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The high accuracy conserved splitting domain decomposition scheme for solving the parabolic equations

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Abstract

In this paper, the high accuracy mass-conserved splitting domain decomposition method for solving the parabolic equations is proposed. In our scheme, the time extrapolation and local multi-point weighted average schemes are used to approximate the interface fluxes on interfaces of sub-domains, while the interior solutions are computed by one dimension high-order implicit schemes in sub-domains. The important feature is that the developed scheme keeps mass conservation and are of second-order convergent in time and fourth-order convergent in space. Numerical experiments confirm the convergence.

Keywords: Parabolic equations, time second-order, space fourth-order, mass-conserved, domain decomposition.
AMS 2010 codes: 65M06; 65M55; 76S05

1 Introduction

Time-dependent parabolic equations are widely used in science and engineering, which are described water head in groundwater modelling, pressure in petroleum reservoir simulation, diffusion phenomena in heat propagation, and atmospheric aerosol transport problems, etc (see [1–3, 7, 8]). Due to the computational complexities and huge computational costs in applications, the non-overlapping explicit/implicit domain decomposition method have been an important tool for solving parabolic equations [4, 6, 7, 10, 11]. Domain decomposition schemes that preserve the mass of the model are important and also required for parallel computations, specially, in long time simulations and for large scale applications. Paper [5] presented an explicit-implicit conservative domain decomposition procedure for parabolic equations, where the fluxes at the sub-domain interfaces were calculated by an average operator from the solutions at the previous time level. Paper [16] studied the cell centered finite difference domain decomposition procedure for the heat equations with constant coefficients

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in one dimension. Papers [13–15] proposed conservative parallel difference schemes for solving 2-dimension (nonlinear) diffusion equation and the theoretical analysis was proved in paper [9]. By the operator splitting technique and the coupling of the solution and its fluxes on staggered meshes, papers [17–19] analyzed the new mass-preserving S-DDM scheme for solving parabolic equations and convection-diffusion equations. However, the above conservative domain decomposition methods are only first-order in time. Recently, papers [20–22] proposed the time second-order mass-conserved domain decomposition methods for parabolic equations with constant coefficients and variable coefficients, respectively.

In this paper, for improving the accuracy and stability of the mass-conserved schemes in papers [20–22], we propose the time second-order and space fourth-order conservative domain decomposition schemes for one-dimension and 2-dimension parabolic equations with Neumann boundary conditions. In the domain decomposition method, we take two steps to solve one-dimension problem. The time extrapolation and local multi-point weighted average schemes are used to approximate the interface fluxes on interfaces of sub-domains, while the interior solutions are computed by the time second-order and space fourth-order implicit schemes in sub-domains. By the operator splitting technique, we take three small time steps (i.e., along $x-$direction, $y-$direction and $x-$direction) to solve 2-dimension parabolic equations at each time interval, successively. The new feature of our schemes is of fourth order accuracy in space step and the stability condition is weaker by increasing properly the value of m. Numerical experiments are given to confirm mass conservation and convergence.

2 Time second-order and space fourth-order conserved DDM for 1-dimension parabolic equations

2.1 One-dimension parabolic model and partition

The one dimensional parabolic equations with variable coefficients are considered as follows,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}(D(x)\dfrac{\partial u}{\partial x}) = f(x,t), \qquad x \in [0,L], \quad t \in [0,\, T], \\ \dfrac{\partial u}{\partial x}\mid_{x=0} = 0,\ \dfrac{\partial u}{\partial x}\mid_{x=L} = 0, \qquad t \in [0,\, T], \\ u(x,0) = u_0(x), \qquad x \in [0,L]. \end{cases} \tag{1}$$

where $D(x)$ is diffusion coefficients and $0 < D_0 \le D(x) \le D_1$. Let $h > 0$ be the space step length, and $\{x_{i+\frac{1}{2}}\}, \{x_i\}$ be the uniform staggered partition points as

$$x_{\frac{1}{2}} = 0, x_{I+\frac{1}{2}} = L, x_{i+\frac{1}{2}} = ih, i = 1,2,\cdots,I-1, x_i = (i-\tfrac{1}{2})h, \quad i = 1,2,\cdots,I. \tag{2}$$

Let $\tau > 0$ be the time step length and t^n be the uniform partition points of [0, T] as

$$t^0 = 0, \quad t^M = T, \quad t^n = n\tau, \quad n = 1,2,\cdots,M-1. \tag{3}$$

For functions $F(x,t)$, define the difference operators as follows,

$$\begin{gathered} \partial_\tau F_i^n = \tfrac{F_i^n - F_i^{n-1}}{\tau}, \quad F_i^{n+\frac{1}{2}} = \tfrac{F_i^n + F_i^{n+1}}{2}, \quad \delta_x F^n_{i+\frac{1}{2}} = \tfrac{F^n_{i+1} - F^n_i}{h}, \\ \Delta_h F_i^n = \frac{\delta_x F^n_{i+\frac{1}{2}} - \delta_x F^n_{i-\frac{1}{2}}}{h} = \tfrac{F^n_{i+1} - 2F^n_i + F^n_{i-1}}{h^2}, \quad \delta_x^3 F^n_{i+\frac{1}{2}} = \Delta_h[\delta_x F^n_{i+\frac{1}{2}}] = \tfrac{F^n_{i+2} - 3F^n_{i+1} + 3F^n_i - F^n_{i-1}}{h^3}. \end{gathered} \tag{4}$$

2.2 mass conserved domain decomposition method

For the stake of simplicity, the domain [0, L] is divided into two sub-domains and $x_{k+\frac{1}{2}}$ is the interface point of the sub-domains. Let $\{U_i^n\}$ and $\{Q^n_{i+\frac{1}{2}}\}$ be the numerical approximations of the exact solutions $\{u_i^n\}$ and $\{D_{i+\frac{1}{2}} \frac{\partial u^n}{\partial x} \mid_{i+\frac{1}{2}}\}$.

