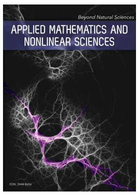
Applied Mathematics and Nonlinear Sciences 3(2) (2018) 447–458



Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

A Comparative Study on Haar Wavelet and Hosaya Polynomial for the numerical solution of Fredholm integral equations

S.C. Shiralashetti[a], H. S. Ramane[a], R.A. Mundewadi[b], R.B. Jummannaver[c]

a. P.G. Department of Mathematics, Karnatak University, Dharwad, India, E-mail: shiralashettisc@gmail.com, hsramane@yahoo.com

b. Department of Mathematics, P.A. College of Engineering, Mangalore, India, E-mail: rkmundewadi@gmail.com

c. Department of Mathematics, Bearys Institute of Technology, Mangalore, India, E-mail: rajesh.rbj065@gmail.com

Submission Info

Communicated by Juan L.G. Guirao
Received 28th april 2018
Accepted 5th September
Available online 5th September

Abstract

In this paper, a comparative study on Haar wavelet method (HWM) and Hosoya Polynomial method(HPM) for the numerical solution of Fredholm integral equations. Illustrative examples are tested through the error analysis for efficiency. Numerical results are shown in the tables and figures.

Keywords: Fredholm integral equation, Hosoya polynomial, Haar Wavelet, path
AMS 2010 codes: 65D30, 45B05

1 Introduction

Integral equations have motivated a large amount of research work in recent years. Integral equations find its applications in various fields of mathematics, science and technology has been studied extensively both at the theoretical and practical level. In particular, integral equations arise in fluid mechanics, biological models, solid state physics, kinetics in chemistry etc. In most of the cases, it is difficult to solve them, especially analytically [1]. Analytical solutions of integral equations, however, either does not exist or are difficult to find. It is precisely due to this fact that several numerical methods have been developed for finding solutions of integral equations.
Consider the Fredholm integral equation:

$$y(x) = f(x) + \int_0^1 k(x,t)y(t)dt \quad 0 \le x,t \le 1 \tag{1}$$

where $f(x)$ and the kernels $k(x,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le x,t \le 1$. We assume that Eq.(1) has a unique solution y to be determined. There are several numerical methods for approximating the solution

ISSN 2444-8656 doi:10.21042/AMNS.2018.2.00035

of Fredholm integral equations are known and many different basic functions have been used. Such as, Lepik et al. [2] applied the Haar Wavelets. Maleknejad et. al [3] applied a combination of Hybrid taylor and block-pulse functions, Rationalized haar wavelet [4], Hermite Cubic splines [5]. Muthuvalu et al. [6] applied Half-sweep arithmetic mean method with composite trapezoidal scheme. In previous paper [35], we compared with [5] for the numerical solution of Fredholm integral equations.

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [7, 8]. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations [2]. In the previous work, system analysis via Haar wavelets was led by Chen and Hsiao [9], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the applications for the Haar analysis into the dynamic systems. Recently, Haar wavelet method is applied for different type of problems. Namely, Bujurke et al. [10–12] used the single term Haar wavelet series for the numerical solution of stiff systems from nonlinear dynamics, nonlinear oscillator equations and Sturm-Liouville problems. Shiralashetti et al. [13–16, 18] applied for the numerical solution of Klein?Gordan equations, multi-term fractional differential equations, singular initial value problems,nonlinear Fredholm integral equations, Riccati and Fractional Riccati Differential Equations. Shiralashetti et al. [17] have introduced the adaptive gird Haar wavelet collocation method for the numerical solution of parabolic partial differential equations.

At present, field of a graph theory is a dynamic field in both theory and applications. Graphs can be utilized as a modeling tool to solve a many problems of practical importance. Structural graph theory is naturally a very wide field itself, and the current state of art is more advanced in some parts than in others.

A graph G consists of a finite nonempty set V of n points(vertex) together with a prescribed set X of m unordered pairs of distinct points of V. Each pair $x = u, v$ of points in X is a edge of G, and x is said to join u and v. We write $x = uv$ and say that u and v are adjacent points (sometimes denoted u adj v).

