Applied Mathematics and Nonlinear Sciences 3(2) (2018) 419–426

Applied Mathematics and Nonlinear Sciences

https://www.sciendo.com

# Hamilton-connectivity of Interconnection Networks Modeled by a Product of Graphs

Donglin Liu[a], Chunxiang Wang[b], Shaohui Wang[c†].

a. *School of Mechanical and Electronic Engineering, Wuhan University of Technology, Wuhan, PR China*

b. *School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P.R. China*

c. *Department of Mathematics, Savannah State University, Savannah, GA 31404, USA*

**Submission Info**

Communicated by Juan L.G. Guirao
Received 15th march 2018
Accepted 30th July 2018
Available online 30th July 2018

## Abstract

The product graph $G_m * G_p$ of two given graphs $G_m$ and $G_p$, defined by J.C. Bermond et al.[J Combin Theory, Series B 36(1984) 32-48] in the context of the so-called $(\triangle, D)$-problem, is one interesting model in the design of large reliable networks. This work deals with sufficient conditions that guarantee these product graphs to be hamiltonian-connected. Moreover, we state product graphs for which provide panconnectivity of interconnection networks modeled by a product of graphs with faulty elements.

**Keywords:** panconnected; fault-hamiltonicity; fault tolerance, interconnection networks
**AMS 2010 codes:** 05C40, 05c45

## 1 Introduction

Linear arrays and rings are two of the most important computational structures in interconnection networks. So, embedding of linear arrays and rings into a faulty interconnected network is one of the important issues in parallel processing. An interconnection network is often modeled as a graph, in which vertices and edges correspond to nodes and communication links, respectively. Thus, the embedding problem can be modeled as finding fault-free paths and cycles in the graph with some faulty vertices and/or edges. In the embedding problem, if the longest path or cycle is required the problem is closely related to well-known hamiltonian problems in graph theory. In the rest of this paper, we will use standard terminology in graphs(see ref.[2]). It is very difficult to determine that a graph is hamiltonian or not. Readers may refer to [4,5,6].

†Corresponding author.
Email addresses: (D. Liu)liudong0@whut.edu.cn; (C. Wang)wcxiang@mail.ccnu.edu.cn; (S. Wang)shaohuiwang@yahoo.com

ISSN 2444-8656 doi:10.21042/AMNS.2018.2.00032

## 2 Definitions and Notation

We follow [2] for graph-theoretical terminology and notation not defined here. A graph $G = (V, E)$ always means a simple graph(without loops and multiple edges), where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. The *degree* of a vertex $v$ is denoted by $d(v) = d_G(v)$, whereas $\delta = \delta(G)$ and $\triangle = \triangle(G)$ stand for the minimum degree and the maximum degree of $G$, respectively. A *path* is a sequence of adjacent vertices, denoted by $x_1x_2\cdots x_\ell$, in which all the vertices $x_1, x_2, \cdots, x_\ell$ are distinct. Let path $P = x_1, x_2, \cdots, x_\ell$ and $P[x_i, x_j]$ denotes the sequence of adjacent vertices from $x_i$ to $x_j$ on path $P$. That is, $P[x_i, x_j] = x_ix_{i+1}\cdots x_{j-1}x_j$. For two vertices $u, v \in V(G)$, a path joining $u$ and $v$ is called a $uv$-path. A path $P$ in graph $G$ is call a Hamiltonian path if $V(P) = V(G)$. A *cycle* is a path such that the first vertex is the same as the last one. A cycle is also denoted by $x_1x_2\cdots x_\ell x_1$. *Length* of a cycle is the number of edges in it. An $\ell$-cycle is a cycle of length $\ell$. Let $c(G) = max\{\ell : \ell\text{-cycle of } G\}$. A cycle of $G$ is called *Hamiltonian cycle* if its length is $|V(G)|$. A graph $G$ is *Hamiltonian* if $G$ contains a Hamiltonian cycle. A graph $G$ is *Hamiltonian-connected* if any two vertices $u, v \in V(G)$ exists a Hamiltonian $uv$-path.

Two edges $xy, uv \in E(G)$ are called *independent* if $\{x, y\} \cap \{u, v\} = \emptyset$. A *matching* is a set of edges that are pairwise independent. A *perfect matching* between two disjoint graphs $G_1$, $G_2$ with the same order $n$ is a matching consisting of $n$ edges such that each of them has one end vertex in $G_1$ and the other one in $G_2$. See [7–14].

The construction of new graphs from two given ones is not unusual at all. Basically, the method consists of joining together several copies of one graph according to the structure of another one, the latter being usually called the main graph of the construction. In this regard, Chartrand and Harary introduced in [3] the concept of permutation graph as follows. For a graph $G$ and a permutation $\pi$ of $V(G)$, the permutation graph $G^\pi$ is defined by taking two disjoint copies of $G$ and adding a perfect matching joining each vertex $v$ in the first copy to $\pi(v)$ in the second. Examples of these graphs include hypercubes, prisms, and some generalized Petersen graphs. The product graph $G_m * G_p$ of two given graphs $G_m$ and $G_p$, defined in [1] by Bermond et al. in the following way.