Define $\pi^n_{i+\frac{1}{2}} = \delta_x U^n_{i+\frac{1}{2}}, q^n_{i+\frac{1}{2}} = D_{i+\frac{1}{2}}\pi^n_{i+\frac{1}{2}}$ and

$$\tilde{\pi}^n_{i+\frac{1}{2}} = \frac{1}{mH}[\frac{5}{4}(\sum_{l=i+1}^{i+m} U^n_l - \sum_{l=i-m+1}^{i} U^n_l) - \frac{1}{12}(\sum_{l=i+m+1}^{i+2m} U^n_l - \sum_{l=i-2m+1}^{i-m} U^n_l)] + \frac{h^2}{12}\delta_x^3 U^n_{i+\frac{1}{2}}, \tag{5}$$

and $\tilde{q}^n_{i+\frac{1}{2}} = D_{i+\frac{1}{2}}\tilde{\pi}^n_{i+\frac{1}{2}}$. Where the large space step $H = mh$ and m is the integer.

Now, we propose the time second-order and space fourth-order conservative domain decomposition scheme of Eqns. (1) in two steps at every time $[t^{n-1}, t^n]$.

Step 1. The interface fluxes $\{Q^{n+1}_{k+\frac{1}{2}}\}$ on the interface are firstly computed by

$$Q^{n+1}_{k+\frac{1}{2}} = \check{q}^{n+1}_{k+\frac{1}{2}} - \tfrac{h^2}{24}\Delta_h \check{q}^{n+1}_{k+\frac{1}{2}} - \tfrac{h^2}{24} D_{i+\frac{1}{2}}\Delta_h \check{\pi}^{n+1}_{k+\frac{1}{2}}, \tag{6}$$

where $\check{q}^{n+1}_{k+\frac{1}{2}}$ and $\check{\pi}^{n+1}_{k+\frac{1}{2}}$ are computed by the time extrapolation and local multi-point weighted average scheme as follows,

$$\check{q}^{n+1}_{k+\frac{1}{2}} = 2\tilde{q}^n_{k+\frac{1}{2}} - \tilde{q}^{n-1}_{k+\frac{1}{2}}, \quad \check{\pi}^{n+1}_{k+\frac{1}{2}} = 2\tilde{\pi}^n_{k+\frac{1}{2}} - \tilde{\pi}^{n-1}_{k+\frac{1}{2}}. \tag{7}$$

Step 2. The interior points $\{U^{n+1}_i\}$ on two sub-domains are computed by the following scheme,

$$\partial_\tau U^{n+1}_i - \tfrac{1}{2}(\delta_x Q^n_i + \delta_x Q^{n+1}_i) = f_i^{n+\frac{1}{2}}, \tag{8}$$

where $\{Q^{n+1}_{i+\frac{1}{2}}\}$ and $\{Q^n_{i+\frac{1}{2}}\}$ are coupled computed as

$$\begin{cases} Q^n_{i+\frac{1}{2}} = q^n_{i+\frac{1}{2}} - \frac{h^2}{24}\Delta_h q^n_{i+\frac{1}{2}} - \frac{h^2}{24} D_{i+\frac{1}{2}}\Delta_h \delta_x U^n_{i+\frac{1}{2}}, \quad \forall i = 1,2,\cdots,I \\ Q^{n+1}_{i+\frac{1}{2}} = q^{n+1}_{i+\frac{1}{2}} - \frac{h^2}{24}\Delta_h q^{n+1}_{i+\frac{1}{2}} - \frac{h^2}{24} D_{i+\frac{1}{2}}\Delta_h \delta_x U^{n+1}_{i+\frac{1}{2}}, \quad i \neq k-1,k,k+1, \\ Q^{n+1}_{i+\frac{1}{2}} = q^{n+1}_{i+\frac{1}{2}} - \frac{h^2}{24}\Delta_h q^{n+1}_{i+\frac{1}{2}} - \frac{h^2}{24} D_{i+\frac{1}{2}}\Delta_h \delta_x U^{n+1}_{i+\frac{1}{2}} + \frac{1}{24}(q^{n+1}_{k+\frac{1}{2}} - \check{q}^{n+1}_{k+\frac{1}{2}}) \\ \qquad + \frac{D_{i+\frac{1}{2}}}{24}(\pi^{n+1}_{k+\frac{1}{2}} - \check{\pi}^{n+1}_{k+\frac{1}{2}}), \quad i = k \pm 1. \end{cases} \tag{9}$$

with the boundary conditions as $Q^n_{\frac{1}{2}} = Q^n_{I+\frac{1}{2}} = 0$. The initial values are given by

$$U^0_i = u^0(x_i), \quad i = 1,2,\cdots,I, \tag{10}$$

and the first time level values $\{U^1_i\}$ are also first computed by the implicit scheme as

$$\partial_\tau U^1_i - \tfrac{1}{2}(\delta_x Q^1_i + \delta_x Q^0_i) = f_i^{\frac{1}{2}}, \quad i = 1,2,\cdots,I. \tag{11}$$

