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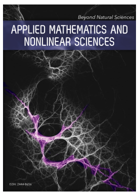

Applied Mathematics and Nonlinear Sciences

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Complex variables approach to the short-axis-mode rotation of a rigid body[†]

Martin Lara[‡]

GRUCACI–University of La Rioja, C/Madre de Dios 53, 26006 Logroño
Spain

Submission Info

Communicated by Juan L.G. Guirao
Received 22nd September 2018
Accepted 22nd December 2018
Available online 31st December 2018

Abstract

Decomposition of the free (triaxial) rigid body Hamiltonian into a "main problem" and a perturbation term provides an efficient integration scheme that avoids the use of elliptic functions and integrals. In the case of short-axis-mode rotation, it is shown that the use of complex variables converts the integration of the torque-free motion by perturbations into a simple exercise of polynomial algebra that can also accommodate the gravity-gradient perturbation when the rigid body rotation is close enough to the axis of maximum inertia.

Keywords: Free rigid body; short-axis-mode; perturbation theory; gravity gradient
AMS 2010 codes: 37J35, 37J40, 70F15, 70H09, 70H15

1 Introduction

The rotation of a triaxial rigid body in the absence of external torques is known to be integrable [1, 2]. In particular, the canonical transformation to Andoyer variables [3] reduces the free rigid body rotation to an integrable, one degree of freedom Hamiltonian, which immediately shows the preservation of the total angular momentum and allows for the representation of the possible solutions by contour plots of the reduced Hamiltonian [4]. However, because the solution to the torque-free motion depends on elliptical integrals and elliptic functions, the closed form solution is replaced in practical applications by corresponding expansions in trigonometric functions truncated to some order —as it is customary for integrable problems in which the solution depends on special functions and may further require series inversion [5, 6]. In particular, useful expansions by Kinoshita apply for the case in which the triaxiality of the rigid body is small [7] and for the case in which the rigid body rotation is close to the axis of either maximum or minimum momentum of inertia [8, 9] —the order of

[†]Main results in this paper were presented in the Journées 2017 des Systèmes de Reference et de la Rotation Terrestre, Alicante, Spain, September 25 – 27, 2017

[‡]Corresponding author.
Email address: mlara0@gmail.com

sciendo ISSN 2444-8656 doi:10.2478/AMNS.2018.2.00042 sciendo

these expansions was later extended by other authors [10]. Alternatively, the expansions in trigonometric functions of the free rigid body solution can be directly constructed using perturbation theory [11, 12], an approach that systematizes the computation of higher orders of the expansions and eases the construction of perturbation solutions in the presence of external torques [13], as opossed to the typical first order approach [14–16].

Perturbation approaches to the torque-free motion of a rigid body start from the decomposition of the free rigid body Hamiltonian in Andoyer variables into a "main problem" and a perturbation term, the solution of the main problem being achievable in elementary functions, contrary to the special functions required in the solution of the full problem. When the triaxiality is small, the selection of either the axisymmetric case or the spherical rotor as the main problem results in a completely reduced zeroth order Hamiltonian that only depends on momenta of the canonical set of Andoyer variables, a fact that simplifies application of the perturbation method because, then, Andoyer variables result to be the action-angle variables of the main problem [11]. Practical application of the perturbative scheme based on the uniaxial model to the case of a non-rigid Earth (rigid mantle and liquid core) have been made in [17].

On the contrary, while the perturbative character of short-axis-mode (SAM) rotation can be shown directly in Andoyer variables, the consequent complete reduction of the main problem by solving the Hamilton-Jacobi equation is required in order to set up an efficient perturbative integration scheme [12]. Indeed, the main problem of SAM rotation is now a simplification of the free rigid body Hamiltonian, contrary to a reduction [18, 19]. That is, it involves the same Andoyer variables as the free rigid body problem, which include coordinates as well as their conjugate momenta, and, in consequence, Andoyer variables are not action-angle variables of the zeroth order Hamiltonian.

As far as the complete reduction of a Hamiltonian is unique [20], the canonical transformation used in finding the completely reduced Hamiltonian is irrelevant to some extent, and the standard approach is to find the action-angle variables of the integrable problem by solving the classical Hamilton-Jacobi equation or variations of it [21]. However, action-angle variables commonly involve singularities for particular configurations in which some angles may be not defined, and become ill-defined for values close to those causing the singularities. The classical instance comes from the orbital motion of a particle, where the action-angle reduction of the Kepler problem by solving the Hamilton-Jacobi equation is achieved in Delaunay variables, in which the argument of the periapsis is not defined for circular orbits, and the argument of the node is not defined in the case of equatorial orbits. Nevertheless, these singularities are of "virtual" nature [22] and can be avoided using non-singular variables. Poincaré canonical coordinates and conjugate momenta avoid the singularities in the orbital motion [23], yet non-canonical sets of variables are commonly preferred in different applications [24–26].

In the case of rotational motion, Andoyer canonical variables, which are the action angle variables of the uniaxial rotation, show singularities in different configurations. Alternative variables [27, 28] remove the singularities of Andoyer variables but at the cost of creating singularities at different locations [29].

Then, in spite of the standard approach used in [12] of finding the action-angle variables that completely reduce the main problem of SAM rotation is perfectly correct, it happens that the variables used share the same deficiencies of Andoyer variables. In particular, they may become ill-defined when the axis of instantaneous rotation evolves very close to the axis of maximum (or minimum) inertia. Therefore, in practice one must to take some care when applying the solution to this kind of motion, which is precisely the case in which the separation of the free rigid body rotation into the main problem of SAM rotation and a perturbation applies.