Let G be a simple connected graph with a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_n\}$, where $|V(G)| = n$ and $|E(G)| = m$. The *degree* of a vertex v in G is the number of edges incident to it and denoted by $deg_G(v)$. The *distance* between the vertices v_i and v_j in G is equal to the length of the shortest path joining them and is denoted by $d(v_i, v_j|G)$.

The *Wiener index* $W = W(G)$ of a graph G is defined as the sum of the distances between all pairs of vertices of G, that is

$$W = W(G) = \sum_{1 \le i < j \le n} d(u_i, v_j|G).$$

This index was putforward by Wiener [21] in 1947 for approximating the boiling points of alkanes. The effect of approximation was surprisingly good. From that point forward, the Wiener index has attracted the attention of chemists.

Recently several modifications of the Wiener index were put forward, of which we mention the following: the hyper–Wiener index [22, 23]:

$$WW = WW(G) = \frac{1}{2} \sum_{u<v} \left[d(u,v)^2 + d(uv)\right] \tag{2}$$

the Harary index [24]:

$$H(G) = \sum_{u<v} \frac{1}{d(u,v)} \tag{3}$$

It is worth noting that all the above structure–descriptors are either special cases of, or are simply related to the graph invariant W_λ , defined as [25, 26]

$$W_\lambda = W_\lambda(G) = \sum_{k\geq 1} d(G,k)\, k^\lambda \tag{4}$$

where $d(G,k)$ is the number of pairs of vertices of the graph G whose distance is k , and where λ is some real (or complex) number. Evidently,

$$\begin{aligned} W &= W_1 \\ WW &= \frac{1}{2}W_2 + \frac{1}{2}W_1 \\ Ha &= W_{-2} \\ RW &= W_{-1} \ . \end{aligned}$$

Another related quantity is the Hosoya polynomial. In the present work, a comparison of a Haar wavelet and Hosoya polynomial methods for the numerical solution of Fredholm integral equations is proposed .

2 Properties of Haar wavelet and Hosoya polynomial

2.1 Haar wavelet

The scaling function $h_1(x)$ for the family of the Haar wavelets is defined as

$$h_1(x) = \begin{cases} 1 \ for \ \ x \in [0,1) \\ 0 \ otherwise \end{cases} \tag{5}$$

The Haar wavelet family for $x \in [0,1)$ is defined as,

$$h_i(x) = \begin{cases} 1 \quad for \ \ x \in [\alpha, \beta) \\ \\ -1 \ for \ \ x \in [\beta, \gamma) \\ \\ 0 \quad otherwise \end{cases} \tag{6}$$

where $\alpha = \dfrac{k}{m}$, $\beta = \dfrac{k+0.5}{m}$, $\gamma = \dfrac{k+1}{m}$
In the above definition the integer, $m = 2^l$, $l = 0,1,...,J$, indicates the level of resolution and integer $k = 0,1,...,m-1$ is the translation parameter. Maximum level of resolution is J. The index i in (6) is calculated using, $i = m+k+1$. In case of minimal values $m = 1$ and $k = 0$ then $i = 2$. The maximal value of i is $N = 2^{J+1}$.
Let us define the collocation points $x_j = \dfrac{j-0.5}{K}$, $j = 1,2,...,N$, discretize the Haar function $h_i(x)$ and the corresponding Haar coefficient matrix $H(i,j) = (h_i(x_j))$, which has the dimension $N \times N$.
If $J = 3 \Rightarrow N = 16$,
Eq. (6) generates the Haar matrix,

$$H(16,16) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

As Haar Wavelets are orthogonal; this mean that any square integrable function over $[0,1]$ can be expanded into Haar wavelets series as:

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \tag{7}$$

where a_i's are Haar wavelet coefficients.
If $f(x)$ be piecewise constant, then sum can be terminated to finite term, that is

$$f(x) = \sum_{i=1}^{N} a_i h_i(x_i) = a^T H \tag{8}$$

$$a^T = (a_1, a_2, ..., a^N), H = (h_1(x), h_2(x), ..., h_N(x))^T. \tag{9}$$

2.2 Hosoya polynomial

The Hosoya polynomial of a graph is a generating function about distance distributing, introduced by Hosoya [27] in 1988. For a connected graph G, the *Hosoya polynomial* denoted by $H(G,\lambda)$ is defined as

$$H(G,\lambda) = \sum_{k\geq 0} d(G,k)\lambda^k \tag{10}$$

where $d(G,k)$ is the number of pairs of vertices of G that are at distance k and λ is the parameter.

