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Solution of two point boundary value problems, a numerical approach: parametric difference method

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Abstract

In this article, we have presented a parametric finite difference method, a numerical technique for the solution of two point boundary value problems in ordinary differential equations with mixed boundary conditions. We have tested proposed method for the numerical solution of a model problem. The numerical results obtained for the model problem with constructed exact solution depends on the choice of parameters. The computed result of a model problem suggests that proposed method is efficient.

Keywords: Boundary Value Problem; Energy Equation; Mixed Boundary Condition; Parametric Difference Method.

AMS 2010 codes: 65L10, 65L12.

1 Introduction

There has been a great deal of interest in developing techniques for the accurate numerical solution of two-point boundary value problems. In this article we have considered a parametric difference method for the numerical solution of the boundary value problems with the form

$$u''(x) = g(x, u, u'), \quad a < x < b \quad (1)$$

subject to the boundary conditions

$$u'(a) = \alpha \quad \text{and} \quad p_1 u(b) + q_1 u'(b) = \beta,$$

or

$$p_0 u(a) + q_0 u'(a) = \alpha_0 \quad \text{and} \quad u'(b) = \beta_0,$$

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where α, β, p_0 etc. are real constants and known. We assume that g is continuous function in $([a, b] \times \mathfrak{R}^2, \mathfrak{R})$.

In many branches of engineering, humanity and sciences ordinary differential equations occur as mathematical models. In general, these equations have solutions that can be expressed in restricted form only. So it is essential to consider approximate solutions by means of numerical techniques. To solve these problems numerically, we have many accurate numerical methods for instance, shooting-projection [1], collocation [2], finite difference methods [3] available in the literature. However the relatively less advanced method, the method of parameter differentiation is known in the literature and some historical development can be found in [4]. The existence and uniqueness of the solution for the problem (1) are assumed. The specific assumption on $g(x, u, u')$ to ensure the existence and uniqueness will not be considered in this article however I may refer literature in [5–8].

In this article we shall develop a hybrid parametric difference method for the approximate numerical solution of problems (1). We have performed a numerical experiment to demonstrate the effectiveness of the method. We have achieved accuracy and bound on the error between the analytical and numerical approximation solution of the problem. To the best of our knowledge, no method similar to the proposed method for the numerical approximate solution has been discussed in the literature to date.

We have presented our work in this article as follows. In the next section we will discuss the parametric difference method and in Section 3 we will derive our propose method. In Section 4 and 5, respectively, we have estimated truncation error and discussed convergence under appropriate condition. The applications of the proposed method to the model problems and numerical results have been produced to show the efficiency in Section 6. Discussion and conclusion on the performance of the method are presented in Section 7.

2 The Parametric Difference Method

Following the ideas in [4], re-formulate the problem (1) either by introducing a physical parameter or identify a physical parameter that appears either in the differential equation or in the boundary conditions as independent variable. Let us introduce λ as physical parameter in differential equation (1), so we have obtained,

$$u''(x) = g(x, u, u') \equiv f(x, \lambda, u, u'), \quad a < x < b \quad (2)$$

and boundary conditions are

$$u'(a) = \alpha \quad \text{and} \quad p_1 u(b) + q_1 u'(b) = \beta,$$

or

$$p_0 u(a) + q_0 u'(a) = \alpha_0 \quad \text{and} \quad u'(b) = \beta_0,$$

Now let introduce new variable $y(x) = \dot{u} = \frac{du}{d\lambda}$, a differentiation of solution of problem (1) with respect to λ . Differentiate problem (2) with respect to λ . Thus we have reduced the problem (2) into following system of equations,

$$\begin{aligned} y'' &= F\left(\frac{\partial f}{\partial \lambda}, y, y'\right), \\ \dot{u} &= y(x) \end{aligned} \quad (3)$$

subject to the boundary conditions,

$$y'(a) = 0 \quad \text{and} \quad p_1 y(b) + q_1 y'(b) = 0,$$

or

$$p_0 y(a) + q_0 y'(a) = 0 \quad \text{and} \quad y'(b) = 0,$$

and $y(x)$ is known for some λ .