Theorem 1. **(Mass conservation)** *The scheme (6) - (11) preserves the global mass conservation over the whole domain, i.e., if $f(x,t) = 0$, then we can obtain*

$$\textstyle\sum_{i=1}^I U^n_i h = \sum_{i=1}^I U^0_i h, \quad n = 1,2,\cdots,M. \tag{12}$$

Proof. When $f(x,t) = 0$, multiplying (8) by h and summing up from $i = 1$ to I, we can obtain that

$$\textstyle\sum_{i=1}^I \partial_\tau U^n_i h - \frac{1}{2}\sum_{i=1}^I(\delta_x Q^n_i + \delta_x Q^{n-1}_i)h = 0, \quad n = 1,2,\cdots,M. \tag{13}$$

Using the boundary condition $Q_{\frac{1}{2}} = Q_{I+\frac{1}{2}} = 0.$, we can obtain that

$$\textstyle\sum_{i=1}^I \delta_x Q^n_i h = 0, \qquad \sum_{i=1}^I \delta_x Q^{n-1}_i h = 0, \quad n = 1,2,\cdots,M. \tag{14}$$

Substituting (14) into (13), we have that $\sum_{i=1}^I \partial_t U^n_i h = 0$. Further, it holds that

$$\textstyle\sum_{i=1}^I U^n_i h = \sum_{i=1}^I U^{n-1}_i h = \cdots = \sum_{i=1}^I U^0_i h. \tag{15}$$

This ends the proof of the theorem.

Theorem 2. *If the exact solution u satisfies the regularity condition $u \in C^0([0,T];C^5(\Omega)) \cap C^2([0,T];C^0(\Omega))$, we have that*

$$\begin{aligned}&\frac{1}{mH}[\frac{5}{4}(\Sigma_{l=i+1}^{i+m} u_l^n - \Sigma_{l=i-m+1}^{i} u_l^n) - \frac{1}{12}(\Sigma_{l=i+m+1}^{i+2m} u_l^n - \Sigma_{l=i-2m+1}^{i-m} u_l^n)] + \frac{h^2}{12}\delta_x^3 u_{i+\frac{1}{2}}^n \\ &= \delta_x u_{i+\frac{1}{2}}^n + O(h^4 + H^4).\end{aligned} \tag{16}$$

and

$$u_l^{n+1} - (2u_l^n - u_l^{n-1}) = O(\tau^2). \tag{17}$$

Proof. For $u_l^n (l = i-2m+1, i-2m+2, \cdots, i+2m)$, by Taylor expansion, it holds that

$$\begin{aligned}u_l^n = u_{i+\frac{1}{2}}^n &+ \frac{\partial u}{\partial x}\mid_{i+\frac{1}{2}} (l-i-\frac{1}{2})h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\mid_{i+\frac{1}{2}} (l-i-\frac{1}{2})^2 h^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\mid_{i+\frac{1}{2}} (l-i-\frac{1}{2})^3 h^3 \\ &+ \frac{1}{24}\frac{\partial^4 u}{\partial x^4}\mid_{i+\frac{1}{2}} (l-i-\frac{1}{2})^4 h^4 + \frac{1}{120}\frac{\partial^5 u}{\partial x^5}\mid_{i+\frac{1}{2}} (l-i-\frac{1}{2})^5 h^5.\end{aligned} \tag{18}$$

Summing with l from $i+1$ to $i+m$, we can obtain that

$$\begin{aligned}\Sigma_{l=i+1}^{i+m} u_l^n = m u_{i+\frac{1}{2}}^n &+ \frac{\partial u}{\partial x}\mid_{i+\frac{1}{2}} \frac{m^2}{2} h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\mid_{i+\frac{1}{2}} \frac{m(4m^2-1)}{12} h^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\mid_{i+\frac{1}{2}} \frac{m^2(2m^2-1)}{8} h^3 \\ &+ \frac{1}{24}\frac{\partial^4 u}{\partial x^4}\mid_{i+\frac{1}{2}} \Sigma_{\kappa=1}^{m}(\kappa-\frac{1}{2})^4 h^4 + \frac{1}{120}\frac{\partial^5 u}{\partial x^5}\mid_{i+\frac{1}{2}} \Sigma_{\kappa=1}^{m}(\kappa-\frac{1}{2})^5 h^5.\end{aligned} \tag{19}$$

where $\kappa = l - i$, and summing with with l from $i-m-1$ to i, we have that

$$\begin{aligned}\Sigma_{l=i-m-1}^{i} u_l^n = m u_{i+\frac{1}{2}}^n &- \frac{\partial u}{\partial x}\mid_{i+\frac{1}{2}} \frac{m^2}{2} h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\mid_{i+\frac{1}{2}} \frac{m(4m^2-1)}{12} h^2 - \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\mid_{i+\frac{1}{2}} \frac{m^2(2m^2-1)}{8} h^3 \\ &+ \frac{1}{24}\frac{\partial^4 u}{\partial x^4}\mid_{i+\frac{1}{2}} \Sigma_{\kappa=1}^{m}(\kappa-\frac{1}{2})^4 h^4 - \frac{1}{120}\frac{\partial^5 u}{\partial x^5}\mid_{i+\frac{1}{2}} \Sigma_{\kappa=1}^{m}(\kappa-\frac{1}{2})^5 h^5.\end{aligned} \tag{20}$$