On the other hand, it will be shown in Section 2.3 that the perturbative arrangement of the free rigid body Hamiltonian in the case of SAM rotation is immediately disclosed when using non-singular variables of Poincaré type, cf. [30]. In these variables, the free rigid body Hamiltonian takes the form of the Hamiltonian of the simple harmonic oscillator disturbed by additional quartic polynomial terms. Hence, since it is well known that the use of complex variables renders very efficient the construction of higher order analytical solutions to perturbed harmonic oscillators (see, for instance, [31–33]), an additional transformation to complex variables is carried out in Section 3 that converts the integration of the free rigid body Hamiltonian by perturbations into a simple exercise of polynomial algebra.

The polynomial structure of the perturbation is not preserved, in general, when perturbation torques are taken into account. However, it is shown in Section 4 that, when the rotation is close enough to the axis of maximum inertia, the gravity-gradient perturbation can be easily tackled within the same perturbative scheme.

Finally, it is worth mentioning that how much fruitful expanding the special functions in which the solution of the torque-free motion inherently depends into trigonometric ones can be, there are specific cases in which approximate analytical solutions to perturbed attitude motion can be achieved directly in closed form [34, 35]. These kinds of solutions widen applicability by avoiding the constraint to particular physical characteristics or dynamical configuration, and have been recently proposed as an appealing alternative for practical application to actual problems of astrodynamics [36].

2 Perturbative arrangement

The free rigid body Hamiltonian is plainly stated in terms of Andoyer [3] canonical variables in the form, cf. Eq. (4) of [37],

$$\mathscr{H}_0 = \left(\frac{1}{A}\sin^2\nu + \frac{1}{B}\cos^2\nu\right)\frac{1}{2}\left(M^2 - N^2\right) + \frac{1}{2C}N^2, \tag{1}$$

where the constants $A \le B \le C$ define the principal moments of inertia, and the variables $(\lambda, \mu, \nu, \Lambda, M, N)$ define the node of the invariable plane on the inertial x, y plane, the node of the equatorial plane of the rigid body over the invariable plane, the component of the rotation of the body around its axis of maximum inertia, the projection of the angular momentum vector along the inertial z axis, the total angular momentum, and the projection of the angular momentum vector along the body axis of maximum inertia, respectively. The fact that the variables λ and μ are cyclic in Hamiltonian (1) immediately shows that the total angular momentum M as well as its projection along the z axis of the inertial frame Λ are integrals. Besides, because Λ is also ignorable in Eq. (1), the node of the invariable plane remains fixed in the inertial x-y plane. Therefore, the free rigid body Hamiltonian in Andoyer variables is only of one degree of freedom in the canonical pair (ν, N).

It belongs to Sadov [38] the merit of finding the transformation to action-angle variables that completely reduces Hamiltonian (1). His approach resorts to a complete solution of the Hamilton-Jacobi equation and involves the elliptic integrals of the first and third kinds, both of them in their incomplete and complete versions. Because the transformation from Andoyer to action-angle variables cannot be obtained explicitly, the completely reduced Hamiltonian must remain also as an implicit function of the action variables. A modern, abridged re-derivation of Sadov's solution, which does not need to deal explicitly with the time when solving the Hamilton-Jacobi equation, can be found in [35]. This solution is summarized in the Appendix for completeness.

Working independently of Sadov, Kinoshita [7] succeeded also in finding the transformation of the free rigid body Hamiltonian to action-angle variables, yet he expressed his solution in terms of the Heuman's Lambda function rather than the elliptic integral of the third kind. However, it must be reminded that the *characteristic* parameter in which the elliptic integral of the third kind depends upon, is defined as a function of the principal moments of inertia of the free rigid body, and is always negative. Because of that, the validity of the elliptic integral of the third kind that appears in the complete reduction of the free rigid body Hamiltonian is constrained to one of the "circular" cases [39], and hence it is customarily evaluated in terms of the Heuman's Lambda function. Therefore, both, Sadov's and Kinoshita's solutions of the torque-free motion must be considered equivalent.

To make the use of the free rigid body solution practical in the study of the rotation under external torques, the closed form solution is customarily expanded in terms of trigonometric functions. Both authors, Sadov and Kinoshita, provided the necessary expansions in powers of the Jacobi's nome, which is a function of the elliptic modulus that improves convergence over the direct use of the elliptic modulus in the expansions. Alternatively, the expansion of the closed form solution can be avoided in the particular cases in which some small quantity is identified in the original Hamiltonian. Then, the free rigid body Hamiltonian can be rearranged as a perturbation

Hamiltonian and the solution of the integrable problem is directly attacked by perturbation methods [6].

Regrettably, because the modulus of the elliptic functions that solve the torque free motion depends on the energy of each particular solution, which is in fact the Hamiltonian, neither the Jacobi's nome nor the elliptic modulus are useful parameters to be identified in the original Hamiltonian in order to apply a perturbation approach. Still, as shown in the Appendix, it happens that this elliptic modulus can be decomposed into a product that splits the physical and dynamical characteristics of the motion, and, luckily, each kind of feature is easily recognized in Hamiltonian (1) after simple rearrangement.

2.1 Small triaxiality

Indeed, Andoyer's [3] original arrangement of the rigid body Hamiltonian

$$\mathscr{H}_0 = \frac{M^2}{2C}\left[1+\alpha\left(1-\frac{N^2}{M^2}\right)-\alpha\,\beta\left(1-\frac{N^2}{M^2}\right)\cos 2\nu\right], \tag{2}$$

where the relations

$$\alpha\,(1+\beta)=\frac{C}{A}-1, \qquad \alpha\,(1-\beta)=\frac{C}{B}-1, \tag{3}$$

define the physical parameters α and β as a function of the principal moments of inertia, shows that when the *triaxiality coefficient* β is small, Eq. (2) admits a perturbative treatment.