We wish to determine the approximate numerical solution $u(x)$ of the problem (2) for different values of λ and $\lambda_0 \leq \lambda \leq \lambda_M$. We define $M - 1$ numbers of nodal points in $[\lambda_0, \lambda_M]$, as $\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_M$ using a uniform step length H such that $\lambda_i = \lambda_0 + jH$, $j = 0, 1, \dots, M$. Also, we define $N - 1$ numbers of nodal points in $[a, b]$, the domain in which the solution of the problem (1) is desired, as $a \leq x_0 < x_1 < x_2 < \dots < x_N = b$ using a uniform step length h such that $x_i = x_0 + ih$, $i = 0, 1, \dots, N$. To simplify the expression we denote the numerical approximation of $u(x)$ at the node $x = x_i$ as u_i . Let us denote F_i as the approximation of the theoretical value of the source function $F(\frac{\partial f}{\partial \lambda}, y(x), y'(x))$ at node $x = x_i$, $i = 0, 1, 2, \dots, N$. Let write a system of equations (3) at nodal point x_i in notational form as,

$$\begin{aligned} y_i'' &= F_i \\ \dot{u}_i &= y_i \end{aligned}$$

subject to the boundary conditions

$$y_0' = 0 \quad \text{and} \quad p_1 y_N + q_1 y_N' = 0,$$

or

$$p_0 y_0 + q_0 y_0' = 0 \quad \text{and} \quad y'(b) = 0,$$

Following the idea in [9], we propose our difference method for the numerical solution of problem (3) as follows,

$$\begin{aligned} y_i - y_{i-1} &= h y_{i-1}' + \frac{h^2}{6} (2F_{i-1} + F_i), & i = 1, 2, \dots, N, \\ y_i' - y_{i-1}' &= \frac{h}{2} (F_i + F_{i-1}), & i = 1, 2, \dots, N, \\ u_{j-1} - u_j &= -H(y_{i-1} + ((\lambda_0 - a) + ((j - 1)H - (i - 1)h))y_{i-1}'), & j = 1, 2, \dots, M. \end{aligned} \tag{4}$$

which is a system of linear/ nonlinear equations depending on forcing function $g(x, u, u')$. We solve system of equations (4) by applying an appropriate iterative method.

3 Derivation of the Difference Method

In this section we outline the derivation and development of the proposed finite difference method. We approximate first problem in (3) as a linear combination of $y(x), y'(x)$ and $F(\frac{\partial f}{\partial \lambda}, y, y')$,

$$a_0 y_{i-1} + a_1 y_i + a_2 h y_{i-1}' + a_3 h y_i' + h^2 (b_0 F_{i-1} + b_1 F_i) = 0 \tag{5}$$

where a_0, \dots, b_1 are constant. To determine these constants, we expand each term in the above expression in a Taylor series about a node x_i and compare the coefficients of $h^p, p = 0, \dots, 3$ in the both sides of the expression. So we obtained a system of linear equations in a_0, \dots, b_1 . Solving the system of equations, we got

$$(a_0, a_1, a_2, a_3, b_0, b_1) = (-1, 1, -1, 0, -\frac{1}{3}, -\frac{1}{6}) \tag{6}$$

Substituting these constants from (6) in (5), we obtained

$$y_i - y_{i-1} - h y_{i-1}' - \frac{h^2}{6} (2F_{i-1} + F_i) = 0 \tag{7}$$

Following the similar idea as above, we derive following equations,

$$\begin{aligned} y_i' - y_{i-1}' &= \frac{h}{2} (F_i + F_{i-1}), \\ u_{j-1} - u_j &= -H(y_{i-1} + ((\lambda_0 - a) + ((j - 1)H - (i - 1)h))y_{i-1}') \end{aligned} \tag{8}$$

4 Local Truncation Error

The local truncation error in the difference method (4) using the exact arithmetic may be calculated by using Taylor series expansion at the nodal points $x = x_i$, $i = 1, 2, \dots, N$. Thus the truncation error T_i in the first equation of the method (4) may be written as :

$$T_{i-1} = -\frac{h^4}{24}y_i^{(4)}, \quad 1 \leq i \leq N. \quad (9)$$

Similarly we can obtain truncation error in second equation of method (4) at $x = x_i$, $i = 1, 2, \dots, N$ as given below,

$$T'_i = -\frac{h^3}{3}y_i^{(4)}, \quad 1 \leq i \leq N. \quad (10)$$

The truncation error in final equation of method (4) at λ_{j-1} , $j = 1, 2, \dots, M$ and x_{i-1} , $i = 1, 2, \dots, N$ is,

$$T_* = \frac{H}{2}\ddot{u}_{j-1} - \frac{((\lambda_0 - a) + ((j-1)H - (i-1)h))^2}{2}y''_{i-1}. \quad (11)$$

Thus we have obtained a truncation error at each node of order $O(H + h^2)$ if $\lambda_0 = a$.