Subtracted (20) from (19), we can obtain that

$$\Sigma_{l=i+1}^{i+m} u_l^n - \Sigma_{l=i-m+1}^{i} u_l^n = \frac{\partial u}{\partial x}\mid_{i+\frac{1}{2}} mH + \frac{1}{24}\frac{\partial^3 u}{\partial x^3}\mid_{i+\frac{1}{2}} mH(2m^2-1)h^2 + O(H^4). \tag{21}$$

Similarly, it holds that

$$\Sigma_{l=i+m+1}^{i+2m} u_l^n - \Sigma_{l=i-2m+1}^{i-m} u_l^n = 3\frac{\partial u}{\partial x}\mid_{i+\frac{1}{2}} mH + \frac{1}{8}\frac{\partial^3 u}{\partial x^3}\mid_{i+\frac{1}{2}} mH(10m^2-1)h^2 + O(H^4). \tag{22}$$

Further, we have that

$$\begin{aligned}&\frac{1}{mH}[\frac{5}{4}(\Sigma_{l=i+1}^{i+m} u_l^n - \Sigma_{l=i-m+1}^{i} u_l^n) - \frac{1}{12}(\Sigma_{l=i+m+1}^{i+2m} u_l^n - \Sigma_{l=i-2m+1}^{i-m} u_l^n)] \\ &= \frac{\partial u^n}{\partial x}\mid_{i+\frac{1}{2}} - \frac{1}{24}\frac{\partial^3 u^n}{\partial x^3}\mid_{i+\frac{1}{2}} h^2 + O(H^4).\end{aligned} \tag{23}$$

Applying the Taylor format, it holds that

$$\frac{\partial u}{\partial x}\mid_{i+\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^n - \frac{h^2}{24}\frac{\partial^3 u}{\partial x^3}\mid_{i+\frac{1}{2}} + O(h^4), \quad \frac{\partial^3 u}{\partial x^3}\mid_{i+\frac{1}{2}} = \delta_x^3 u_{i+\frac{1}{2}}^n + O(h^2). \tag{24}$$

Substituting (23) into (24), we can obtain (16). Similarly, it leads (17).

3 Time second-order and space fourth-order splitting conserved DDM for 2-dimension parabolic equations

3.1 2-dimension parabolic problem

The two-dimensional parabolic equations are considered as

$$\begin{cases}\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(a^1(x,y)\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(a^2(x,y)\frac{\partial u}{\partial y}) = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \\ \nabla u \cdot \vec{n} = 0, \quad (x,y,t) \in \partial\Omega \times (0,T], \\ u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega,\end{cases} \tag{25}$$

where $\Omega = [0,1]\times[0,1]$, $a^1(x,y)$ and $a^2(x,y)$ are the diffusion coefficients. Assume that $0 < a_0 \le \{a^1(x,y), a^2(x,y)\} \le a_1$ are the known smooth functions. Define $h_x = \frac{1}{I}$ and $h_y = \frac{1}{J}$ be spatial step size along x-directional and y-directional, respectively. I and J are the positive integers. Introducing the following staggered meshes as

$$\begin{aligned} &x_{i+\frac{1}{2}} = ih_x,\ i = 0,1,\cdots,I, \quad x_i = (i+\tfrac{1}{2})h_x,\ i = 0,1,\cdots,I-1, \\ &y_{j+\frac{1}{2}} = jh_y,\ j = 0,1,\cdots,J, \quad y_j = (j+\tfrac{1}{2})h_y,\ j = 0,1,\cdots,J-1. \end{aligned} \tag{26}$$

For simplicity, we assume that $a^1(x,y)$ and $a^2(x,y)$ are constants and $f \equiv 0$ as below, when $a^1(x,y)$ and $a^2(x,y)$ are variable coefficients, the schemes are modified as similar as 1-dimension problem.

3.2 Conserved splitting domain decomposition scheme

For simplicity of description, we assume that the domain Ω be divided into 2×2 block sub-domains (see Fig. 1). Let $\{(x_{i+\frac{1}{2}}, y_j)\}$ and $\{(x_i, y_{j+\frac{1}{2}})\}$ be the nodes for the fluxes while $\{(x_i, y_j)\}$ are the nodes used for the