Thus

$$\mathscr{H}_0 = \mathscr{H}_{0,0}+\beta\,\mathscr{H}_{1,0}$$

in which the zeroth order term

$$\mathscr{H}_{0,0} = \frac{M^2}{2C}\left[1+\alpha\left(1-\frac{N^2}{M^2}\right)\right]$$

corresponds to an oblate axisymmetric body ($A=B \Rightarrow \beta=0$), whose Hamiltonian is completely reduced in Andoyer variables $\mathscr{H}_{0,0}=\mathscr{H}_{0,0}(M,N)$. On the other hand, the perturbation,

$$\mathscr{H}_{1,0} = -\frac{M^2}{2C}\alpha\left(1-\frac{N^2}{M^2}\right)\cos 2\nu,$$

which is due to the lack of axial symmetry of the rigid body, depends on the angle ν in addition to N and M. This perturbative arrangement eases the computation of a perturbation solution in trigonometric functions that can be easily extended to any order of β, and that, of course matches Kinoshita's [7] series expansion of the closed form solution in powers of β [11].

2.2 SAM rotation

An alternative perturbative arrangement has been recently proposed for rigid bodies rotating close to its axis of maximum inertia, irrespective of their triaxiality [12]. In that case $N\approx M$ and, therefore, $\frac{1}{2}(1-N/M)=\sin^2\frac{1}{2}J \ll 1$, a fact that motivates reorganization of Eq. (2) in the form

$$\mathscr{H}_0 = \mathscr{M}+\varepsilon\,\mathscr{P}, \tag{4}$$

where

$$\mathscr{M} = \frac{M^2}{2C}\left[1+2\alpha\left(1-\frac{N}{M}\right)(1-\beta\cos 2\nu)\right], \tag{5}$$

is taken as the integrable part, ε is a formal small parameter, and

$$\mathscr{P} = -\frac{M^2}{2C}\alpha\left(1-\frac{N}{M}\right)^2(1-\beta\cos 2\nu), \tag{6}$$

is a perturbation $|\mathscr{P}| \ll \mathscr{M}$.

Now, the zeroth order Hamiltonian (5), which has been dubbed the main problem of SAM rotation in [12], is not a completely reduced Hamiltonian, and, on the contrary, it involves the same variables as the free rigid body Hamiltonian (2). Therefore, to approach the solution of the torque-free motion by perturbations in the case of SAM rotation, it is convenient first to find the action-angle variables that completely reduce Eq. (5). This transformation has been achieved in [12] by solving the Hamilton-Jacobi equation. However, it will be shown in the next Section that the complete reduction of Eq. (5) can be immediately obtained without need of solving the Hamilton-Jacobi equation when using non-singular variables of the Poincaré type.

2.3 Non singular variables

Andoyer variables are singular for $N = M$, a case in which ν is not defined. However, this singularity is virtual [22], and is easily avoided using non-singular variables of the Poincaré type, cf. [30].

Proposition 1. *The transformation $T_1 : (x,g,X,G) \longrightarrow (\mu,\nu,M,H)$, given by*

$$x = -\sqrt{2(M-N)}\sin\nu, \tag{7}$$

$$X = \sqrt{2(M-N)}\cos\nu, \tag{8}$$

$$g = \mu+\nu, \tag{9}$$

$$G = M, \tag{10}$$

is completely canonical.

Proof. Straightforward computations show that the differential form $w = M\mathrm{d}\mu + N\mathrm{d}\nu - (X\mathrm{d}x + G\mathrm{d}g)$ is an exact differential. More precisely, $w = \mathrm{d}W$ with $W \equiv \frac{1}{2}(M-N)\sin 2\nu = -\frac{1}{2}xX$. Therefore, the transformation is canonical with null reminder and multiplier 1 (see, [40], for instance).

The inverse transformation to Eqs. (7)–(10) is

$$M = G, \tag{11}$$

$$N = G - \tfrac{1}{2}(x^2+X^2), \tag{12}$$

$$\sin\nu = -\frac{x}{\sqrt{x^2+X^2}}, \qquad \cos\nu = \frac{X}{\sqrt{x^2+X^2}}, \tag{13}$$

$$\mu = g-\nu, \tag{14}$$

and, by direct replacement in Eq. (5), it is easy to see that it converts the Hamiltonian of the main problem of the SAM rotation into

$$\mathscr{M}\circ T_1 = \frac{G^2}{2C} + \frac{G}{C}\frac{2\alpha}{1+\omega^2}\frac{1}{2}\left(X^2+\omega^2 x^2\right), \tag{15}$$

in which

$$\omega = \sqrt{\frac{1+\beta}{1-\beta}}, \tag{16}$$

is a function of the rigid body's moments of inertia. Alternatively

$$\beta = \frac{\omega^2-1}{\omega^2+1}.$$

One easily recognizes in Eq. (15) the Hamiltonian of a harmonic oscillator of (non-dimensional) frequency ω, and it is well known that the classical Poincaré transformation $T_2 : (\ell, L, \omega) \longrightarrow (x,X)$ given by

$$x = \sqrt{2L/\omega}\sin\ell, \qquad X = \sqrt{2\omega L}\cos\ell, \tag{17}$$

completely reduces the Hamiltonian of the harmonic oscillator to a function of only the momentum L. Indeed, replacing Eq. (17) into Eq. (15),

$$\mathscr{M}\circ T_1\circ T_2=\frac{G^2}{2C}+\frac{G}{C}\frac{2\alpha}{\omega^2+1}\omega L, \tag{18}$$

which, in view of the definition of ω in Eq. (16), matches the completely reduced Hamiltonian in Eq. (24) of [12], as expected.