5 Convergence of the Difference Method

Let us assume the following conditions [10],

- (i) $u(x)$ and $u'(x)$ are finite
- (ii) $g(x, u(x), u'(x))$ is continuous in $([a, b] \times \mathfrak{R}^2, \mathfrak{R})$
- (iii) $\frac{\partial g}{\partial u}$ and $\frac{\partial g}{\partial u'}$ exist and are continuous in $[a, b]$
- (iv) $\frac{\partial g}{\partial u} > 0$ and $|\frac{\partial g}{\partial u'}| < w$ in $[a, b]$ for some positive real number w .

Under above conditions problem (1) will possess unique solution [7]. However the forcing function $F(\frac{\partial f}{\partial \lambda}, y, y')$ in (3) may be written as,

$$F(\frac{\partial f}{\partial \lambda}, y, y') \equiv \frac{\partial f}{\partial u}y + \frac{\partial f}{\partial u'}y' + \frac{\partial f}{\partial \lambda} \quad (12)$$

Let us assume $M_i = (\frac{\partial f}{\partial y})_i$, $M'_i = (\frac{\partial f}{\partial y'})_i$ and using (12), we can write difference method (4) as follows,

$$\begin{aligned} & -\left(1 + \frac{h^2}{3}M_{i-1}\right)y_{i-1} + \left(1 - \frac{h^2}{6}M_i\right)y_i - \frac{h^2}{6}M'_i y'_i - h\left(1 + \frac{h}{3}M'_{i-1}\right)y'_{i-1} = \frac{h^2}{6}\left(2\left(\frac{\partial f}{\partial \lambda}\right)_{i-1} + \left(\frac{\partial f}{\partial \lambda}\right)_i\right), \\ & -\frac{h}{2}M_{i-1}y_{i-1} - \frac{h}{2}M_i y_i + \left(1 - \frac{h}{2}M'_i\right)y'_i - \left(1 - \frac{h}{2}M'_{i-1}\right)y'_{i-1} = \frac{h}{2}\left(\left(\frac{\partial f}{\partial \lambda}\right)_{i-1} + \left(\frac{\partial f}{\partial \lambda}\right)_i\right), \quad i = 1, 2, \dots, N \end{aligned} \quad (13)$$

Let Y_i and y_i be respectively an exact solution and approximate solution of the system of equations (3). Let us define $\varepsilon_i = y_i - Y_i$, an error between the exact analytical and numerical approximation solution of the problem (3) and similarly we can define $\varepsilon'_i = y'_i - Y'_i$. Let us write (13) in matrix form

$$\mathbf{D}\boldsymbol{\varepsilon} + \mathbf{T} = \mathbf{0}, \quad (14)$$

where matrix \mathbf{D} is of order $2(N - 1) \times 2(N - 1)$, moreover we have partitioned the \mathbf{D} in the following manner,

$$\mathbf{D} = \begin{pmatrix} \mathbf{A}_{11} & \vdots & \mathbf{A}_{12} \\ \dots & \vdots & \dots \\ \mathbf{A}_{21} & \vdots & \mathbf{A}_{22} \end{pmatrix}$$

where each matrix \mathbf{A} is of order $(N - 1) \times (N - 1)$ and these matrices are,

$$\mathbf{A}_{11} = \begin{pmatrix} -1 - \frac{h^2}{3}M_0 & 1 - \frac{h^2}{6}M_1 & & 0 \\ & -1 - \frac{h^2}{3}M_1 & 1 - \frac{h^2}{6}M_2 & \\ & & \ddots & \ddots \\ & & & -1 - \frac{h^2}{3}M_{N-2} & 1 - \frac{h^2}{6}M_{N-1} \\ 0 & & & & -1 - \frac{h^2}{3}M_{N-1} \end{pmatrix}_{N-1 \times N-1}$$

