t + \int_t^T \rho(s,\mathbf{E}\left[e^{\beta A(s)}|Y_s^{n-1}|^2\right])ds, \quad t\in[0,T]. \tag{3.16}$$

Finally let
$$M = 2\Lambda_0 + 2\int_0^T a(s)ds \geq 0. \tag{3.17}$$

Arguing as in [[9], Lemma 2], we choose T_1 such that

$$\Lambda_0 + \int_t^T \rho(s,M)ds \leq M, \qquad t\in[T_1,T]. \tag{3.18}$$

By inequality (3.16) and (3.18), for $t\in[T_1,T]$, we have

$$\begin{aligned}
\mathbf{E}\left[e^{\beta A(t)}\left|Y_t^1\right|^2\right] &\leq \Lambda_t + \int_t^T \rho(s,0)\,ds \leq \Lambda_0 \leq M,\\
\mathbf{E}\left[e^{\beta A(t)}\left|Y_t^2\right|^2\right] &\leq \Lambda_t + \int_t^T \rho\left(s,\mathbf{E}[e^{\beta A(s)}|Y_s^1|^2]\right)ds \leq \Lambda_0 + \int_t^T \rho(s,M)\,ds \leq M,\\
\mathbf{E}\left[e^{\beta A(t)}\left|Y_t^3\right|^2\right] &\leq \Lambda_t + \int_t^T \rho\left(s,\mathbf{E}[e^{\beta A(s)}|Y_s^2|^2]\right)ds \leq \Lambda_0 + \int_t^T \rho(s,M)\,ds \leq M.
\end{aligned}$$

Hence, by induction, one can prove that for all $n\geq 1$,

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^n|^2\right] \leq M, \quad T_1 \leq t \leq T.$$

□

The main result of this section is the following theorem:

Theorem 3.7. *Let the assumptions* **(H2)** *be satisfied. Then, the fractional BSDE* (3.2) *has a unique solution* (Y,Z_1,Z_2) *in the space* $\mathscr{B}^2([0,T],\mathbf{R})$.

Proof. We split the proof into two parts.
(i) Existence: Using the constant M given by (3.17), we consider the sequence $(\varphi_n)_{n\geq 1}$ given by

$$\varphi_0(t) = \int_t^T \rho(s,M)\,ds, \qquad \varphi_{n+1}(t) = \int_t^T \rho(s,\varphi_n(s))\,ds, \quad n\geq 0, \quad t\in[T_1,T].$$

Then for all $t\in[T_1,T]$, from the proof of Lemma 3.6, one can deduce that

$$\begin{aligned}
\varphi_0(t) &= \int_t^T \rho(s,M)\,ds \leq M,\\
\varphi_1(t) &= \int_t^T \rho(s,\varphi_0(s))\,ds \leq \int_t^T \rho(s,M)\,ds = \varphi_0(t) \leq M,\\
\varphi_2(t) &= \int_t^T \rho(s,\varphi_1(s))\,ds \leq \int_t^T \rho(s,\varphi_0(s))\,ds = \varphi_1(t) \leq M.
\end{aligned}$$

By induction, one can prove that for all $n\geq 1$, $\varphi_n(t)$ satisfies

$$0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq \cdots \leq \varphi_1(t) \leq \varphi_0(t) \leq M.$$

Then $\{\varphi_n(t), t \in [T_1,T]\}_{n\geq 1}$ is uniformly bounded. On the other hand, for all $n \geq 1$ and $t_1, t_2 \in [T_1,T]$, we obtain

$$|\varphi_n(t_1) - \varphi_n(t_2)| = \left| \int_{t_1}^{t_2} \rho\left(s, \varphi_{n-1}(s)\right) ds \right| \leq \left| \int_{t_1}^{t_2} \rho\left(s, M\right) ds \right|.$$

Since, for fixed v, $\int_0^T \rho\left(s, v\right) ds < +\infty$. So

$$\sup_n |\varphi_n(t_1) - \varphi_n(t_2)| \to 0 \quad as \quad |t_1 - t_2| \to 0,$$

which means that $\{\varphi_n(t), t \in [T_1,T]\}_{n\geq 1}$ is an equicontinuous family of function. Therefore, by the Arzelá-Ascoli theorem, we can define by $\varphi(t)$ the limit function of $(\varphi_n(t))_{n\geq 1}$.
By (3.10), one knows that $\varphi(t) = 0$, $t \in [T_1,T]$.
Now for all $t \in [T_1,T]$, $n,m \geq 1$, in view of Lemmas 3.5 and 3.6, we have