In this way —formulation of the main problem Hamiltonian in the nonsingular variables in Eqs. (7)–(10) followed by the Poincaré transformation in Eq. (17)— the computation of the action-angle variables of the main problem of the SAM rotation carried out in [12] is dramatically abridged to the composition of the canonical transformations T_2 and T_1.

On the other hand, the use of action-angle variables, while customary, is not a requirement in perturbation theory. Indeed, in view of Eq. (6) takes the form of a quartic polynomial after applying the transformation to nonsingular variables in Eq. (7)–(10), viz.

$$\mathscr{P}\circ T_1=-\frac{\alpha}{8C}\left(\frac{2\omega^2}{\omega^2+1}x^4+2x^2X^2+\frac{2}{\omega^2+1}X^4\right), \tag{19}$$

the perturbation solution can be directly constructed in Cartesian variables.

3 Perturbation solution in complex variables

For perturbed harmonic oscillators, it is well known that the use of complex variables makes the whole perturbation approach very efficient, cf. [31–33]. Therefore, the perturbation Hamiltonian given by Eqs. (15) and (19) is next reformulated in complex variables.

Proposition 2. *The transformation $T_3:(x,g,X,G)\longrightarrow(u,v,U,V;\Omega,\kappa)$, given by*

$$x=\frac{1}{\sqrt{2\Omega}}(u-iU), \tag{20}$$

$$X=\sqrt{\frac{\Omega}{2}}(U-iu), \tag{21}$$

$$g=\sqrt{\kappa}\,v, \tag{22}$$

$$G=\frac{1}{\sqrt{\kappa}}V, \tag{23}$$

where $i=\sqrt{-1}$, and Ω and κ are constant, is completely canonical.

Proof. As before, the differential form $w'=X\mathrm{d}x+G\mathrm{d}g-(U\mathrm{d}u+V\mathrm{d}v)$ is exact. More precisely, straightforward computations show that $w'=\mathrm{d}W'$ with $W'\equiv\frac{1}{4}(iu^2+2uU+iU^2)$.

If, besides, the choices $\Omega=\omega$, as given in Eq. (16), and

$$\kappa=\alpha\sqrt{1-\beta^2}=\sqrt{\left(\frac{C}{A}-1\right)\left(\frac{C}{B}-1\right)}, \tag{24}$$

are made, then Eq. (15) is rewritten in the real (v,V) and complex variables (u,U) in the form

$$\mathscr{M}\circ T_1\circ T_3=\frac{V^2}{2C\kappa}-\frac{V\sqrt{\kappa}}{C}iuU, \tag{25}$$

whereas Eq. (19) takes the form

$$\mathscr{P}\circ T_1\circ T_3=\frac{\alpha}{4C}\left[2u^2U^2-i\beta\left(u^3U-uU^3\right)\right]. \tag{26}$$

Then, an analytical approximation to the flow stemming from Eqs. (25) and (26) can be computed with the Lie transforms method [41]. Since this method is standard these days, details are not provided here and interested readers are referred to textbooks in the literature, as, for instance [42, 43].

3.1 The Lie operator

The Lie derivative $\mathcal{L}_{\mathcal{M}}$ associated to the Eq. (25), is given by the Poisson bracket operator $\mathcal{L}_{\mathcal{M}} = \{\quad ; \mathcal{M}\}$, viz.

$$\mathcal{L}_{\mathcal{M}} = \frac{\sqrt{\kappa}}{C}\left[Vi\left(U\frac{\partial}{\partial U} - u\frac{\partial}{\partial u}\right) + \left(\frac{V}{\kappa^{3/2}} + iuU\right)\frac{\partial}{\partial v}\right], \tag{27}$$

and the partial differential equation $\mathcal{L}_{\mathcal{M}}(W_n) = \widetilde{M}_n - M_{0,n}$ must be solved at each order n of the perturbation theory to compute the corresponding term W_n of the generating function. Terms $\widetilde{M}_n$ are known from previous computations whereas terms $M_{0,n}$ are chosen to pertain to the kernel of the Lie derivative, viz. $\mathcal{L}_{\mathcal{M}}(W_n) = 0$.

However, because Eq. (26) does not depend on v, when dealing with the torque free motion as a perturbation problem one can assume that the generating function is independent of v. Hence,

$$\mathcal{L}_{\mathcal{M}} = \frac{\sqrt{\kappa}}{C}Vi\left(U\frac{\partial}{\partial U} - u\frac{\partial}{\partial u}\right). \tag{28}$$

Then, for any integers $j \geq 0$ and $k \geq 0$,

$$\mathcal{L}_{\mathcal{M}}(u^j U^k) = \frac{\sqrt{\kappa}}{C}Vi(k-j)u^j U^k,$$

and, therefore, $\mathcal{L}_{\mathcal{M}}(u^j U^k) = 0$ requires that $j = k$. That is, the kernel of the Lie operator is composed of monomials of the form $(uU)^j$, whereas all other monomials $u^j U^k$, $j \neq k$, pertain to the image.

In consequence, the solution of the homological equation becomes a trivial operation in complex variables. Indeed, any monomial $q_{j,k}u^j U^k$, $j \neq k$, where $q_{j,k}$ is a numeric coefficient, contributes a term

$$\frac{C}{V\sqrt{\kappa}}i\frac{q_{j,k}}{j-k}u^j U^k, \tag{29}$$

to the generating function. In this way the need of solving partial differential equations is completely avoided.