an upper diagonal matrix with bandwidth of 1,

$$\mathbf{A}_{12} = \begin{pmatrix} -\frac{h^2}{6}M'_1 & & & 0 \\ -h - \frac{h^2}{3}M'_1 & -\frac{h^2}{6}M'_2 & & \\ \ddots & \ddots & & \\ & -h - \frac{h^2}{3}M'_{N-2} & -\frac{h^2}{6}M'_{N-1} & \\ 0 & & -h - \frac{h^2}{3}M'_{N-1} & -\frac{h^2}{6}M'_N \end{pmatrix}_{N-1 \times N-1}$$

a lower diagonal matrix with bandwidth of 1,

$$\mathbf{A}_{21} = \begin{pmatrix} -\frac{h}{2}M_0 & -\frac{h}{2}M_1 & & 0 \\ & -\frac{h}{2}M_1 & -\frac{h}{2}M_2 & \\ & & \ddots & \ddots \\ & & & -\frac{h}{2}M_{N-2} & -\frac{h}{2}M_{N-1} \\ 0 & & & & -\frac{h}{2}M_{N-1} \end{pmatrix}_{N-1 \times N-1}$$

an upper diagonal matrix with bandwidth of 1 and finally,

$$\mathbf{A}_{22} = \begin{pmatrix} 1 - \frac{h}{2}M'_1 & & & 0 \\ -1 - \frac{h}{2}M'_1 & 1 - \frac{h}{2}M'_2 & & \\ \ddots & \ddots & & \\ & -1 - \frac{h}{2}M'_{N-2} & 1 - \frac{h}{2}M'_{N-1} & \\ 0 & & -1 - \frac{h}{2}M'_{N-1} & 1 - \frac{h}{2}M'_N \end{pmatrix}_{N-1 \times N-1}$$

a lower diagonal matrix with bandwidth of 1,

$$\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{N-1}, \varepsilon'_1, \dots, \varepsilon'_N)^t,$$

and

$$\mathbf{T} = (T_0, T_1, \dots, T_{N-1}, T'_1, \dots, T'_N)^t.$$

are $1 \times 2(N - 1)$ matrices.

Let us assume that either $M'_i < 0$ or $M' \neq \frac{2}{h}$ for $i = 1, 2, \dots, N$. These either assumptions ensure that diagonal block matrices \mathbf{A}_{11} and \mathbf{A}_{22} are invertible. Moreover $\mathbf{A}_{11}^{-1} < 0$ a negative definite but $\mathbf{A}_{22}^{-1} > 0$ a positive definite if $|M'_i| < \frac{2}{h}$. Let us assume

$$|\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}| < 1 \tag{15}$$

hence block matrix \mathbf{D} is invertible [11].

Let us define,

$$a_2^* = \|\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\|_\infty \quad \text{and} \quad a_{*1} = \|\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\|_\infty.$$

and

$$S_i^* = \begin{cases} -\frac{h^2}{6}(2M_{i-1} + M_i), & 1 \leq i \leq N - 1 \\ -1 - \frac{h^2}{3}M_{i-1}, & i = N \end{cases}$$

and

$$S_{*i} = \begin{cases} 1 - \frac{h}{2}M'_i, & i = 1 \\ -\frac{h}{2}(M'_{i-1} + M'_i), & 2 \leq i \leq N \end{cases}$$

sum of the elements in a row of the matrices \mathbf{A}_{11} and \mathbf{A}_{22} respectively. Let $S_{*min}^* = \min\{S_i^*\}, 1 \leq i \leq N$ and $S_{*max} = \max\{S_{*i}\}, 1 \leq i \leq N$. Thus

$$\|\mathbf{A}_{11}^{-1}\|_\infty \leq \frac{1}{|S_{*min}^*|} \quad \text{and} \quad \|\mathbf{A}_{22}^{-1}\|_\infty \leq \frac{1}{|S_{*max}|}.$$

So, we have

$$\frac{1}{S_{*min}^*} = \max\left\{\frac{1}{|S_{*min}^*|}, \frac{1}{|S_{*max}|}\right\} \tag{16}$$