$$\begin{aligned}
&\mathbf{E}\left[e^{\beta A(t)}|Y_t^n|^2\right] \leq M, \\
&\mathbf{E}\left[e^{\beta A(t)}\left|Y_t^{1+m} - Y_t^1\right|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}[e^{\beta A(s)}|Y_s^m|^2]\right) ds \leq \int_t^T \rho\left(s, M\right) ds = \varphi_0(t) \leq M, \\
&\mathbf{E}\left[e^{\beta A(t)}\left|Y_t^{2+m} - Y_t^2\right|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta A(s)}\left|Y_s^{1+m} - Y_s^1\right|^2\right]\right) ds \leq \varphi_1(t) \leq M, \\
&\mathbf{E}\left[e^{\beta A(t)}\left|Y_t^{3+m} - Y_t^3\right|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta A(s)}\left|Y_s^{2+m} - Y_s^2\right|^2\right]\right) ds \leq \varphi_2(t) \leq M.
\end{aligned}$$

By induction, we can derive that

$$m \geq 1, \quad \mathbf{E}\left[e^{\beta A(t)}\left|Y_t^{n+m} - Y_t^n\right|^2\right] \leq \varphi_{n-1}(t), \quad t \in [T_1,T].$$

Therefore we have

$$\sup_{t\in[T_1,T]} \mathbf{E}\left[e^{\beta A(t)}\left|Y_t^{n+m} - Y_t^n\right|^2\right] \leq \sup_{t\in[T_1,T]} \varphi_{n-1}(t) = \varphi_{n-1}(T_1) \to 0 \quad n \to \infty.$$

Exploiting the argument developed in [[1], Theorem 3.9] we prove that the sequence (Y^n, Z_1^n, Z_2^n) is a Cauchy sequence in $\mathscr{B}^2([T_1,T],\mathbf{R})$. Letting $n \to +\infty$ in eq.(3.11), we obtain

$$Y_t = \xi + \int_t^T f(s,\eta_s,Y_s,Z_{1,s},Z_{2,s})ds - \int_t^T Z_{1,s}dB_{1,s}^{H_1} - \int_t^T Z_{2,s}dB_{2,s}^{H_2}, \quad T_1 \leq t \leq T.$$

In other words, we have shown the existence of the solution (Y,Z_1,Z_2) to fractional BSDE (3.2) on $[T_1,T]$. Finally, by iteration, one can deduce the existence on $[T-\lambda(T-T_1),T]$, for each λ, and therefore the existence on the whole $[0,T]$.

(ii) Uniqueness:
Let $\left(Y_t^i, Z_{1,t}^i, Z_{2,t}^i\right)_{0\leq t\leq T}$, $i = 1,2$, be two solutions of fractional BSDE (3.2).
Using the same method as in the proof of Lemma (3.5), we have

$$\begin{aligned}
\mathbf{E}\left[e^{\beta A(t)}|Y_t^1 - Y_t^2|^2\right] + C_0\mathbf{E}\int_t^T e^{\beta A(s)}|Z_{1,s}^1 - Z_{1,s}^2|^2 ds + C_0\mathbf{E}\int_t^T e^{\beta A(s)}|Z_{2,s}^1 - Z_{2,s}^2|^2 ds \\
\leq \int_t^T \rho(s, \mathbf{E}\left[e^{\beta A(s)}|Y_s^1 - Y_s^2|^2\right])ds, \quad t \in [0,T]. \qquad (3.19)
\end{aligned}$$

Therefore

$$\mathbf{E}\left[e^{\beta A(t)}|Y_t^1 - Y_t^2|^2\right] \leq \int_t^T \rho\left(s, \mathbf{E}\left[e^{\beta A(s)}|Y_s^1 - Y_s^2|^2\right]\right) ds, \quad t \in [0,T] \tag{3.20}$$

From the comparison theorem of ODE, we know that $\mathbf{E}\left[e^{\beta t}|Y_t^1 - Y_t^2|^2\right] \leq r(t)$, where $r(t)$ is the maximum of solution of (3.10) on $[0,T]$. As a consequence, we have $Y_t^1 = Y_t^2$ for $t \in [0,T]$. From (3.19), we deduce $\left(Z_{1,t}^1, Z_{2,t}^1\right) = \left(Z_{1,t}^2, Z_{2,t}^2\right)$ for $t \in [0,T]$. This completes the proof. □

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