**Definition 1.** Let $G_m = (V(G_m), E(G_m))$ and $G_p = (V(G_p), E(G_p))$ be two graphs. Let us give an arbitrary orientation to the edges of $G_m$, in such a way that an arc from vertex $x$ to vertex $y$ is denoted by $e_{xy}$. For each arc $e_{xy}$, let $\pi_{e_{xy}}$ be a permutation of $V(G_p)$. Then the product graph $G_m * G_p$ has $V(G_m) \times V(G_p)$ as vertex set, with two vertices $(x, x'), (y, y')$ being adjacent if either

$$x = y \qquad \textit{and} \qquad x'y' \in E(G_p)$$

or

$$xy \in E(G_m) \qquad \textit{and} \qquad y' = \pi_{e_{xy}}(x').$$

The product graph $G_m * G_p$ can be viewed as formed by $|V(G_m)|$ disjoint copies of $G_p$, each arc $e_{xy}$, indicating that some perfect matching between the copies $G_p^x, G_p^y$( respectively generated by the vertices $x$ and $y$ of $G_m$) is added. So the graph $G_m$ is usually called the *main graph* and $G_p$ is called the *pattern graph* of the product graph $G_m * G_p$ . Moreover, every edge of $G_m * G_p$ that belongs to any of the $|E(G_m)|$ perfect matchings between copies of $G_p$ is an *cross* edge of $G_m * G_p$.

Observe that if we choose $\pi_{e_{xy}}(x') = x'$ for any arc $e_{xy}$ then $G_m * G_p = G_m \Box G_p$. Furthermore, if $G_m$ is $K_2$ we have $K_2 * G = G^\pi$, a permutation graph. Hence, $G_m * G_p$ can be considered as a generalized permutation graph.

**Definition 2.** A graph $G$ is called $f-$fault hamiltonian (resp. $f$-fault hamiltonian-connected) if there exist a hamiltonian cycle (resp. if each pair of vertices are joined by a hamiltonian path) in $G - F$ for any set $F$ of faulty elements with $|F| \leq f$.

If a graph $G$ is $f-$fault hamiltonian (resp. $f-$fault hamiltonian-connected), then it is necessary that $f \leq \delta(G) - 2$ (resp. $f \leq \delta(G) - 3$), where $\delta(G)$ is the minimum degree of $G$.

**Definition 3.** A graph $G$ is call $f-$fault $q$-panconnected if each pair of fault-free vertices are joined by a path in $G-F$ of every length from $q$ to $|V(G-F)|-1$ inclusive for any set $F$ of fault elements with $|F| \leq f$.

When we are to construct a path from $s$ to $t$, $s$ and $t$ are called a source and a sink, respectively, and both of them are called terminals. When we find a path/cycle, sometimes we regard some fault-free vertices and edges as faulty elements. They are called *virtual* faults.

## 3 Main Results

For the sake of convenience, we write $m(p)$ for the order of $G_m$( $G_p$, respectively.) Suppose $V(G_m) = \{x_1, x_2, \cdots, x_m\}$ and $V(G_p) = \{y_1, y_2, \cdots, y_p\}$.

**Theorem 1.** *If $G_m$ is hamiltonian, and $G_p$ is hamiltonian-connected, then $G_m * G_p$ is also hamiltonian-connected.*

*Proof.* Choose two vertices in $u, v \in V(G_m * G_p)$. We have the following cases:

Case 1. $u, v$ belong to the same subgraph, say, $G_P^{x_1}$. Since $G_m$ is hamiltonian, suppose hamiltonian-cycle $C = x_1 x_2 \cdots x_m$. Since $G_P^{x_1}$ is hamiltonian-connected, $u, v$ are connected by a hamiltonian-path $P_1$ in $G_P^{x_1}$, suppose $P_1 = (x_1, y_1)(x_1, y_2)\cdots(x_1, y_p)$, where $(x_1, y_1) = u, (x_1, y_p) = v$. Since $x_1 x_m \in E(G_m)$, suppose $(x_1, y_2)(x_m, y_2) \in E(G_m * G_p)$. Then $u, v$ are connected by a hamiltonian-path $P'$ in $G_m * G_p$ by choosing the endvertex, say, $(x_2, y_1)$ of cross edge $(x_1, y_1)(x_2, y_1)$ and have a hamiltonian-cycle $C_2$ passing $(x_2, y_1)$ in $G_P^{x_2}$. Suppose the last vertex is $(x_2, y_p)$ in $C_2$, and choosing the endvertex, say, $(x_3, y_p)$ of cross edge $(x_3, y_p)(x_2, y_p)$ and have a hamiltonian-cycle $C_3$ passing $(x_3, y_p)$ in $G_P^{x_3}$. Suppose the last vertex is $(x_3, y_l)$ in $C_3$, and so on, choosing the endvertex, say, $(x_m, y_j)$ $(1 \leq j \leq p\ )$ of cross edge $(x_{m-1}, y_1)(x_m, y_j)$ and choosing the endvertex $(x_m, y_2)$ of cross edge $(x_1, y_2)(x_m, y_2)$, have a hamiltonian-path $P_m$ passing $(x_m, y_j)$ and $(x_m, y_2)$ in $G_P^{x_m}$. Let $P' = u(x_2, y_1) \cup C_2[(x_2, y_1), (x_2, y_p)] \cup (x_2, y_p)(x_3, y_p) \cup C_3[(x_3, y_p), (x_3, y_l)] \cup \cdots \cup (x_{m-1}, y_1)(x_m, y_j) \cup P_m[(x_m, y_j), (x_m, y_2)] \cup (x_m, y_2)(x_1, y_2) \cup P_1[(x_1, y_2), v]$. Then we obtain the desired result.