But we know in [11] that

$$\|\mathbf{D}^{-1}\|_\infty \leq \frac{\max\{\|\mathbf{A}_{11}^{-1}\|_\infty, \|\mathbf{A}_{22}^{-1}\|_\infty\}(1 + a_2^*)(1 + a_{*1})}{(1 + a_2^*) + (1 + a_{*1}) - (1 + a_2^*)(1 + a_{*1})} \tag{17}$$

Each term in (17) is well defined, so from (14)

$$\|\boldsymbol{\epsilon}\|_\infty \leq \frac{\max\{\|\mathbf{A}_{11}^{-1}\|_\infty, \|\mathbf{A}_{22}^{-1}\|_\infty\}(1 + a_2^*)(1 + a_{*1})}{(1 + a_2^*) + (1 + a_{*1}) - (1 + a_2^*)(1 + a_{*1})} \|\mathbf{T}\|_\infty \tag{18}$$

It follows from (13), (14) and (18) that $\|\boldsymbol{\epsilon}\| \rightarrow 0$ as $h \rightarrow 0$. Thus we conclude that method (4) will converge.

6 Numerical Results

In this section we have considered model problems to perform the numerical experiment. In these model problems, we consider different number of noddle points for both x_i and λ_j . In computation of maximum absolute error MAE between the analytical solution $u(\lambda_j)$ and computed numerical solution u_j of the problem, we have used the following formula,

$$MAE = \max_{1 \leq j \leq M} |u(\lambda_j) - u_j|.$$

We have respectively applied Gauss-Seidel and Newton-Raphson method to solve the system of linear and nonlinear equations those arise from the method (4). The solutions are computed on different values of N and M . The iteration is continued until either the maximum difference between two successive iterates is less than 10^{-8} or the number of iterations reached 10^3 . All computations were performed on a Windows 2007

Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC.

Problem 1. The model nonlinear problem that arises in mathematical modeling of the isothermal packed-bed reactor [4],

$$\frac{1}{N_{pe}} u''(x) + u'(x) - \lambda u^n = 0, \quad 0 < x < 1$$

subject to boundary conditions

$$u'(0) = 0, \quad \text{and} \quad u(1) + \frac{1}{N_{pe}} u'(1) = 1.$$

The problem is to find solutions corresponding to a given value of N_{pe} and n for a range of values of λ . In this problem we identify λ as a parameter. Let the constructed analytical solution of the problem is $u(x) = \frac{N_{pe}}{N_{pe}-1} \exp(x^2 - x^3)$. The MAE for $n = 2$, different values of N, M and range of values of λ is presented in Tables 1-4.

Problem 2. The model nonlinear boundary value problem that arises in analysis of the confinement of a plasma column by radiation pressure [12, 13] with different boundary conditions,

$$u''(x) = \lambda \sinh(\lambda u(x)), \quad 0 < x < 1$$

subject to boundary conditions

$$u'(0) = 1, \quad \text{and} \quad u(1) = 0.$$

The problem is to find solutions for a range of values of λ . In this problem we identify λ as a parameter. Let the constructed analytical solution of the problem is $u(x) = \sinh(x)$. The MAE for different values of N, M and range of values of λ are presented in Table 5.

Table 1 Maximum absolute error (Problem 1).

M	λ	MAE				
		N				
		16	32	64	128	256
8	.19	.82138918(-1)	.75354241(-1)	.69477819(-1)	.93044417(-8)	.57223819(-1)
	.199	.86510330(-1)	.79471337(-1)	.73395520(-1)	.66898733(-1)	.60657050(-1)
	.1999	.87102056(-1)	.79883814(-1)	.93044417(-8)	.67304015(-1)	.61004162(-1)
	.20	.93044417(-8)	.93044417(-8)	.93044417(-8)	.93044417(-8)	.93044417(-8)
	.201	.87558843(-1)	.80390550(-1)	.74223734(-1)	.67704059(-1)	.61429124(-1)
	.21	.91972984(-1)	.84560312(-1)	.93044417(-8)	.71391739(-1)	.93044417(-1)
16	.19	.96539810(-1)	.89871243(-1)	.84503368(-1)	.78302220(-1)	.71302131(-1)
	.199	.10187446(0)	.94899289(-1)	.89246385(-1)	.82724325(-1)	.75411789(-1)
	.1999	.10245341(0)	.95427811(-1)	.90004981(-1)	.83180606(-1)	.75825751(-1)
	.20	.93044417(-8)	.93044417(-8)	.93044417(-8)	.93044417(-8)	.93044417(-8)
	.201	.10313483(0)	.96064523(-1)	.90328887(-1)	.83708838(-1)	.76336697(-1)
	.21	.10851809(0)	.10116192(0)	.95119081(-1)	.88291369(-1)	.80537640(-1)

Table 2 Maximum absolute error (Problem 1).