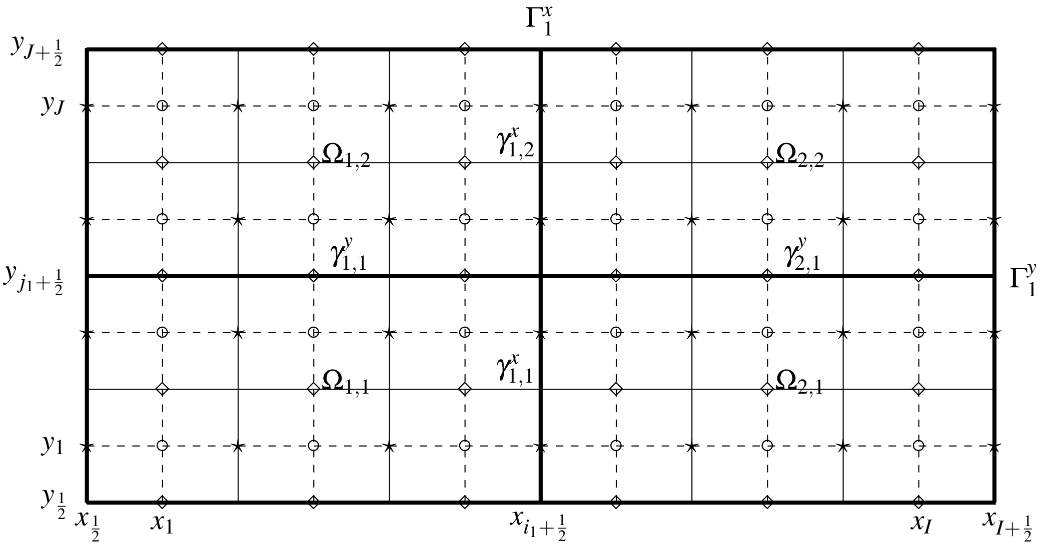


Fig. 1 The staggered meshes in 2×2 sub-domains: $\circ,\star,\diamond$ - the points of $(i,j), (i+\frac{1}{2},j), (i,j+\frac{1}{2})$.

solution. The line $\Gamma_1^x : x = x_{i_1+\frac{1}{2}}$ is the interface of $\Omega_{1,q}$ and $\Omega_{2,q}$, $q = 1,2$, where i_1 denotes the mesh point index of interface location of Γ_1^x along $x-$direction. The line $\Gamma_1^y : y = y_{j_1+\frac{1}{2}}$ is the interface of sub-domains $\Omega_{p,1}$ and $\Omega_{p,2}$, $p = 1,2$, and j_1 denotes the mesh point index of interface location of Γ_1^y along $y-$direction.

Let $q^x_{i+\frac{1}{2},j} = a^1\delta_x U_{i+\frac{1}{2},j}$, $q^y_{i,j+\frac{1}{2}} = a^2\delta_y U_{i,j+\frac{1}{2}}$, and

$$\begin{aligned} \tilde{q}^x_{i+\frac{1}{2},j} &= \frac{a^1}{m_1H_1}\Big[\frac{5}{4}\Big(\sum_{l=i+1}^{i+m} U_{l,j} - \sum_{l=i-m+1}^{i} U_{l,j}\Big) - \frac{1}{12}\Big(\sum_{l=i+m+1}^{i+2m} U_{l,j} - \sum_{l=i-2m+1}^{i-m} U_{l,j}\Big)\Big] + \frac{{h_x}^2}{12}a^1\delta_x^3 U_{i+\frac{1}{2},j}, \\ \tilde{q}^y_{i,j+\frac{1}{2}} &= \frac{a^2}{m_2H_2}\Big[\frac{5}{4}\Big(\sum_{l=i+1}^{i+m} U_{i,l} - \sum_{l=i-m+1}^{i} U_{i,l}\Big) - \frac{1}{12}\Big(\sum_{l=i+m+1}^{i+2m} U_{i,l} - \sum_{l=i-2m+1}^{i-m} U_{i,l}\Big)\Big] + \frac{{h_y}^2}{12}a^2\delta_y^3 U_{i,j+\frac{1}{2}}, \end{aligned} \tag{27}$$

where $H_1 = m_1h_x$, $H_2 = m_2h_y$.

Now we describe the algorithm of our time second-order and space fourth-order conserved splitting domain decomposition scheme on $\Omega_{1,1}$ at each time $[t^{n-1}, t^n]$ in details as

Step 1: Along x-direction.

(a) The intermediate interface fluxes $\{Q^{x,n^*}_{i_1+\frac{1}{2},j}\}$ on the interface are firstly computed by

$$Q^{x,n^*}_{i_1+\frac{1}{2},j} = \check{q}^{x,n^*}_{i_1+\frac{1}{2},j} - \frac{{h_x}^2}{12}\Delta_{h_x}\check{q}^{x,n^*}_{i_1+\frac{1}{2},j}, \tag{28}$$

where $\breve{q}^{x,n^*}_{i_1+\frac{1}{2},j}$ are computed by the time extrapolation and local multi-point weighted average scheme as follows,

$$\breve{q}^{x,n^*}_{i_1+\frac{1}{2},j} = \begin{cases} \frac{1}{4}(5\tilde{q}^{x,1}_{i_1+\frac{1}{2},j} - \tilde{q}^{x,0}_{i_1+\frac{1}{2},j}), & n=1, \\ 2\tilde{q}^{x,n}_{i_1+\frac{1}{2},j} - \tilde{q}^{x,n-1^{**}}_{i_1+\frac{1}{2},j}, & n \geq 2. \end{cases} \tag{29}$$

(b) The intermediate variables $\{U^{n^*}_{i,j}\}$ are computed by the x-directional splitting implicit scheme.

$$\begin{cases} \frac{U^{n^*}_{i,j}-U^{n}_{i,j}}{\tau} = \frac{1}{4}(\delta_x Q^{x,n^*}_{i,j} + \delta_x Q^{x,n}_{i,j}), & (x_i,y_j) \in \Omega_{1,1}, \\ Q^{x,n}_{i+\frac{1}{2},j} = q^{x,n}_{i+\frac{1}{2},j} - \frac{h_x{}^2}{12}\Delta_{h_x} q^{x,n}_{i+\frac{1}{2},j}, & (x_i,y_j) \in \Omega_{1,1}, \\ Q^{x,n^*}_{i+\frac{1}{2},j} = q^{x,n^*}_{i+\frac{1}{2},j} - \frac{h_x{}^2}{12}\Delta_{h_x} q^{x,n^*}_{i+\frac{1}{2},j}, & i=1,2,\cdots i_1-2, \\ Q^{x,n^*}_{i+\frac{1}{2},j} = q^{x,n^*}_{i+\frac{1}{2},j} - \frac{h_x{}^2}{12}\Delta_{h_x} q^{x,n^*}_{i+\frac{1}{2},j} + \frac{1}{12}(q^{x,n^*}_{i_1+\frac{1}{2},j} - \breve{q}^{x,n^*}_{i_1+\frac{1}{2},j}), & i=i_1-1. \end{cases} \tag{30}$$

Step 2: Along y-direction.