Case 2. $u, v$ belong to two subgraphs, suppose $G_P^{x_1}$, $G_P^{x_i}$ $(1 \leq i \leq m)$. and $u \in G_P^{x_1}$ and $v \in G_P^{x_i}$. Suppose $u = (x_1, y_1)$, $v = (x_i, y_j)$ $(1 \leq j \leq p)$ . Since $G_m$ is hamiltonian, suppose hamiltonian-cycle $C = x_1 x_2 \cdots x_m x_1$. Since $G_P^{x_1}$ is hamiltonian-connected, $u$ belongs to a hamiltonian-cycle $C_1$ in $G_P^{x_1}$, suppose $C_1 = (x_1, y_1)(x_1, y_2)\cdots(x_1, y_p)(x_1, y_1)$, where $(x_1, y_1) = u$.

Then $u, v$ are connected by a hamiltonian-path $P'$ in $G_m * G_p$ by choosing the endvertex, say, $(x_2, y_1)$ and $(x_2, y_2)$ of cross edges $(x_1, y_1)(x_2, y_1)$ and $(x_1, y_2)(x_2, y_2)$. Then there exist a hamiltonian-path $P_2$ connecting $(x_2, y_1)$ and $(x_2, y_2)$ in $G_p^{x_2}$ since $G_p^{x_2}$ is Hamilton-connected. Taking adjacent vertices $(x_2, y_t)$ and $(x_2, y_{t+1})$ in path $P_2$, then $P_2 = P_{21} \cup (x_2, y_t)(x_2, y_{t+1}) \cup P_{22}$, where $P_{21} = P_2[(x_2, y_1), (x_2, y_t)]$ and $P_{22} = P_2[(x_2, y_{t+1}), (x_2, y_2)]$. Choosing the endvertices , say, $(x_3, y_t)$ and $(x_3, y_{t+1})$ of cross edges $(x_3, y_t)(x_2, y_t)$ and $(x_3, y_{t+1})(x_2, y_{t+1})$. Then there exist a hamiltonian-path $P_3$ connecting $(x_3, y_t)$ and $(x_3, y_{t+1})$ in $G_p^{x_3}$ since $G_p^{x_3}$ is Hamilton-connected. Taking adjacent vertices $(x_3, y_j)$ and $(x_3, y_{j+1})$ in path $P_3$, then $P_3 = P_{31} \cup (x_3, y_j)(x_3, y_{j+1}) \cup P_{32}$, where $P_{31} = P_3[(x_3, y_t), (x_3, y_j)]$ and $P_{32} = P_3[(x_3, y_{j+1}), (x_3, y_{t+1})]$, and so on. Until we take adjacent vertices $(x_{i-2}, y_\mu)$ and $(x_{i-2}, y_{\mu+1})$ in path $P_{i-2}$, and the endvertices, say, $(x_{i-1}, y_\mu)$ and $(x_{i-1}, y_{\mu+1})$ of cross edges $(x_{i-1}, y_\mu)(x_{i-2}, y_\mu)$ and $(x_{i-1}, y_{\mu+1})(x_{i-2}, y_{\mu+1})$. Then there exist a hamiltonian-path $P_{i-1}$ connecting $(x_{i-1}, y_\mu)$ and $(x_{i-1}, y_{\mu+1})$ in $G_p^{x_{i-1}}$ since $G_p^{x_{i-1}}$ is Hamilton-connected. Let path $P'_1 = u(x_2, y_1) \cup P_{21} \cup (x_2, y_t)(x_3, y_t) \cup P_{31} \cup \cdots \cup P_{(i-2)1} \cup (x_{i-2}, y_\mu)(x_{i-1}, y_\mu) \cup P_{i-1} \cup (x_{i-1}, y_{\mu+1})(x_{i-2}, y_{\mu+1}) \cup P_{(i-2)2} \cup \cdots \cup P_{32} \cup (x_3, y_{t+1})(x_2, y_{t+1}) \cup P_{22} \cup (x_2, y_2)(x_1, y_2) \cup C_1[(x_1, y_2), (x_1, y_p)]$.

Choosing the endvertex, say, $(x_m, y_p)$ of cross edge $(x_1, y_p)(x_m, y_p)$. Then there exist a hamiltonian-cycle $C_m$ passing $(x_m, y_p)$ in $G_p^{x_m}$ since $G_p^{x_m}$ is Hamilton-connected. Suppose the last vertex is $(x_m, y_1)$ in $C_m$, taking the endvertex, say, $(x_{m-1}, y_p)$ of cross edge $(x_m, y_1)(x_{m-1}, y_p)$. Then there exist a hamiltonian-cycle $C_{m-1}$ passing $(x_{m-1}, y_p)$ in $G_p^{x_{m-1}}$ since $G_p^{x_{m-1}}$ is Hamilton-connected, and so on. Until we take the last vertex, say, $(x_{i+1}, y_1)$ in $C_{i+1}$, taking the endvertex, say, $(x_i, y_p)$ of cross edge $(x_{i+1}, y_1)(x_i, y_p)$.