3.2 Free rigid body solution by perturbations

The perturbation procedure starts writing the free rigid body Hamiltonian in the form of the Taylor series expansion, viz.

$$\mathcal{H}_0 = \sum_{n\geq 0}\frac{\varepsilon^n}{n!}H_{n,0}(u,U),$$

where $H_{0,0} \equiv \mathcal{M}$ as given in Eq. (25), $H_{1,0} \equiv \mathcal{P}$ as given by Eq. (26), $H_{n,0} = 0$ for $n \geq 2$, and ε is a formal small parameter. Straightforward computations based on the previously discussed properties of the Lie operator, yield the normalized Hamiltonian, in new, prime variables

$$\mathcal{K} = \frac{\sqrt{\kappa}}{2C}V^2\left[\frac{1}{\kappa^{3/2}} + \sum_{n\geq 0}\frac{(-\alpha)^n}{\kappa^{n/2}}p_n\left(\frac{iu'U'}{V}\right)^{n+1}\right], \tag{30}$$

where p_n are polynomials in the triaxiality coefficient β. The first few triaxiality polynomials are $p_0 = 2$, and

$$p_1 = 1,$$
$$p_2 = \beta^2$$

$$
\begin{aligned}
p_3 &= \frac{5}{8}\beta^2,\\
p_4 &= \frac{3}{32}\beta^2\left(3\beta^2+8\right),\\
p_5 &= \frac{7}{32}\beta^2\left(5\beta^2+4\right),\\
p_6 &= \frac{1}{128}\beta^2\left(45\beta^4+354\beta^2+128\right),\\
p_7 &= \frac{9}{1024}\beta^2\left(265\beta^4+650\beta^2+128\right),\\
p_8 &= \frac{5}{8192}\beta^2\left(953\beta^6+14888\beta^4+17120\beta^2+2048\right),\\
p_9 &= \frac{11}{8192}\beta^2\left(4075\beta^6+20212\beta^4+13104\beta^2+1024\right),
\end{aligned}
$$

which, as expected, are the same as those in Table 2 of [12] after adjusting subindices and scaling by β^2.

The transformation from prime to original variables

$$
u = u' + \sum_{n\geq 1}\frac{\varepsilon^n}{n!}u_{0,n}(u',U'), \qquad U = U' + \sum_{n\geq 1}\frac{\varepsilon^n}{n!}U_{0,n}(u',U'),
$$

is obtained by successive evaluations of Deprit's triangle

$$
\xi_{n,q} = \xi_{n+1,q-1} + \sum_{0\leq m\leq n}\binom{n}{m}\{\xi_{n-m,q-1};S_{m+1}\}, \tag{31}
$$

using the generating function $\mathscr{S} = \sum_{n\geq 0}(\varepsilon^n/n!)\,S_{n+1}$ where terms

$$
S_n = \frac{\alpha^n}{\kappa^{n/2}V^n}\beta\left(u^2+U^2\right)s_n, \tag{32}
$$

are computed using Eq. (29). The first few s_n are

$$
\begin{aligned}
s_1 &= \frac{1}{8}uU,\\
s_2 &= -\frac{i}{4}u^2U^2,\\
s_3 &= \frac{i}{64}\left[24i\left(\beta^2+2\right)uU+5\beta(U^2-u^2)\right]u^2U^2,\\
s_4 &= \frac{3}{64}\left[2i\left(57\beta^2+32\right)uU+\beta\left(9\beta^2+20\right)\left(U^2-u^2\right)\right]u^3U^3,\\
s_5 &= \frac{1}{64}\left[2\left(343\beta^4+2024\beta^2+480\right)u^2U^2-11\beta^2\left(\beta^2+2\right)\left(u^4+U^4\right)\right.\\
&\quad\left.-6i\beta\left(147\beta^2+100\right)uU\left(U^2-u^2\right)\right]u^3U^3,\\
s_6 &= \frac{15}{64}i\left[-2\left(909\beta^4+1588\beta^2+192\right)u^2U^2+\beta^2\left(63\beta^2+44\right)\left(u^4+U^4\right)\right.\\
&\quad\left.+2i\beta\left(91\beta^4+653\beta^2+200\right)uU\left(U^2-u^2\right)\right]u^4U^4,\\
s_7 &= \frac{45i}{2048}\left[\beta\left(124067\beta^4+272282\beta^2+44800\right)u^2U^2\left(U^2-u^2\right)-\beta^3\left(283\beta^2+186\right)\left(U^6-u^6\right)\right.\\
&\quad\left.-32i\beta^2\left(116\beta^4+917\beta^2+308\right)uU\left(u^4+U^4\right)+64i\left(720\beta^6+8345\beta^4+6496\beta^2+448\right)u^3U^3\right]u^4U^4,
\end{aligned}
$$

$$
\begin{aligned}
s_8 = \frac{105}{2048} \Big[& 3\beta \left(51619\beta^6 + 725628\beta^4 + 730320\beta^2 + 71680\right) u^2 U^2 \left(U^2 - u^2\right) \\
& - 3\beta^3 \left(539\beta^4 + 4396\beta^2 + 1488\right) \left(U^6 - u^6\right) - 4i\beta^2 \left(42087\beta^4 + 104446\beta^2 + 19712\right) uU \left(u^4 + U^4\right) \\
& + 4i \left(391113\beta^6 + 1385374\beta^4 + 578816\beta^2 + 24576\right) u^3 U^3 \Big] u^5 U^5.
\end{aligned}
$$

The construction of the perturbation solution by Lie transforms results extremely efficient in this way. As a demonstration of the performance, it has been extended to the order 35 in the small parameter ε. Thanks to the use of Eqs. (30) and (32), the most computationally expensive part of the procedure at each perturbation order is the evaluation of the Poisson brackets, which are needed for the computation of the known terms of the homological equation and also for filling Deprit's triangle.