Comparing (4) and (10) we see that both W_λ and $H(\lambda)$ are determined by the numbers $d(G,k)$, $k = 1,2,\dots$. Indeed, from knowing $H(G,\lambda)$ one can deduce $W_\lambda(G)$ and vice versa. Thus, from the Hosoya polynomial it is possible to calculate a large variety of distance–based molecular structure–descriptors.

The connection between the Hosoya polynomial and the Wiener index is elementary [27,28]:

$$W(G) = H'(G,1)$$

where $H'(G,\lambda)$ is the first derivative of $H(G,\lambda)$. The hyper–Wiener index can be computed from the first and second derivatives as [29]

$$WW(G) = H'(G,1) + \frac{1}{2}H''(G,1) .$$

The *path* on n vertices (or of length $n-1$) is the graph with n vertices-say, $1,2,.,n$ and with $n-1$ edges, such that vertices i and $i+1$ are adjacent, $i=1,2,...,n-1$. For any positive integer n we denote path as P_n, then Hosoya polynomial of path is:

$$
\begin{aligned}
H(P_1,\lambda) &= \sum_{k\geq 0} d(P_1,k)\lambda^k = 1 \\
H(P_2,\lambda) &= \sum_{k\geq 0} d(P_2,k)\lambda^k = \lambda+2 \\
H(P_3,\lambda) &= \sum_{k\geq 0} d(P_3,k)\lambda^k = \lambda^2+2\lambda+3 \\
&. \\
&. \\
&. \\
H(P_n,\lambda) &= \sum_{k\geq 0} d(P_n,k)\lambda^k = n+(n-1)\lambda+(n-2)\lambda^2+...+[n-(n-2)]\lambda^{n-2}+[n-(n-1)\lambda^{n-1}]
\end{aligned}
$$

For more information about the *Hosoya polynomial* one can refer [30–34].
A function $f(x) \in L^2[0,1]$ is expanded as:

$$f(x) = \sum_{i=1}^{n} c_i H(P_i,x) = C^T H_P(x), \tag{11}$$

where C and $H_P(x)$ are $n\times 1$ matrices given by:

$$C = [c_1,c_2,...,c_n]^T, \tag{12}$$

and

$$H_P(x) = [H(P_1,x),H(P_2,x),...,H(P_n,x)]^T. \tag{13}$$

3 Method of Solution

3.1 Haar Wavelet Method(HWM)

Let us consider the equation

$$y(x) - \int_0^1 k(x,t)y(t)dt = f(x), \quad x,t \in [0,1] \tag{14}$$

and seek the solution in the form

$$y(x) = \sum_{i=1}^{N} a_i h_i(x) \tag{15}$$

where a_i's are Haar wavelet coefficients.
Substituting (15) in (14), we get

$$\sum_{i=1}^{N} a_i h_i(x) - \sum_{i=1}^{N} a_i G_i(x) = f(x), \tag{16}$$

where

$$G_i(x) = \int_0^1 K(x,t)h_i(t)dt$$

Substituting the collocation points in Eq. (16), we get a system of algebraic equations

$$\sum_{i=1}^{N} [a_i h_i(x_l) - a_i G_i(x_l)] = f(x_l), \quad l = 1,2,...,N \tag{17}$$

Matrix form of this system of Eq. (17) is as follows,

$$a(H-G) = F,$$

Substituting the Coefficients a_i in Eq. (15), we get $y = (y(x_l))$ is the approximate solution

$$y = aH$$

where, $F = f(f(x_l)), G = (G_i(x_l))$
In the present case

$$G_i(x_l) = \int_0^1 K(x,t)h_i(t)dt = \begin{cases} x+0.5 & for\ i=1 \\ \frac{-1}{4m^2} & for\ i>1 \end{cases} \tag{18}$$

where $K(x,t) = x+t$.