If $(x_i,y_p)=(x_i,y_j)$. Then there exist a hamiltonian-cycle $C_i$ passing $(x_i,y_p)$ in $G_p^{x_i}$ since $G_p^{x_i}$ is Hamilton-connected, taking the adjacent vertex of $(x_i,y_j)$, say, $(x_i,y_{j-1})$ in cycle $C_i$ and taking the endvertex, say, $(x_{i+1},y_{j-1})$ of cross edge $(x_{i+1},y_{j-1})(x_i,y_{j-1})$ (if $(x_{i+1},y_{j-1})=(x_{i+1},y_p)$, then taking the other adjacent vertex of $(x_i,y_j)$ such that $(x_{i+1},y_{j-1})\neq(x_{i+1},y_p)$). Then there exist a hamiltonian-path $P_{i+1}$ connecting $(x_{i+1},y_p)$ and $(x_{i+1},y_{j-1})$ in $G_p^{x_{i+1}}$ since $G_p^{x_{i+1}}$ is Hamilton-connected. Let $P_2'=(x_1,y_p)(x_m,y_p)\cup C_m[(x_m,y_p),(x_m,y_1)]\cup(x_m,y_1)(x_{m-1},y_p)\cup C_{m-1}[(x_{m-1},y_p),(x_{m-1},y_1)]\cup\cdots\cup P_{i+1}\cup(x_{i+1},y_{j-1})(x_i,y_{j-1})\cup C_i[(x_i,y_{j-1}),v]$.

If $(x_i,y_p)\neq(x_i,y_j)$. Then there exist a hamiltonian-path $P_i$ connecting $(x_i,y_p)$ and $(x_i,y_j)$ in $G_p^{x_i}$ since $G_p^{x_i}$ is Hamilton-connected. Let $P_2''=(x_1,y_p)(x_m,y_p)\cup C_m[(x_m,y_p),(x_m,y_1)]\cup(x_m,y_1)(x_{m-1},y_p)\cup C_{m-1}[(x_{m-1},y_p),(x_{m-1},y_1)]\cup\cdots\cup C_{i+1}[(x_{i+1},y_p),(x_{i+1},y_1)]\cup(x_{i+1},y_1)(x_i,y_p)\cup P_i[(x_i,y_p),v]$.

Let $P'=P_1'\cup P_2'$ or $P'=P_1'\cup P_2''$. Then we obtain the desired result. □

**Corollary 2.** *If $G_m$ and $G_p$ are hamiltonian-connected, then $G_m*G_p$ is also hamiltonian-connected.*

**Corollary 3.** *If $G_m$ and $G_p$ are hamiltonian, then,*

*(i)$c(G_m*G_p)\geq mp-p$;*

*(ii)There exists a hamiltonian path in $G_m*G_p$.*

*Proof.* Since $G_m$ is Hamiltonian, suppose Hamiltonian-cycle $C'=x_1x_2\ldots x_mx_1$. Since $G_p$ is Hamiltonian, suppose Hamiltonian-cycle $C''=y_1y_2\ldots y_py_1$.

(i) We obtain a cycle $C$ as follows: Choosing any vertex, say, $(x_1,y_1)$, in $V(G_p^{x_1})$. Taking the endvertex, say, $(x_2,y_1)$, of cross edge $(x_1,y_1)(x_2,y_1)$ and have a hamiltonian-cycle $C_2$ passing $(x_2,y_1)$ in $G_p^{x_2}$. Suppose the last vertex is $(x_2,y_p)$ in $C_2$. Taking the endvertex, say, $(x_3,y_1)$, of cross edge $(x_2,y_p)(x_3,y_1)$ and have a hamiltonian-cycle $C_3$ passing $(x_3,y_1)$ in $G_p^{x_3}$. The last vertex is $(x_3,y_p)$ in $C_3$, $\cdots$, and so on, until taking the endvertex, say, $(x_m,y_1)$, of cross edge $(x_m,y_1)(x_{m-1},y_p)$ and have a hamiltonian-cycle $C_m$ passing $(x_m,y_1)$ in $G_p^{x_m}$. The last vertex is $(x_m,y_p)$ in $C_m$. Taking the endvertex, say, $(x_1,y_j)$, of cross edge $(x_m,y_p)(x_1,y_j)$ and have a hamiltonian-cycle $C_1$ passing $(x_1,y_j)$ in $G_p^{x_1}$. So the length of $((x_1,y_j),(x_1,y_1))$-path in $G_p^{x_1}$ is at least $\lfloor\frac{p}{2}\rfloor$, but if $(x_1,y_j)=(x_1,y_1)$, then (i) holds.

(ii) We obtain a Hamiltonian path as follows: Choosing any vertex, say, $(x_1,y_1)$, in $V(G_p^{x_1})$, and have a hamiltonian-cycle $C_1$ passing $(x_1,y_1)$ in $G_p^{x_1}$. The last vertex is $(x_1,y_p)$ in $C_1$. Taking the endvertex, say, $(x_2,y_1)$, of cross edge $(x_1,y_p)(x_2,y_1)$ and have a hamiltonian-cycle $C_2$ passing $(x_2,y_1)$ in $G_p^{x_2}$. The last vertex is $(x_2,y_p)$ in $C_2$. Taking the endvertex, say, $(x_3,y_1)$, of cross edge $(x_2,y_p)(x_3,y_1)$ and have a hamiltonian-cycle $C_3$ passing $(x_3,y_1)$ in $G_p^{x_3}$. The last vertex is $(x_3,y_p)$ in $C_3$, $\cdots$, and so on, until taking the endvertex, say, $(x_m,y_1)$, of cross edge $(x_m,y_1)(x_{m-1},y_p)$ and have a hamiltonian-cycle $C_m$ passing $(x_m,y_1)$ in $G_p^{x_m}$. The last vertex is $(x_m,y_p)$ in $C_m$. Then we obtain hamiltonian-path $((x_1,y_1),(x_m,y_p))$-path. □

**Corollary 4.** *If there exists hamiltonian path in $G_m$, $G_p$ is hamiltonian, then there exists a hamiltonian path in $G_m*G_p$*

$F_i$ denote the sets of faulty elements in $G_p^{x_i}$, $1\leq i\leq m$. $F_0$ denotes the set of faulty edges in cross edge-set. $|F_i|=f_i$, $0\leq i\leq m$. We denote by $f_v^{x_1},f_v^{x_2},\cdots,f_v^{x_m}$ the number of faulty vertices in $G_p^{x_1},G_p^{x_2},\cdots,G_P^{x_m}$, respectively, and by $f_v$ the number of faulty vertices in $G_m*G_p$, so that $f_v=f_v^{x_1}+f_v^{x_2}+\cdots+f_v^{x_m}$. Note that the length of a hamiltonian path in $G_m*G_p-F$ is $mp-f_v-1$.