This fact is illustrated in Fig. 1 where the label H_n means the time spent in evaluating each Hamiltonian term n in Eq. (30), S_n the time spent in evaluating each term of the generating function using Eq. (29), "known" means the time spent in computation of the known terms of the homological equation at each step of the perturbation approach, "triangle" means the time spent in filling the Deprit's triangle with the new terms computable at each step, and "total" is the total time spent at each step, which is roughly the sum of the previous times plus some minor operations required by the algorithm. The computations have been carried out with Wolfram *Mathematica* 9 in a MacBook Pro running under macOS Mojave, version 10.14, with a 2.8 GHz Intel Core i7 processor and 16 GB of RAM.

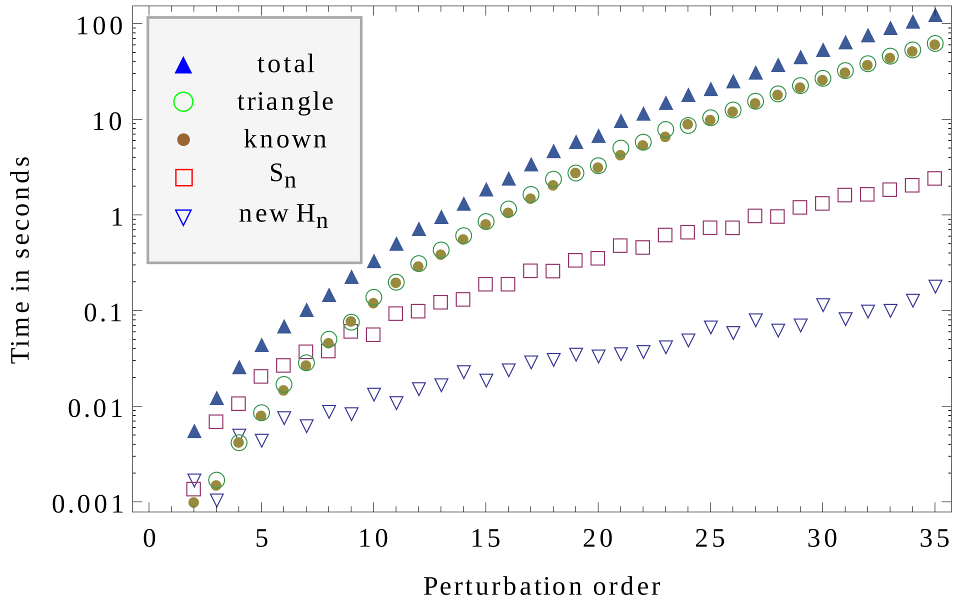

Fig. 1 Absolute times (in logarithmic scale) spent in the computation of each order of the Lie transforms algorithm.

To further emphasize the efficiency of the selection procedure used with the complex variables approach, as opposite to the usual, time consuming, averaging procedures, these times are shown in Fig. 2 relative to the time required in choosing the different terms of the transformed Hamiltonian in prime variables in Eq. (30). In this figure it is noted that the time spent in the computation of the generating function remains between 5 and 10 times higher than the time spent in the computation of the new Hamiltonian term except for the lower orders, where the ratio can be lower. On the other hand, the time spent in the computation of the known terms of the homological equation is similar to the time spent in filling Deprit's triangle at each step, and it grows high with the order of the perturbation theory. In particular, we checked that the computation of H_n grows roughly at a quadratic rate with n, while the computation of S_n grows approximately at a cubic rate with n, and the other times grow close to a quintic rate with the order of the perturbation theory.

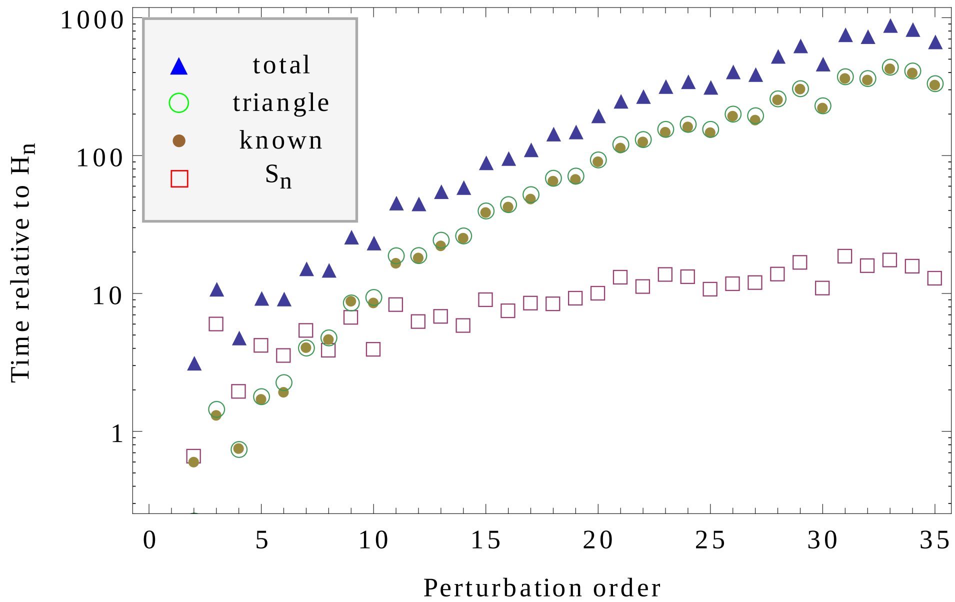

Fig. 2 Ratios S_n, "known", "triangle", and "total" (in logarithmic scale) of the times spent at each perturbation order of the Lie transforms algorithm to the time spent in the computation of the new Hamiltonian term $t = H_n$ using Eq. (30).