λ	MAE				
	N=M				
	16	32	64	128	256
.19	.96539810(-1)	.98467313(-1)	.10002618(0)	.93044417(-8)	.93044417(-8)
.199	.10187446(0)	.10409938(0)	.10581468(0)	.10681983(0)	.10737519(0)
.1999	.10245141(0)	.10459718(0)	.10633237(0)	.10736809(0)	.10792833(0)
.20	.93044417(-8)	.93044417(-8)	.93044417(-8)	.93044417(-8)	.93044417(-8)
.201	.10313483(0)	.10540088(0)	.10712849(0)	.10813649(0)	.10870439(0)
.21	.10851809(0)	.11108328(0)	.11298846(0)	.93044417(-8)	.93044417(-8)

Table 3 Maximum absolute error (Problem 1).

N	λ	MAE			
		M			
		4	8	16	32
16	.19	.58880460(-1)	.82138918(-1)	.96539810(-1)	.10463663(0)
	.199	.61741766(-1)	.86510330(-1)	.10187446(0)	.11056767(0)
	.1999	.93044417(-8)	.87102056(-1)	.10245341(0)	.11112830(0)
	.20	.93044417(-8)	.93044417(-8)	.93044417(-8)	.93044417(-8)
	.201	.62708415(-1)	.87558843(-1)	.10313483(0)	.11192413(0)
	.21	.65576933(-1)	.91972984(-1)	.10851809(0)	.11790539(0)

Table 4 Maximum absolute error (Problem 1).

N	λ	MAE		
		M		
		64	128	256
16	.19	.10895841(0)	.11119245(0)	.11232845(0)
	.199	.11521111(0)	.11761030(0)	.11882679(0)
	.1999	.11581678(0)	.11824594(0)	.11947899(0)
	.20	.93044417(-8)	.93044417(-8)	.93044417(-8)
	.201	.12292860(0)	.12553489(0)	.12686017(0)
	.21	.82138918(-1)	.75354241(-1)	.69477819(-1)

Table 5 Maximum absolute error (Problem 2).

M	λ	MAE		
		N		
		4	8	16
8	.10	.11055857(0)	.11045857(0)	.11032658(0)
	.15	.17244926(0)	.17190818(0)	.17155264(0)
	.20	.23749650(0)	.23588327(0)	00000

We considered a nonlinear model problem to test the computational result of the proposed method. The numerical results for model problem 1 which is presented in table 1-4, for different values of N , M and range of values of λ . The results are presented in table 1, errors do not grow as either N or M increases for $\lambda = .20$. Thus we conclude that proposed method is convergent for $\lambda = .20$. However, such convergence is slow for small value of parameter λ . The convergence of the proposed method depends on the value of the parameter λ . Also proposed method produces growing error as either M or λ increase while N remains fixed. Thus we observed that the proposed method does not converge for same parameter λ and fixed N as M increases. We have drawn this observation from the result presented in table 3-4 for considered problem 1. It seems the rate of the convergence is substantially very low. Under these observations, we have found computational result in numerical experiment for the example 2.

7 Conclusion

A parametric finite difference method to find the numerical solution of two point boundary value problems with uniform mesh has been developed and discussed. The method discretizes the problem (1) at the discrete nodal points and is transformed into a system of algebraic equations given by (4). The propose method produces an approximate numerical solution of the model problems with the uniform step size. We have discussed the convergence of the proposed method, but we have not estimated exact rate of convergence. The numerical results for the model problems showed that the proposed method is convergent for large values of N . The idea presented in this article leads to the possibility to develop efficient computational method for the numerical solution of higher order boundary value problems. Works in these directions are in progress.

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