**Theorem 5.** *Let $G_m$ have a hamiltonian-path, $G_p$ is $f-$fault hamiltonian-connected and $f+1-$fault hamiltonian, $p\geq 3f+6$, then,*

*(a) for any $f\geq 1$, $G_m*G_p$ is $f+2-$fault hamiltonian, and*

*(b) for $f=0$, $G_m*G_p$ with 2 faulty elements has a hamiltonian cycle unless one faulty element is contained in $G_p^{x_i}$ and the other faulty element is contained in $G_p^{x_j}$, $i\neq j$, $1\leq i,j\leq m$.*

*Proof.* (a) Suppose $C=x_1x_2\cdots x_m$ is hamiltonian-path in $G_m$. Assuming the number of faulty elements $|F|\leq f+2$, we will construct a cycle of length $\ell$, $\ell=mp-f_v$, in $G_m*G_p-F$.

Case 1. $f_i \le f, 0 \le i \le m$. Taking two adjacent vertices $u, v \in V(G_p^{x_1} - F_1)$. Since $G_p$ is $f-$fault hamiltonian-connected, there exists hamiltonian $uv$-path, say, $P_1$, in $G_p^{x_1} - F_1$. We first claim that there exist an edge $(x,y)$ on $P_1$ such that all of $\bar{x}, (x,\bar{x}), \bar{y}$, and $(y,\bar{y})$ do not belong to $F$, where $\bar{x}, \bar{y} \in G_p^{x_2}$, $(x,\bar{x})$, $(y,\bar{y})$ are two cross edges. Since there are $p - f_v^{x_1} - 1$ candidate edges on $P_1$ and at most $f+2$ faulty elements can "block" the candidates, at most two candidates per one faulty element. By assumption $p \ge 3f+6$, and the claim is proved. The path $P_{12}$ can be obtained by merging $P_1$ and a hamiltonian-path $P_2$ in $G_p^{x_2} - F_2$ between $\bar{x}, \bar{y}$ with the edges $(x,\bar{x})$, $(y,\bar{y})$, of course the edge $(x,y)$ is discarded. Similarly, there exist an edge $(x',y')$ on $P_2$ such that all of $\bar{x'}, (x',\bar{x'}), \bar{y'}$, and $(y',\bar{y'})$ do not belong to $F$, where $\bar{x'}, \bar{y'} \in G_p^{x_3}$, $(x',\bar{x'})$, $(y',\bar{y'})$ are two cross edges. The path $P_{123}$ can be obtained by merging $P_{12}$ and a hamiltonian-path $P_3$ in $G_p^{x_3} - F_3$ between $\bar{x'}, \bar{y'}$ with the edges $(x',\bar{x'})$, $(y',\bar{y'})$, of course the edge $(x',y')$ is discarded, and so on, at least we obtain the hamiltonian-path $P_{12\cdots m}$ in $G_m * G_P - F$. Therefore $P_{12\cdots m} \cup uv$ is hamiltonian-cycle in $G_m * G_P - F$.

Case 2. There exists some $i, j$ such that $f_i = f+1$, $f_j = 1$. Then $f_t = 0, 0 \le t \le m, t \ne i, j$. Since $f \ge 1$.

Subcase 2.1. $i = 0$. Taking two adjacent vertices $u, v \in V(G_p^{x_1} - F_1)$. Since $G_p$ is $f-$fault hamiltonian-connected, there exists hamiltonian $uv$-path, say, $P_1$, in $G_p^{x_1} - F_1$. Similar to Case 1, it follows that $G_m * G_p$ is $f+2-$fault hamiltonian.

Subcase 2.2. $i = 1$. Since $G_p$ is $f+1-$fault hamiltonian, Taking hamiltonian cycle in $G_P^{x_1} - F_1$. Similar to Case 1, it follows that $G_m * G_p$ is $f+2-$fault hamiltonian.