3.2 Hosoya Polynomial Method (HPM)

Consider the Fredholm integral equation,

$$y(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \quad 0 \le x\, t \le 1 \tag{19}$$

To solve Eq. (19), the procedure is as follows:
Step 1: We first approximate $y(x)$ as truncated series defined in Eq. (11). That is,

$$y(x) = C^T H_P(x) \tag{20}$$

where C and $H_P(x)$are defined similarly to Eqs. (12) and (13).
Step 2: Then substituting Eq. (20) in Eq. (19), we get,

$$C^T H_P(x) = f(x) + \int_0^1 K(x,t)[C^T H_P(t)]dt \tag{21}$$

Step 3: Substituting the collocation point $x_i = \frac{i-0.5}{n}$, $i = 1,2,...,n$ in Eq. (21). Then we obtain,

$$C^T H_P(x_i) = f(x_i) + C^T \int_0^1 K(x_i,t)H_P(t)dt \tag{22}$$

$$C^T (H_P(x_i) - Z) = f, \text{ where } Z = \int_0^1 K(x_i,t)H_P(t)dt$$

Step 4: Now, we get the system of algebraic equations with unknown coefficients.

$$C^T K = f, \text{ where } K = (H_P(x_i) - Z)$$

Solving the above system of equations, we get the Hosoya coefficients 'C' and then substituting these coefficients in Eq. (20), we get the required approximate solution of Eq. (19).

4 Numerical Examples

In this section, we consider the some of the demonstrate the capability of the method and error function is presented to verify the accuracy and efficiency of the following numerical results:

$$Errorfunction = \|y_e(x_i) - y_a(x_i)\|_\infty = \sqrt{\sum_{i=1}^{n} (y_e(x_i) - y_a(x_i))^2}$$

where y_e and y_a are the exact and approximate solution respectively.
Example 1. Consider Fredholm integral equation of the second kind [20],

$$y(x) = x^6 log(x) + \int_0^1 (x+t)y(t)dt, \;\; 0 \leq x \leq 1. \tag{23}$$

which has the exact solution $y(x) = x^6 log(x) + 0.3096x + 0.1752$.

Table 1: Numerical result of Example 1.

x	Exact solution	HWM (N = 8)	Abs. Error (HWM)	HPM (n = 8)	Abs. Error (HPM)
0.0625	0.1945	0.2046	1.00e-02	0.1947	1.94e-04
0.1875	0.2332	0.2453	1.20e-02	0.2334	2.36e-04
0.3125	0.2709	0.2850	1.41e-02	0.2711	2.78e-04
0.4375	0.3049	0.3211	1.62e-02	0.3052	3.19e-04
0.5625	0.3311	0.3495	1.83e-02	0.3315	3.61e-04
0.6875	0.3485	0.3689	2.04e-02	0.3489	4.03e-04
0.8125	0.3670	0.3895	2.25e-02	0.3675	4.45e-04
0.9375	0.4216	0.4462	2.45e-02	0.4221	4.86e-04

Example 2. Consider the Fredholm integral equation of the second kind [2],

$$y(x) = x^2 + \int_0^1 (x+t)y(t)dt, \;\; 0 \leq x \leq 1. \tag{24}$$

which has the exact solution $y(x) = x^2 - 5x - (17/6)$. We solved the Eq. (24) by Hosoya polynomial method for $n = 3$, we get the Hosoya coefficient $C_1 = \frac{49}{6}, C_2 = -7, C_3 = 1$. Substituting these coefficients in Eq. (20) we obtain,

$$y(x) = \frac{49}{6} H_P(x_1) + (-7) H_P(x_2) + (1) H_P(x_3)$$
$$y(x) = x^2 - 5x - (17/6).$$

Solved by analyticaly, which gives the exact solution of this problem.
Example 3. Consider the Fredholm integral equation of the second kind [3],

$$y(x) = exp(x) + (1 - exp(1))x - 1 + \int_0^1 (x+t)y(t)dt, \;\; 0 \leq x \leq 1. \tag{25}$$

which has the exact solution $y(x) = exp(x)$.