4 Gravity gradient perturbation

The perturbation approach based on the main problem of SAM rotation is also feasible for motion under external torques. In the particular case of gravity-gradient perturbations due to a distant body, one can make the simplifying assumption that the disturbing body moves with Keplerian motion. If, besides, the orbital and inertial planes are assumed to match, the gravity-gradient effect is derived from the potential

$$\mathscr{D} = -\frac{n^2 a^3}{2r^3}\left[(C-B)(1-3\gamma_3^2)-(B-A)(1-3\gamma_1^2)\right], \tag{33}$$

where r, a, and n, are the radius, semi-major axis, and mean motion, respectively, of the Keplerian orbit in which the distant body evolves, and γ_i, $i = 1,2,3$, are the direction cosines of the line joining the centers of mass of the rotating body and the distant body, which are expressed in Andoyer variables by simple rotations. Namely,

$$\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = R_3(\nu)\circ R_1(J)\circ R_3(\mu)\circ R_1(I)\circ R_3(\vartheta)\circ\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{34}$$

where $I = \arccos(\Lambda/M)$, $J = \arccos(N/M)$, and $\vartheta = \lambda - \phi$, ϕ is the true anomaly of the distant body, and the simplifying assumption that the inertial plane is the same as the orbital plane has been made.

After carrying out the rotations involved in Eq. (34), it is found that the majority of perturbation terms that comprise Eq. (33) are factored by $\sin J$ or $\sin^2 J$ (see [13], for instance). Besides, typical values of the gravity-gradient perturbation for solar system bodies are small when compared with the torque-free rotation, say below 10^{-6}. Therefore, in those cases in which the inclination angle J is small, terms $\mathscr{O}(\sin\frac{1}{2}J)$ can be neglected. In this approximation, the only relevant terms of the gravity-gradient perturbation, in the new variables, are simply

$$\begin{aligned}\mathscr{D} = -\frac{n^2}{4}\frac{a^3}{r^3}\Big\{&\left(C-\frac{A+B}{2}\right)\left(2-3\sin^2 I+3\sin^2 I\cos 2\vartheta\right)\\ &+\frac{3}{4}(B-A)\left[(1-\cos I)^2\cos(2\vartheta-2g)+2\sin^2 I\cos 2g+(1+\cos I)^2\cos(2\vartheta+2g)\right]\Big\}.\end{aligned} \tag{35}$$

Notably, to this order of approximation, Eq. (35) does not depend either on ν or J. Hence, it is free also from x and X, as follows from Eqs. (12)–(13), and, in consequence, it is free from the complex variables u and U, as follows from Eqs. (20) and (21).

Now, the full Lie derivative in Eq. (27) is involved in the solution of the homological equation because g, and, therefore v is present in the perturbation. Still, the first summand in the square brackets of Eq. (27) vanishes for terms of the form $F(v,V,uU)$. Hence, because Eq. (35) is made precisely of terms of this type, and in view of the form of the second summand in the square brackets of Eq. (27), which only includes a factor uU, the homological equation is easily solved. Indeed, if we choose the new Hamiltonian term

$$\langle \mathscr{D} \rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathscr{D}\,\mathrm{d}g = -\frac{n^2}{4}\frac{a^3}{r^3}\left(C-\frac{A+B}{2}\right)\left(2-3\sin^2 I+3\sin^2 I\cos 2\vartheta\right),$$

and assume that there is not coupling with the previous terms of the perturbation theory, a particular solution of the homological equation for the order n corresponding to this term, is

$$S_n = -\frac{9n^2}{256}\frac{a^3}{r^3}\frac{(B-A)C\sqrt{\kappa}}{i\kappa^{3/2}uU-V}\left[(1+c)^2\sin(2\vartheta+2\sqrt{\kappa}v)+2s^2\sin 2\sqrt{\kappa}v-(1-c)^2\sin(2\vartheta-2\sqrt{\kappa}v)\right]. \quad (36)$$

5 Conclusions

Short-axis mode rotation of a free rigid body is naturally decomposed into a main problem and a perturbation, a fact that leads to the straightforward integration of the rotation by perturbation series. When using non-singular variables of the Poincaré type, the main problem has the form of a harmonic oscillator, whose frequency is related to the triaxiality of the rigid body, whereas the perturbation is a quartic polynomial. Then, the use of complex variables makes the construction of the perturbation solution trivial. The polynomial character of the perturbation does not persist, in general, when the motion is affected by external torques. However, when the rotation is close to the axis of maximum inertia, the gravity-gradient perturbation can also be approached in complex variables.

Acknowledgements

Partial support by the Spanish State Research Agency and the European Regional Development Fund under Projects ESP2016-76585-R and ESP2013-41634-P (AEI/ERDF, EU) are recognized. Discussions with A. Escapa, University of León, motivated the development of the current research.

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APPENDIX

Sadov's [38] solution of the torque free motion is summarized here for convenience of prospective readers. Note that the solution is constrained to the case of rotation about the axis of maximum inertia. Full details on its derivation can be found in [35].