Subcase 2.3. $i \ne 1, 0$. Since $G_p$ is $f+1-$fault hamiltonian, Taking hamiltonian cycle $C_i$ in $G_P^{x_i} - F_i$. Since $f_i = f+1$, $f_j = 1$. Then $f_t = 0$, $0 \le t \le m, t \ne i, j$, then there exist two adjacent edges $(x,y), (y,z)$ on $C_i$ such that all of $\bar{x}, (x,\bar{x}), \bar{y}$, and $(y,\bar{y})$ , $z'$, and $(z,z')$, and $(y,y')$ do not belong to $F$, where $\bar{x}, \bar{y} \in G_P^{x_{i+1}}$,and $y', z' \in G_P^{x_{i-1}}$, $(x,\bar{x})$, $(y,\bar{y})$, $(y,y')$, $(z,z')$ are cross edges. Similar to Case 1, $G_P^{x_{i-1}}$ has hamiltonian path $P_{i-1}$ connecting the vertex $y', z'$ and $G_P^{x_{i+1}}$ has hamiltonian path $P_{i+1}$ connecting the vertex $\bar{y}, \bar{x}$. We obtain a cycle $C_{(i-1)i(i+1)}$ by merging $C_i$ and hamiltonian path $P_{i-1}, P_{i+1}$ with the edges $(x,\bar{x})$, $(y,\bar{y})$,$(z,z')$, $(y,y')$, of course the edges $(x,y)$ and $(y,z)$ are discarded. Taking an edge $\alpha, \beta$ in $P_{i-1}$ and $P_{i+1}$, respectively. Similar to Case 1, at least we obtain cycle $C_{12\cdots m}$, then $G_m * G_p$ is $f+2-$fault hamiltonian.

Case 3. There exists some $f_i$ such that $f_i = f+2$. Then $f_j = 0$, $0 \le j \le m, j \ne i$.

Subcase 3.1. $i = 0$. Taking two adjacent vertices $u, v \in V(G_p^{x_1} - F_1)$. Since $G_p$ is $f-$fault hamiltonian-connected, there exists hamiltonian $uv$-path, say, $P_1$, in $G_p^{x_1} - F_1$. Since $p \ge 3f+6$, then there exist an edge $(x,y)$ on $P_1$ such that all of $\bar{x}, (x,\bar{x}), \bar{y}$, and $(y,\bar{y})$ do not belong to $F$, where $\bar{x}, \bar{y} \in G_p^{x_2}$, $(x,\bar{x})$, $(y,\bar{y})$ are two cross edges. Similar to Case 1, it follows that $G_m * G_p$ is $f+2-$fault hamiltonian.

Subcase 3.2. $i = 1$. Since $G_p$ is $f+1-$fault hamiltonian, we select an arbitrary faulty element $\alpha$ in $G_p^{x_1}$. Regarding $\alpha$ as a virtual fault-free element. Taking hamiltonian cycle $C_1$ in $G_P^{x_1} - F_1$. If $\alpha$ is a faulty vertex on $C_1$, let $x$ and $y$ be two vertices on $C_1$ next to $\alpha$; else if $C_1$ passes through the faulty edge $\alpha$, let $x$ and $y$ be the endvertices of $\alpha$. The cycle $C_{12}$ is obtained by merging $C_1 - \alpha$ and a hamiltonian-path in $G_P^{x_2}$ joining $\bar{x}$ and $\bar{y}$ with cross edges $(x,\bar{x}), (y,\bar{y})$, where $\bar{x}, \bar{y} \in V(G_p^{x_2})$. Similar to Case 1, it follows that $G_m * G_p$ is $f+2-$fault hamiltonian.

Subcase 3.3. $i \ne 1, 0$. Since $G_p$ is $f+1-$fault hamiltonian, we select an arbitrary faulty element $\alpha$ in $G_p^{x_i}$. Regarding $\alpha$ as a virtual fault-free element. Taking hamiltonian cycle $C_i$ in $G_P^{x_1} - F_i$. If $\alpha$ is a faulty vertex on $C_i$, let $x$ and $y$ be two vertices on $C_i$ next to $\alpha$; else if $C_i$ passes through the faulty edge $\alpha$, let $x$ and $y$ be the endvertices of $\alpha$. Let $z$ be the vertices on $C_i$ next to $y$. The cycle $C_{(i-1)i(i+1)}$ is obtained by merging $C_i - \alpha$ and a hamiltonian-path in $G_P^{x_{i+1}}$ joining $\bar{x}$ and $\bar{y}$ and a hamiltonian-path in $G_P^{x_{i-1}}$ joining $z'$ and $y'$ with cross edges $(z,z'), (y,y'), (x,\bar{x}), (y,\bar{y})$, where $z', y' \in V(G_P^{x_{i-1}})$ and $\bar{x}, \bar{y} \in V(G_P^{x_{i+1}})$. Similar to Subcase 2.3, it follows that $G_m * G_p$ is $f+2-$fault hamiltonian.

(b) If $f = 0$. Since $G_p$ is $0-$fault hamiltonian-connected and $1-$fault hamiltonian, we have similar proof subcase 3.2 by regarding a faulty element as virtual faulty-free element if the two faulty elements is contained in subgraph $G_p^{x_i}$. This completes the proof. □

**Corollary 6.** *Let $G_m = K_2$ , $G_p$ is $f-$fault hamiltonian-connected and $f+1-$fault hamiltonian, $p \ge 3f+6$, then,* *(a) for any $f \geq 1$, $K_2 * G_p$ is $f+2-$fault hamiltonian.*

*(b) for $f = 0$, $K_2 * G_p$ with 2 faulty elements has a hamiltonian cycle unless one faulty element is contained in $G_p^{x_1}$ and the other faulty element is contained in $G_p^{x_2}$.*

**Theorem 7.** *Let $G_m$ have a hamiltonian-path, $G_p$ is $f-$fault $q-$panconnected and $f+1$-fault hamiltonian, $p \geq 2q+f+2$, $q \geq 2f+5$, then,*

*(a) for any $f \geq 2$, $G_m * G_p$ is $f+1-$fault $q+m-$panconnected,*

*(b) for $f = 1$, $G_m * G_p$ with 2 faulty elements has a path of every length $q+m$ or more joining $s$ and $t$ unless $s$ and $t$ are contain in the same subgraph $G_p^{x_i}$ and their neighbors are the faulty element in $G_p^{x_j}$, $i \neq j$,*

*(c) for $f = 0$, $G_m * G_p$ with 1 faulty element has a path of every length $q+m$ or more joining $s$ and $t$ unless $s$ and $t$ are contained in the same subgraph $G_p^{x_i}$ and one of their neighbors is the faulty element in $G_p^{x_j}$, $i \neq j$.*

*Proof.*

(a) Suppose $P = x_1x_2\cdots x_m$ is a hamiltonian-path of $G_m$. Assuming the number of faulty elements $|F| \leq f+1$, we will construct a path of every length $\ell$, $q+m \leq \ell \leq mp-f_v-1$, in $G_m * G_p - F$ joining any pair of vertices $s$ and $t$.