Table 2: Numerical result of Example 3.

x	Exact solution	HWM (N = 8)	Abs. Error (HWM)	HPM (n = 8)	Abs. Error (HPM)
0.0625	1.0645	1.0892	2.47e-02	1.0645	5.50e-09
0.1875	1.2062	1.2362	2.99e-02	1.2062	6.56e-09
0.3125	1.3668	1.4021	3.52e-02	1.3668	7.63e-09
0.4375	1.5488	1.5893	4.04e-02	1.5488	8.69e-09
0.5625	1.7551	1.8008	4.57e-02	1.7551	9.77e-09
0.6875	1.9887	2.0397	5.09e-02	1.9887	1.08e-08
0.8125	2.2535	2.3098	5.62e-02	2.2535	1.19e-08
0.9375	2.5536	2.6151	6.14e-02	2.5536	1.30e-08

Table 3: Error Analysis.

Haar Wavelet Method (HWM)				Hosoya Polynomial Method (HPM)		
N	Example 1	Example 2	Example 3	n	Example 1	Example 3
8	2.46e-02	7.2e-02	2.17e-02	3	1.25e-01	1.83e-02
16	6.70e-03	1.7e-02	1.55e-02	4	6.89e-02	2.15e-03
32	2.10e-03	4.3e-03	3.90e-03	6	5.64e-03	6.86e-06
64	9.25e-04	1.3e-02	9.82e-04	8	4.86e-04	1.30e-08

Example 4. Consider the Fredholm integral equation of the second kind [19],

$$y(x) = e^{2x+(\frac{1}{3})} + \int_0^1 \frac{-1}{3} e^{2x-(\frac{5}{3})t} y(t)dt, \ \ 0 \le x \le 1. \tag{26}$$

which has the exact solution $y(x) = e^{2x}$. Applying the proposed method to solve Eq. (26) for $n = 8$. We obtain the approximate solution $y(x)$ as shown in table 4. Error analysis is shown in figure 1.

Table 4: Numerical result of Example 4.

x	Exact solution	Method [19] ($m = 128$)	Abs. Error (Method [19])	HPM (n = 8)	Abs. Error (HPM)
0	1.00000	1.00004	4.00e-05	0.99999	7.39e-06
0.1	1.22140	1.22147	6.72e-05	1.22140	6.61e-07
0.2	1.49182	1.49180	2.46e-05	1.49182	1.35e-09
0.3	1.82211	1.82218	6.12e-05	1.82211	4.61e-08
0.4	2.22554	2.22551	3.09e-05	2.22554	1.29e-07
0.5	2.71828	2.71822	6.18e-05	2.71828	6.17e-08
0.6	3.32011	3.32017	5.30e-05	3.32011	1.73e-07
0.7	4.05519	4.05529	9.00e-05	4.05519	1.31e-07
0.8	4.95303	4.95306	2.75e-05	4.95303	1.25e-07
0.9	6.04964	6.04961	3.74e-05	6.04964	9.70e-07
1	7.38905	7.38901	4.60e-05	7.38905	8.99e-06

Example 5. Consider the Fredholm integral equation of the second kind [4],

$$y(x) = e^x - \frac{e^{x+1}-1}{x+1} + \int_0^1 e^{xt} y(t)dt, \ \ 0 \le x \le 1. \tag{27}$$

which has the exact solution $y(x) = e^x$. Applying the proposed method to solve Eq. (27) for N = 8. We obtain the approximate solution $y(x)$ as shown in table 5. Error analysis is shown in figure 2.