(a) The interface fluxes $\{Q^{y,n^{**}}_{i,j_1+\frac{1}{2}}\}$ on interface are computed explicitly by

$$Q^{y,n^{**}}_{i,j_1+\frac{1}{2}} = \breve{q}^{y,n^{**}}_{i,j_1+\frac{1}{2}} - \frac{h_y{}^2}{12}\Delta_{h_y}\breve{q}^{y,n^{**}}_{i,j_1+\frac{1}{2}}, \tag{31}$$

and define $\breve{q}^{y,n^{**}}_{i,j_1+\frac{1}{2}} = 3\tilde{q}^{y,n^*}_{i,j_1+\frac{1}{2}} - 2\tilde{q}^{y,n}_{i,j_1+\frac{1}{2},j}$.

(b) The numerical solutions $\{U^{**}_{i,j}\}$ are solved by the y-directional splitting implicit scheme.

$$\begin{cases} \frac{U^{n^{**}}_{i,j}-U^{n^*}_{i,j}}{\tau} = \frac{1}{2}(\delta_y Q^{y,n^{**}}_{i,j} + \delta_y Q^{y,n^*}_{i,j}) & (x_i,y_j) \in \Omega_{1,1}, \\ Q^{y,n^*}_{i,j+\frac{1}{2}} = q^{y,n^*}_{i,j+\frac{1}{2}} - \frac{h_y{}^2}{12}\Delta_{h_y} q^{y,n^*}_{i,j+\frac{1}{2}}, & (x_i,y_j) \in \Omega_{1,1}, \\ Q^{y,n^{**}}_{i,j+\frac{1}{2}} = q^{y,n^{**}}_{i,j+\frac{1}{2}} - \frac{h_y{}^2}{12}\Delta_{h_y} q^{y,n^{**}}_{i,j+\frac{1}{2}}, & j=1,2,\cdots,j_1-2 \\ Q^{y,n^{**}}_{i,j+\frac{1}{2}} = q^{y,n^{**}}_{i,j+\frac{1}{2}} - \frac{h_y{}^2}{12}\Delta_{h_y} q^{y,n^{**}}_{i,j+\frac{1}{2}} + \frac{1}{12}(q^{y,n^{**}}_{i,j_1+\frac{1}{2}} - \breve{q}^{y,n^{**}}_{i,j_1+\frac{1}{2}}), & j=j_1-1. \end{cases} \tag{32}$$

Step 3: Along x-direction.

(a) The intermediate interface fluxes $\{Q^{x,n+1}_{i_1+\frac{1}{2},j}\}$ on the interface are re-computed explicitly as

$$Q^{x,n+1}_{i_1+\frac{1}{2},j} = \breve{q}^{x,n+1}_{i_1+\frac{1}{2},j} - \frac{h_x{}^2}{12}\Delta_{h_x}\breve{q}^{x,n+1}_{i_1+\frac{1}{2},j}, \tag{33}$$

where $\breve{q}^{x,n+1}_{i_1+\frac{1}{2},j} = \frac{1}{2}(3\tilde{q}^{x,n^{**}}_{i_1+\frac{1}{2},j} - \tilde{q}^{x,n^*}_{i_1+\frac{1}{2},j})$.

(b) The intermediate variables $\{U^{n+1}_{i,j}\}$ are computed by the x-directional splitting implicit scheme.

$$\begin{cases} \frac{U^{n+1}_{i,j}-U^{n^{**}}_{i,j}}{\tau} = \frac{1}{4}(\delta_x Q^{x,n+1}_{i,j} + \delta_x Q^{x,n^{**}}_{i,j}), & (x_i,y_j) \in \Omega_{1,1}, \\ Q^{x,n^{**}}_{i+\frac{1}{2},j} = q^{x,n^{**}}_{i+\frac{1}{2},j} - \frac{h_x{}^2}{12}\Delta_{h_x} q^{x,n^{**}}_{i+\frac{1}{2},j}, & (x_i,y_j) \in \Omega_{1,1}, \\ Q^{x,n+1}_{i+\frac{1}{2},j} = q^{x,n+1}_{i+\frac{1}{2},j} - \frac{h_x{}^2}{12}\Delta_{h_x} q^{x,n+1}_{i+\frac{1}{2},j}, & i=1,2,\cdots,i_1-2, \\ Q^{x,n+1}_{i+\frac{1}{2},j} = q^{x,n+1}_{i+\frac{1}{2},j} - \frac{h_x{}^2}{12}\Delta_{h_x} q^{x,n+1}_{i+\frac{1}{2},j} + \frac{1}{12}(q^{x,n+1}_{i_1+\frac{1}{2},j} - \breve{q}^{x,n+1}_{i_1+\frac{1}{2},j}), & i=i_1-1. \end{cases} \tag{34}$$

The boundary conditions are approximated by

$$Q^x_{\frac{1}{2},j} = 0,\ Q^y_{i,\frac{1}{2}} = 0,\ \{(x_{\frac{1}{2}},y_j),(x_i,y_{\frac{1}{2}})\} \in \partial\Omega_h, \tag{35}$$

The initial values are computed by $U^0_{i,j} = u_0(x_i,y_j)$, and the first time level values $\{U^1_{i,j}\}$ are need to compute by splitting scheme.