Case 1. $f_i \leq f$, $0 \leq i \leq m$.

Subcase 1.1. When both $s,t$ is contained in $G_p^{x_1}$. There exists a path $P_1$ of length $\ell_1$ in $G_p^{x_1}$ joining $s$ and $t$ for every $q \leq \ell_1 \leq p - f_v^{x_1} - 1$. We are to construct a longer path $P_{12}$ that passes through vertices in $G_p^{x_1}$ and $G_p^{x_2}$. We first claim that there exist one edge $(x,y)$, on $P_1$ such that all of $\bar{x},(x,\bar{x}),\bar{y}$, and $(y,\bar{y})$, do not belong to $F$, where $\bar{x},\bar{y} \in G_p^{x_2}$, $(x,\bar{x})$, $(y,\bar{y})$ are two cross edges. Since there are $\ell_1$ candidate edges on $P_1$ and at most $f+1$ faulty elements can "block" the candidates, at most two candidates per one faulty element. By assumption $q \geq 2f+5$, and the claim is proved. The path $P_{12}$ can be obtained by merging $P_1$ and a path $P_2$ in $G_p^{x_2} - F_2$ between $\bar{x},\bar{y}$ with the edges $(x,\bar{x})$, $(y,\bar{y})$, of course the edge $(x,y)$ is discarded. Let $\ell_2$ be the length of $P_2$, the length $\ell_{12}$ of $P_{12}$ can be anything in the range $2q+1 \leq \ell_{12} \leq \ell_1+\ell_2+1 \leq 2p - f_v^{x_1} - f_v^{x_2} - 1$. Since $p \geq 2q+f+2, f_v^{x_i} \leq f$, we have $2q+1 \leq p - f_v^{x_1} - 1$. So there exist path connecting the vertex $s$ and $t$ such that the length of path can be anything in the range $[q, 2p - f_v^{x_1} - f_v^{x_2} - 1]$. Similarly, we claim that there exist one edge $(u,v)$, on $P_2$ such that all of $\bar{u},(u,\bar{u}),\bar{v}$, and $(v,\bar{v})$, do not belong to $F$, where $\bar{u},\bar{v} \in G_p^{x_3}$, $(u,\bar{u})$, $(v,\bar{v})$ are two cross edges. The path $P_{123}$ can be obtained by merging $P_{12}$ and a path $P_3$ in $G_p^{x_3} - F_3$ between $\bar{u},\bar{v}$ with the edges $(u,\bar{u})$, $(v,\bar{v})$, of course the edge $(u,v)$ is discarded. Let $\ell_3$ be the length of $p_3$, the length $\ell_{123}$ of $P_{123}$ can be anything in the range $3q+2 \leq \ell_{123} \leq \ell_{12}+\ell_3+1 \leq 3p - f_v^{x_1} - f_v^{x_2} - f_v^{x_3} - 1$. Since $p \geq 2q+f+2, f_v^{x_i} \leq f$, we have $2q+1 \leq p - f_v^{x_1} - 1$ and $3q+2 \leq 2p - f_v^{x_1} - f_v^{x_2} - 1$, and so on, at least the path $P_{12\cdots m}$, and $q \leq \ell_{12\cdots m} \leq mp - f_v - 1$.

Subcase 1.2. When both $s,t$ is contained in $G_p^{x_i}$, $i \neq 1$. We first claim that there exist two edges $(x,y),(u,v)$ on $P_i$ such that all of $\bar{x},(x,\bar{x}),\bar{y}$, and $(y,\bar{y})$, $\bar{u},(u,\bar{u}),\bar{v}$, and $(v,\bar{v})$ do not belong to $F$, where $\bar{x},\bar{y} \in G_p^{x_{i+1}}$, $\bar{u},\bar{v} \in G_p^{x_{i-1}}$, and $(x,\bar{x})$, $(y,\bar{y})$, $(u,\bar{u})$, $(v,\bar{v})$ are four cross edges. Since there are $\ell_1$ candidate edges on $P_1$ and at most $f+1$ faulty elements can "block" the candidates, at most two candidates per one faulty element. By assumption $q \geq 2f+5$, and the claim is proved. The path $P_{(i-1)i(i+1)}$ can be obtained by merging $P_i$ and two paths $P_{i-1}, P_{i+1}$ in $G_p^{x_{i-1}} - F_{i-1}$, $G_p^{x_{i+1}} - F_{i+1}$, respectively, between $\bar{u},\bar{v}$ with the edges $(u,\bar{u})$, $(v,\bar{v})$, and $\bar{x},\bar{y}$ with the edges $(x,\bar{x})$, $(y,\bar{y})$, of course the edge $(u,v)$, $(x,y)$ is discarded. Similar to Subcase 1.1, it follows that the path $P_{12\cdots m}$, and $q \leq \ell_{12\cdots m} \leq mp - f_v - 1$.