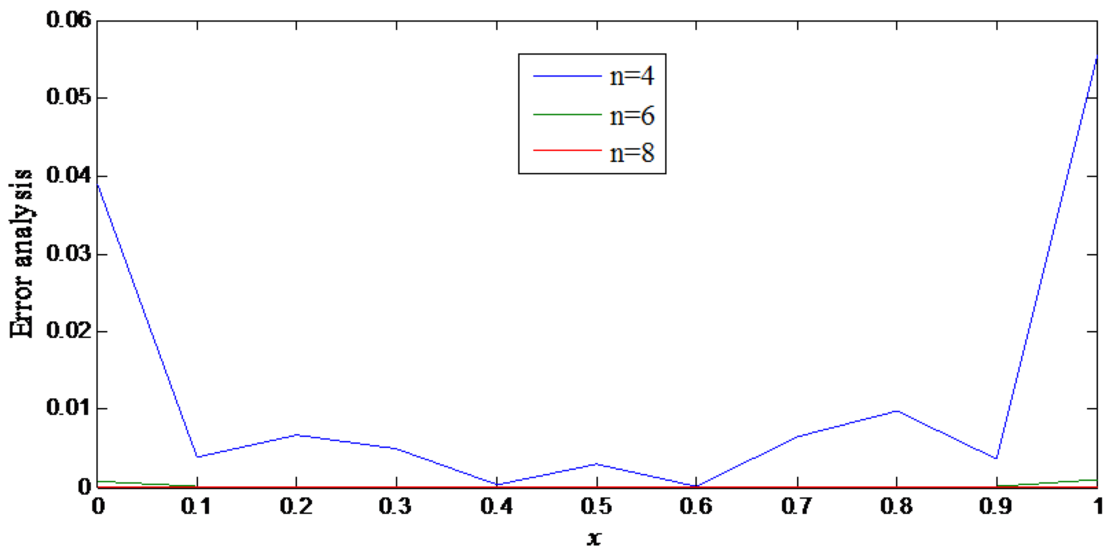


Fig. 1 Error analysis of Example 4.

Table 5: Numerical result of Example 5.

x	Exact solution	Method [4] ($k = 32$)	Abs. Error (Method [4])	HPM ($n = 8$)	Abs. Error (HPM)
0	1.00000	1.01642	1.64e-02	1.00000	3.79e-06
0.1	1.10517	1.11627	1.10e-02	1.10517	1.82e-07
0.2	1.22140	1.22593	4.52e-03	1.22140	4.30e-07
0.3	1.34985	1.34637	3.48e-03	1.34985	4.56e-07
0.4	1.49182	1.47864	1.31e-03	1.49182	4.88e-07
0.5	1.64872	1.62391	2.48e-02	1.64872	5.39e-07
0.6	1.82211	1.84004	1.79e-02	1.82211	5.80e-07
0.7	2.01375	2.02082	7.06e-03	2.01375	6.34e-07
0.8	2.22554	2.21936	6.18e-03	2.22554	6.90e-07
0.9	2.45960	2.43742	2.21e-02	2.45960	7.18e-07
1	2.71828	2.67690	4.13e-02	2.71828	1.10e-06

5 Conclusion

A newly developed method Hosoya polynomial method (HPM) [35] is applied for the numerical solution of Fredholm integral equations and compared with the Haar Wavelet Method (HWM)). Both the methods reduces an integral equation into a set of algebraic equations. Numerical results of the Hosoya polynomial method (HPM) gives higher accuracy with exact ones and existing method (HWM). Some illustrative examples are tested through the error analysis. This shows that, the accuracy improves with increasing the n number of vertices of polynomial for better accuracy. Error analysis justifies the comparative study of a new developed method is effective, validity and applicability with existing method.