Remark 1. *The conserved splitting domain decomposition scheme (28)-(35) is proposed over block-divided domain decompositions for solving 2-dimension parabolic equations. The three steps are used to compute the solutions $\{U_{i,j}^{n+1}\}$ at each time. At Step 1 (along $x-$direction), it leads to symmetric and penta-diagonal matrix systems of $\{U_{i,j}^{*}\}$ over $\Omega_{1,1}$ by substituting the intermediate fluxes $\{Q_{i+\frac{1}{2},j}^{x,n}\}$, $\{Q_{i_1+\frac{1}{2},j}^{x,n^*}\}$ and $\{Q_{i+\frac{1}{2},j}^{x,n^*}\}$ into the first equation of (30), which is solved by Thomas method [12]. Similarly, we can obtain $\{U_{i,j}^{**}\}$ along $y-$direction and $\{U_{i,j}^{n+1}\}$ along $x-$direction again.*

Theorem 3. **(Mass conservation)** *The scheme (28) - (35) preserves the global mass conservation over the whole domain, i.e.,*

$$\sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n+1} h_x h_y = \sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n} h_x h_y, \quad \forall n = 0, 2, \cdots, M-1. \tag{36}$$

Proof. Similar proof as (12), we can obtain the mass along x-direction in Step 1,

$$\sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n^*} h_x h_y = \sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n} h_x h_y, \tag{37}$$

along y-direction in Step 2,

$$\sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n^{**}} h_x h_y = \sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n^*} h_x h_y, \tag{38}$$

and along x-direction in Step 3,

$$\sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n+1} h_x h_y = \sum_{i=1}^{I}\sum_{j=1}^{J} U_{i,j}^{n^{**}} h_x h_y. \tag{39}$$

Adding (37), (38) and (39). We complete the proof.

4 Numerical experiments

In the section, we present numerical experiments to illustrate the performance of the scheme such as mass conservation, orders of convergence and stability. The domains $\Omega = [0,1] \times [0,1]$ and are divided into 2×2 sub-domains. Take uniform mesh steps $h_x = h_y = h$ and $m = m_1 = m_2$. Let $u(x,y,t^n)$ be the exact solution and $\{U_{i,j}^n\}$ be the approximate solution of the problem. Define solution errors in discrete L^2 norm as

$$e_h^n = h\sqrt{\sum_{i,j}(u(x_i,y_j,t^n) - U_{i,j}^n)^2}.$$

and mass errors $\mathrm{MassErr} = |\mathrm{Mass_n} - \mathrm{Mass_0}|$, where $\mathrm{Mass_0} = \sum_{i,j} U_{i,j}^0 h^2$ and

$$\mathrm{Mass_n} = \sum_{i,j} U_{i,j}^n h^2 + \tau \sum_{l=1}^{n}\sum_{i,j} f_{i,j}^l h^2,$$

Assume that $D = a^1 = a^2$, and the the exact solution of Eqns. (25) is $u = e^{-2D\pi^2 t}\cos\pi x\cos\pi y$. Table 1 presents the errors and the order of convergence in space step at $t = 0.1$. The space step size h is selected from $1/10$ to $1/80$, while the time step size is taken as $\tau = 1/10000$ and $m = 2$.

The time order of convergence of the scheme at time $t = 0.1$ is presented in Table 2. Taking $\tau = 0.1h^2$ and $D = 1E-2, 1E-1, D = 1$.

From Table 1 and 2, we can see clearly that our scheme are of fourth-order convergence in spatial step and second-order convergence in time step for the cases of different diffusions.

Take the space step $h = 1/40$ and the time step τ from $1/800$ to $1/2000$ and $m = 3$ in Table 3. It is clear that our scheme is conserved for the cases of different diffusions and different time step, since the errors of mass have reached the machine accuracy 10^{-17}.

The effect of m on the stability of our scheme for the solutions is presented in Table 4 at $t = 0.01$. Take $D = 1$, $r = \frac{\tau}{h^2} = 2$ and $h = 1/200$. From Table 4, we can find that when $r = 2$, our scheme is still stable, conservative and has very good accurate results by increasing properly the value of $m \geq 5$.

In Figure 2, we take $h = 1/100$, $\tau = 1/10000$, $D = 1$, and $m = 5$. From the contour and surface plots of concentration, it is clear that the shape of solution moves smoothly without any numerical oscillation.

Table 1 Errors and ratios of convergence in space for different diffusion D.

$D \setminus h$		1/10	1/20	1/40	1/80
1E-2	e_h	1.0572E-5	6.8466E-7	2.9351E-8	9.9707E-10
	ratio	-	3.9487	4.5439	4.8796
1E-1	e_h	4.6522E-5	1.7552E-6	6.7088E-8	2.9600E-9
	ratio	-	4.7282	4.7094	4.5024
1	e_h	4.2444E-5	1.7928E-6	8.0280E-8	1.5923E-9
	ratio	-	4.5653	4.4810	5.6559

Table 2 Errors and ratios of convergence in time for different diffusion D.