Subcase 1.3. When $s$ is in $G_p^{x_i}$, $t$ is in $G_p^{x_j}$, $i \neq j$. Since $G$ is $f-$fault hamiltonian-connected, then $f \leq \delta(G) - 3$, where $\delta(G)$ is the minimum degree of $G$, then there exist one adjacent vertex $s' \in G_p^{x_i}$ of $s$ such that the cross edges sequence $T \notin F$, where $T = s'(x_{i+1}, y^{i+1})$, $(x_{i+1}, y^{i+1})(x_{i+2}, y^{i+2}), \cdots, (x_{j-1}, y^{j-1})(x_j, y^j)$, where $(x_{i+1}, y^{i+1}) \in G_p^{x_{i+1}}$, $(x_{i+2}, y^{i+2}) \in G_p^{x_{i+2}}$, $(x_{j-1}, y^{j-1}) \in G_p^{x_{j-1}}$, $(x_j, y^j) \in G_p^{x_j}$. There exists a path $P_1$ of length $\ell_1$ in $G_p^{x_j}$ joining $t$ and $(x_j, y^j)$ for every $q \leq \ell_1 \leq p - f_v^{x_j} - 1$. Hence a path $P_1'$ joining $s$ and $t$ can be obtained by merging $P_1$ and the cross edges sequence $T$ with the length $\ell_1'$ of path $P_1$ for every integer in the range $q+j-i+1 \leq \ell_1' \leq p - f_v^{x_j} + j - i - 1$. Similarly to Subcase 1.2, it follows that the result completes.

Case 2. There exists some $f_i$ such that $f_i = f+1$. Then $f_j = 0$, $1 \leq j \leq m, j \neq i$. Since $f \geq 2$. Similar to case 1, it follows that the result completes.

It immediately follows from Case 1, where the assumption $f \geq 2$ is never used, that $f = 0,1$, $G_m * G_p$ with $f+1$ faulty elements has a path of every length $q+m$ or more joining $s$ and $t$ unless $s$ and $t$ are contained in the same subgraph $G_p^{x_i}$ and one of their neighbors is the faulty element in $G_p^{x_j}$, $i \neq j$. Thus, the proof of $(c)$ is done. we testify the case (b), note that $G_p$ is 1-fault $q$-panconnected. This completes the proof.

## References


[1] Bermond J. C. , Delorme C. , and Farhi G., Large graphs with given degree and diameter II , J Combin. Theory, Series B 36 (1984) 32-48. 10.1109/TC.1984.1676504.

[2] BondyJ. A. and Murty U. S. R., Graph Theory with Applications , 5th printing, American Elsevier Publishing Co., Inc., 1976.

[3] Chartrand G. and Harary F., Planar permutation graphs , Ann Inst H Poincaré Sec B 3 (1967) 433-438. http://eudml.org/doc/76875

[4] D. Bauer and E. Schmeichel: On a theorem of Häggkvist and Nicoghossian, In Y. Alavi, F.R.K. Chung, R.L. Graham and D. S. Hsu, eds., Graph Theory, Combinatorics, Algorithms, and Applications-Proceedings $2^{nd}$ China-USA Graph Theory Conference, SIAM(1991)20-25.

[5] B. Wei, A Generalization of a result of Bauer and Schmeichel, Graphs and Combin. 9 (1993) 383-389. 10.1007/BF02988325

[6] D. Bauer, H.J. Broersma, and H.J. Veldman, On generalizing a theorem of Jung, Ars Combinatoria 40 (1995) 207-218.

[7] S. Wang, C. Wang, J. Liu, On extremal multiplicative Zagreb indices of trees with given domination number, Applied Mathematics and Computation, 332 (2018) 338-350. 10.1016/j.amc.2018.03.058.

[8] Z. Shao, J. Xu, S. Sheikholeslami, S. Wang, The domination complexity and related extremal values of large 3-D torus, Complexity, Volume 2018, Article ID 3041426, 8 pages. 10.1155/2018/3041426.

[9] S. Wang, B. Wei, A note on the independent domination number versus the domina- tion number in bipartite graphs, Czechoslovak Mathematical Journal, 67 (142) (2017), 533-536. 10.21136/CMJ.2017.0068-16.

[10] S. Ji, S. Wang, On the sharp lower bounds of Zagreb indices of graphs with given number of cut vertices, Journal of Mathematical Analysis and Applications, 458 (1) 2018, 21-29. 10.1016/j.jmaa.2017.09.005.

[11] Y. Ahmad, U. Ali, M. bilal, S. Zafar, Z. Zahid, Some new standard graphs labeled by 3-total edge product cordial labeling 2 (2017)(1) 61-72. 10.21042/AMNS.2017.1.00005.

[12] S. Sudhakar, S. Francis, V. Balaji, Odd mean labeling for two star graph, (2) 2017 (1) 195-200, 10.21042/AMNS.2017.1.00016.

[13] B. Basavanagoud, V.R. Desai, S. Patil, $(\beta,\alpha)$-Connectivity Index of Graphs (2) 2017 (1) 21-30, 10.21042/AMNS.2017.1.00003.

[14] S. Zhou, L. Xu, Y. Xu, A sufficient condition for the existence of a k-factor excluding a given r-factor (2) 2017 (1) 13-20, 10.21042/AMNS.2017.1.00002.