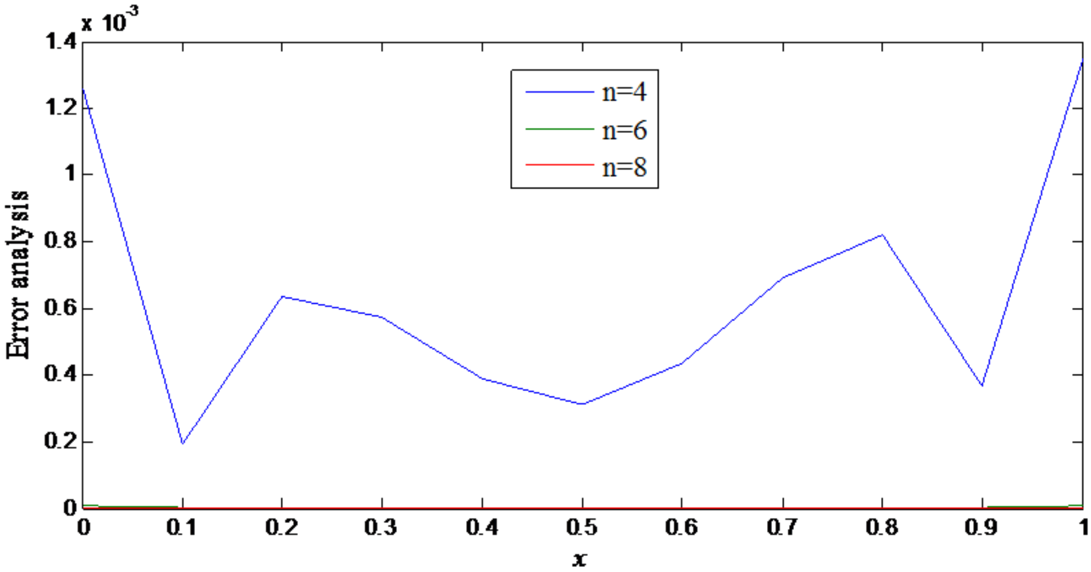


Fig. 2 Error analysis of Example 5.

Acknowledgements

The authors thank for the financial support of UGC's UPE Fellowship vide sanction letter D. O. No. F. 14-2/2008(NS/PE), dated-19/06/2012 and F. No. 14-2/2012(NS/PE), dated 22/01/2013.

References

[1] A.M. Wazwaz. A First Course in Integral Equations. WSPC. New Jersey. 1997.
[2] U. Lepik, E. Tamme, Application of the Haar wavelets for solution of linear integral equations, in: Dynamical Systems and Applications, Antala. Proce. (2004) 494-507.
[3] K. Maleknejad, Y. Mahmoudi, Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block-Pulse functions, App. Math. Comp. 149 (2004) 799-806.
[4] K. Maleknejad, F. Mirzaee, Using rationalized Haar wavelet for solving linear integral equations, App. Math. Comp. 160 (2005) 579-587.
[5] K. Maleknejad, M. Yousefi, Numerical solution of the integral equation of the second kind by using wavelet bases of hermite cubic splines, App. Math. Comp. 183 (2006) 134-141.
[6] M. S. Muthuvalu, J. Sulaiman, Half-Sweep Arithmetic Mean method with composite trapezoidal scheme for solving linear fredholm integral equations, App. Math. Comp. 217 (2011) 5442-5448.
[7] G. Beylkin, R. Coifman, V. Rokhlin, Fast wavelet transforms and numerical algorithms I, Commun. Pure Appl. Math. 44 (1991) 141-183.
[8] C. K. Chui, Wavelets: A Mathematical Tool for Signal Analysis, SIAM, Philadelphia, PA, 1997.
[9] C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed parameter systems, IEEE Proc. Pt. D. 144 (1) (1997) 87-94.
[10] N. M. Bujurke, S. C. Shiralashetti, C. S. Salimath, Numerical solution of stiff systems from non-linear dynamics using single term haar wavelet series, Inter. Jour. Nonlin. Dynam. 51 (2008) 595-605.
[11] N. M. Bujurke, S. C. Shiralashetti, C. S. Salimath, An Application of Single Term Haar Wavelet Series in the Solution of non-linear oscillator Equations, Jour. Comp. Appl. Math. 227 (2010) 234-244.
[12] N. M. Bujurke, S. C. Shiralashetti, C. S. Salimath, Computation of eigenvalues and solutions of regular Sturm-Liouville problems using Haar wavelets, Jour. Comp. Appl. Math. 219 (2008) 90-101.
[13] S.C. Shiralashetti, L. M. Angadi, A. B. Deshi, M. H. Kantli, Haar wavelet method for the numerical solution of Klein?Gordan equations, Asian-European J. Math. 9(01) (2016) 1650012.
[14] S.C. Shiralashetti, A.B. Deshi, An efficient haar wavelet collocation method for the numerical solution of multi-term fractional differential equations. Nonlinear Dyn. 83 (2016) 293-303.