Solution of the torque-free motion in action-angle variables

Starting from initial conditions in Andoyer variables (μ_0, ν_0, M, N_0), the time history of the rotation solution is computed as follows. First, compute the *characteristic*

$$f = \frac{C}{A}\frac{B-A}{C-B}, \tag{A1}$$

the energy integral

$$\Upsilon = \left(\frac{1}{A}\sin^2\nu + \frac{1}{B}\cos^2\nu\right)\frac{1}{2}\left(M^2 - N^2\right) + \frac{1}{2C}N^2, \tag{A2}$$

the moment of inertia-type auxiliary variable

$$\Delta = M^2/(2\Upsilon), \tag{A3}$$

the elliptic parameter

$$m = \frac{(C-\Delta)}{(\Delta-A)}\frac{(B-A)}{(C-B)}, \tag{A4}$$

and the elliptic argument ψ, from

$$\cos\psi = \frac{\sqrt{1+f}\sin\nu}{\sqrt{1+f\sin^2\nu}}, \qquad \sin\psi = \frac{\cos\nu}{\sqrt{1+f\sin^2\nu}}. \tag{A5}$$

Next, compute the constant frequencies

$$n_\ell = \frac{M}{2A}\frac{C-A}{C}\frac{\pi}{\mathrm{K}(m)}\sqrt{\frac{f}{(1+f)(f+m)}} \tag{A6}$$

where $\mathrm{K}(m)$ is the complete elliptic integrals of the first kind, and

$$n_g = -\frac{L}{G}n_\ell + \frac{G}{A}\left(1 - \frac{C-A}{C}\frac{f}{f+m}\right) \tag{A7}$$

Then, the initial conditions in action angle variables (ℓ_0, g_0, L_0, G) are evaluated from the sequence

$$\ell = -\frac{\pi}{2\mathrm{K}(m)}\mathrm{F}(\psi|m) \tag{A8}$$

$$g = \mu + \sqrt{1+f}\sqrt{\frac{f+m}{f}}\left[\frac{\Pi(-f|m)}{\mathrm{K}(m)}\mathrm{F}(\psi|m) - \Pi(-f,\psi|m)\right] \tag{A9}$$

$$L = \frac{2M}{\pi}\sqrt{1+f}\sqrt{\frac{f+m}{f}}\left[\Pi(-f|m) - \frac{m}{f+m}\mathrm{K}(m)\right] \tag{A10}$$

$$G = M. \tag{A11}$$

where $\mathrm{F}(\psi|m)$ and $\Pi(-f,\psi|m)$ are the incomplete elliptic integrals of the first and third kinds, respectively and $\Pi(-f|m)$ is the complete integral of the third kind.

Now, the solution to Hamilton equations

$$\ell = \ell_0 + n_\ell t, \qquad g = g_0 + n_g t, \tag{A12}$$

can be evaluated at any desired time. After that, the time history $(\mu(t), \nu(t), M, N(t))$ is recovered by first inverting Eq. (A8) to compute

$$\psi = \mathrm{am}\left(-2\frac{\mathrm{K}(m)}{\pi}\ell\,|m\right) \tag{A13}$$

where am notes the Jacobi's elliptic amplitude. Next, inverting Eq. (A5), $\nu(t)$ is unambiguously computed from

$$\cos\nu = \frac{\sqrt{1+f}\sin\psi}{\sqrt{1+f\sin^2\psi}}, \qquad \sin\nu = \frac{\cos\psi}{\sqrt{1+f\sin^2\psi}}, \tag{A14}$$

and $\mu(t)$ is computed by inverting Eq. (A9), viz.

$$\mu = g - \sqrt{1+f}\sqrt{(f+m)/f}\left[\frac{\Pi(-f|m)}{\mathrm{K}(m)}\mathrm{F}(\psi|m) - \Pi(-f,\psi|m)\right] \tag{A15}$$

Finally, $M = G$ and

$$N = G\sqrt{\frac{f}{f+m}}\sqrt{1-m\sin^2\psi}. \tag{A16}$$

It is worth mentioning that m is constant only in the case of torque-free motion, in which both L and G remains constant. However, this is not the case in the presence of external torques, where both L and G may vary, and hence the computation of the inverse transformation should start from the computation of m from the implicit equation

$$L = \frac{2G}{\pi}\sqrt{1+f}\sqrt{\frac{f+m}{f}}\left[\Pi(-f|m) - \frac{m}{f+m}\mathrm{K}(m)\right]. \tag{A17}$$

Note that the elliptic parameter, as defined in Eq. (A4), combines physical features, related to the principal moments of inertia, as well as dynamical properties, related to the energy. Still, it admits the decomposition $m = pf = p\beta\sigma$, where $0 \le \beta \le 1$ is the triaxiality coefficient

$$\beta = \frac{C(B-A)}{(C-B)A + (C-A)B},$$

previously defined implicitly in Eq. (3),

$$\sigma = \frac{B}{A}\frac{C-A}{C-B} > 1,$$

and

$$p = \frac{1/\Delta - 1/C}{1/A - 1/\Delta} = \frac{\Upsilon_{\max} - \Upsilon_{\min}}{\Upsilon_{\max} - \Upsilon} - 1,$$

in which $\Upsilon = \frac{1}{2}M^2/\Delta$ notes the energy—that is, the value to which the Hamiltonian evaluates for given initial conditions. Note that when the rotation happens close to the axis of maxima inertia, then $\Upsilon \to \Upsilon_{\min} \Rightarrow p \to 0$.

Thus, for a given value of σ we find two cases in which the elliptic modulus is small. Namely, when the triaxiality is small, or when the rotation takes place close to the axis of maximum inertia —the combination of both circumstances leading, of course, to the more favorable case.

It must be noted, however, that both cases are not completely independent, and care must be taken on the validity of the expansion. Indeed, in spite of a small triaxiality, if the rotation happens close to the separatrix $\Upsilon \approx \Upsilon_{\text{separatrix}}$ and the corresponding energy is close to the maximum $\Upsilon_{\text{separatrix}} \sim \Upsilon_{\max}$, then p could grow very high preventing convergence in m. On the other hand, even when the rotation takes place close to the axis of maximum inertia, if $B \sim C$, then σ can be very large, also preventing convergence in m.