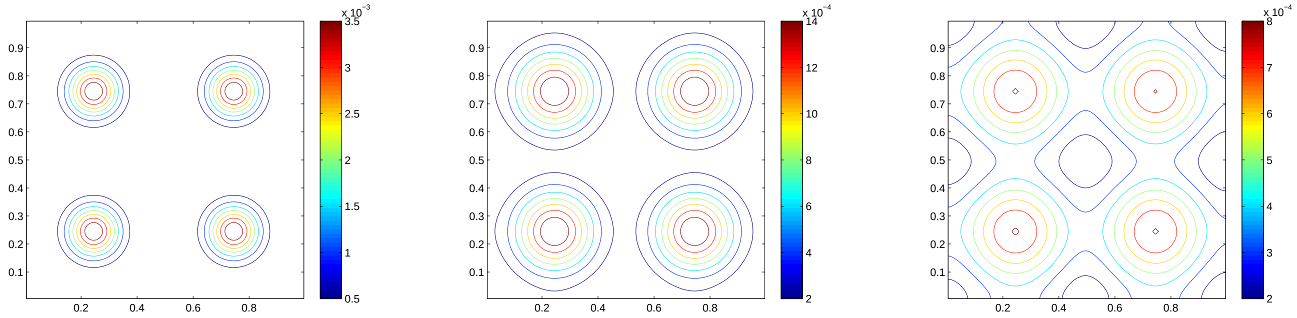
$D \setminus \tau$		1/1000	1/4000	1/9000	1/16000
1E-2	e_h	3.3475E-5	1.6835E-5	1.1157E-5	8.3407E-6
	ratio	-	1.9695	2.2111	2.3563
1E-1	e_h	4.6382E-5	1.7548E-6	2.5642E-7	6.7091E-8
	ratio	-	2.3621	2.3717	2.3303
1	e_h	4.2394E-5	1.7795E-6	2.9232E-7	8.3107E-8
	ratio	-	2.2872	2.2273	2.1860

Table 3 Errors and mass errors for different diffusion D and different τ.

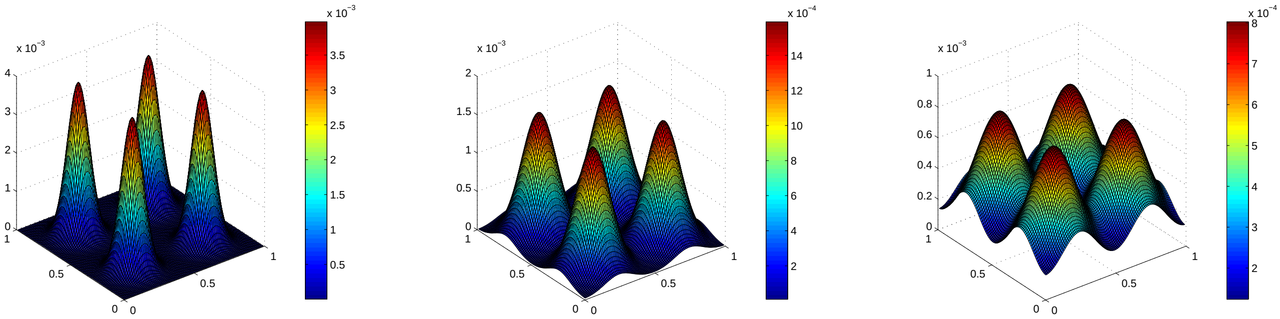
$\tau \setminus D$		0.01	0.1	0.5	1
1/1000	e_h	1.7193E-7	3.1160E-7	2.7226E-7	2.7227E-7
	MassErr	2.9837E-18	1.1241E-17	4.0246E-18	4.2986E-17
1/2000	e_h	1.7211E-7	3.0987E-7	3.1648E-7	1.4113E-7
	MassErr	3.7401E-17	2.6368E-17	2.3835E-17	4.1113E-18
1/3000	e_h	1.7218E-7	3.0962E-7	3.2929E-17	1.8791E-7
	MassErr	3.1850E-17	4.4409E-17	4.4548E-17	6.9042E-18
1/4000	e_h	1.7221-7	3.0956E-7	3.3409E-7	2.0642E-7
	MassErr	4.8503-17	1.1796E-18	2.4876E-17	1.7781E-17

Table 4 The effect of m on the stability.

$r \setminus m$		2	3	5	10	20
2	e_h	4.2699E+19	3.2818E+02	1.5482E-6	1.6600E-8	2.5564E-7
	MassErr	1.4909E+02	6.2177E-16	8.2406E-18	9.7145E-19	2.0761E-17



(a) The contour plots of concentration at $t = 0.002, 0.005, 0.01$.



(b) The surface plots of concentration at $t = 0.002, 0.005, 0.01$.

Fig. 2 The numerical simulation for heat propagation.

5 Conclusion

In this paper, the time second-order and space fourth-order conserved splitting domain decomposition scheme is developed for solving 2-dimension parabolic equations. In our splitting domain decomposition method, the time extrapolation and local multi-point weighted average schemes are used to approximate the interface fluxes on interfaces of sub-domains, while the interior solutions are computed by the splitting high-order implicit schemes in sub-domains. The analysis of stability and convergence will be studied in further work.

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