[15] S.C. Shiralashetti, A. B. Deshi, P. B Mutalik Desai, Haar wavelet collocation method for the numerical solution of singular initial value problems, Ain Shams Eng. J. 7(2) (2016) 663-670.
[16] S.C. Shiralashetti, A.B. Deshi, Haar Wavelet Collocation Method for Solving Riccati and Fractional Riccati Differential Equations, Bulletin. Math. Sci. Appl. 17 (2016) 46-56.
[17] S.C. Shiralashetti, L. M. Angadi, M. H. Kantli, A. B. Deshi, Numerical solution of parabolic partial differential equations using adaptive gird Haar wavelet collocation method, Asian-European J. Math. (2016), 1750-026.
[18] S.C. Shiralashetti, R. A. Mundewadi, Leibnitz-Haar Wavelet Collocation Method for the Numerical Solution of Nonlinear Fredholm Integral Equations. Inter. J. Eng. Sci. Res. Tech. 5(9) (2016) 264-273.
[19] M. H. Reihani, Z. Abadi, Rationalized Haar functions method for solving Fredholm and Volterra integral equations, J. Comp. Appl. Math. 200 (2007) 12?20.
[20] V. Mishra, H. Kaur and R. C. Mittal. Haar wavelet algorithm for solving certain differential, integral and integro-differential equations. Int. J. of Appl. Math. Mech., 8 (6) (2012): 69-82.
[21] H. Wiener, Structural determination of paraffin boiling points, J. Amer. Chem. Soc. 69 (1947) 17-20.
[22] I. Gutman, Relation between hyper–Wiener and Wiener index, Chem. Phys. Lett. 364 (2002) 352-356.
[23] D. J. Klein, I. Lukovits, I. Gutman, *On the definition of the hyper–Wiener index for cycle–containing structures* J. Chem. Inf. Comput. Sci. 35 (1995) 50-52.
[24] D. Plavšić, S. Nikolić, N. Trinajstić, Z. Mihalić, On the Harary index for the characterization of chemical graphs, J. Math. Chem. 12 (1993) 235-250.
[25] I. Gutman, A. A. Dobrynin, S. Klavžar, L. Pavlović, Wiener–type invariants of trees and their relation, Bull. Inst. Comb. Appl. 40 (2004) 23-30.
[26] D. J. Klein, I. Gutman, Wiener–number–related sequences, J. Chem. Inf. Comput. Sci. 39 (1999) 534-536.
[27] H. Hosoya, On some counting polynomials in chemistry, Discrete Applied Mathematics, 19 (1-3)(1988) 239-257.
[28] E. V. Konstantinova, M. V. Diudea, *The Wiener polynomial derivatives and other topological indices in chemical research*, Croat. Chem. Acta 73 (2000) 383-403.
[29] G. Cash, S. Klavžar, M. Petkovšek, *Three methods for calculation of the hyper–Wiener index of molecular graphs*, J. Chem. Inf. Comput. Sci. 43 (2002) 571-576.
[30] I. Gutman, Hosoya polynomial and the distance of the total graph of a tree, Publ. Elektrotehn. Fak. (Beograd) Ser. Mat., 10 (1999) 53-58.
[31] I. Gutman, S. Klavžar, M. Petkovšek, P. Žigert, On Hosoya polynomials of benzenoid graphs, MATCH Commun. Math. Comput. Chem., 43 (2001) 49-66.
[32] M. Lepović, I. Gutman, A collective property of trees and chemical trees, J. Chem. Inf. Comput. Sci., 38 (1998) 823-826.
[33] K. P. Narayankar, S. B. Lokesh, V. Mathad, I. Gutman, Hosoya polynomial of Hanoi graphs, Kragujevac J. Math., 36 (2012) 51-57.
[34] H. S. Ramane, K. P. Narayankar, S. S. Shirkol, A. B. Ganagi, Terminal Wiener index of line graphs, MATCH Commun. Math. Comput. Chem., 69 (2013) 775-782.
[35] H. S. Ramane, S.C. Shiralashetti, R. A. Mundewadi, R. B. Jummannaver, Numerical Solution of Fredholm Integral Equations Using Hosoya Polynomial of Path Graphs, American Journal of Numerical Analysis, 2017, Vol. 5, No. 1, 11-15.