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Equivalent Analytical Functions of Sums of Sigmoid like Transcendental Functions

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Abstract

There is no mathematical solution to adding up transcendental functions other than numerical process. This paper put forward analytical method to model the sum of sigmoid like functions with an equivalent function. The *Brillouin* and *Langevin* as well as the *error* function, the *tanh*, *sigmoid* and the tan^{-1} functions are investigated, their equivalent functions are calculated for four components and the error between the numerical (computer assisted) result and the equivalent function is tested for accuracy. The best modelling function, the most useful to include into mathematical operations, is pointed out finally, based on its performance and convenience. The paper intends to help people involved mostly in modelling hysteresis in Magnetism and other field of research in physics.

Keywords: transcendental functions, sigmoid functions, modelling.

1 Introduction

One of the unsolved problems of mathematics is to add up transcendent functions analytically to be used in calculations. Functions used for calculating magnetisation and for its mathematical modelling of hysteresis, almost exclusively are falling into this category. The different models usually provide close approximations to the *Brillouin* $B_j(x)$ [1] and *Langevin* $L(x)$ [2] functions, at the same time, giving approximate solutions to Everett integral [3,4] as a replacement of the Preisach distribution [5]. The most well known and most frequently used analytical functions for this kind of modelling, leading to the most likely distributions, are the *Erf* (error function) [6], *Tanh* (hyperbolic tangent) [7, 8], *Si* (sigmoid) [9] and the tan^{-1} (inverse tangent) [10] functions. Although other functions are also used and the method described here may apply to those functions as well, we concentrate on these four functions to demonstrate the useful nature of the process for mathematicians and physics researchers alike. These can be a useful approximating replacement to other numerical methods in mathematical calculations. These may be applicable in other applied mathematical calculations as well. The sigmoid like character of the magnetisation process obviated the choice of the subject area [11].

To remind the reader, the functions investigated in this paper are as follows:

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The Brillouin function:

$$B_J(x) = \frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J}x\right) - \frac{1}{2J}\coth\left(\frac{x}{2J}\right) \tag{1}$$

The Langevin function:

$$L(x) = \coth(x) - \frac{1}{x} \tag{2}$$

The error function:

$$a_k f_E(\propto_k x) = \frac{a_k}{\sqrt{\pi}}\int_{-\propto_k x}^{+\propto_k x} e^{t^2}\,dt \tag{3}$$

The tangent hyperbolic function:

$$a_k f_T(\propto_k x) = a_k \tanh(\propto_k x) \tag{4}$$

The sigmoid function:

$$a_k f_S(\propto_k x) = \frac{a_k}{1+e^{-\propto_k x}} - \frac{a_k}{2} \tag{5}$$

The inverse tangent function:

$$a_k f_A(\propto_k x) = a_k tan^{-1}(\propto_k x) \tag{6}$$

2 Equivalent function

In Magnetics, in most occasions, composite materials are used, which unavoidably calls in modelling, for the summation of the functions above [11]. Single phase materials are, almost solely, used only in research and calibration purposes. When $a_k\ f_z(\alpha_k x)$ is representing any of the investigated functions and $a_0 f_{0z}(\alpha_0 x)$ describes the equivalent function of their sum respectively, then the numerical values of a_0 and α_0 parameters need to be determined to complete the characteristic formulation of the equivalent function in a normalized analytical form. For the calculation of these two parameters two criteria will be used. The first criterion calls for the equivalence of a_0 and the sum of the amplitudes of the n number of component functions. The second criterion prescribes that a_0, the inclination of the curve with the horizontal axis should be the same as that of composite curve at $x = 0$.

The mathematical formulations of the two criteria are as follows:
The first criteria in mathematical form:

$$a_0 f_{0z}(\propto_0 x) = \sum_k^n a_k f_z(\propto_k x) \tag{7}$$

At $x = x_m$

$$a_0 = \sum_k^n a_k \tag{8}$$

Here n is the maximum number of k functions to be added up and x_m is the maximum numerical value of x where the functions satisfy the second criterion. The user has the freedom to select x_m within the saturation region.

This choice will slightly change numerical values of a_0 and α_0 without effecting the mathematical formulations described above.

To satisfy the second criteria the first derivatives of the functions have to be taken:

$$\frac{d}{dx}[a_0 f_{0z}(\propto_0 x)] = \frac{d}{dx}\sum_k^n a_k f_z(\propto_k x) \tag{9}$$

At $x = 0$

$$a_0 \propto_0 = \sum_k^n a_k \propto_k \tag{10}$$

Therefore

$$\propto_0 = \frac{1}{a_0}\sum_k^n a_k \propto_k \tag{11}$$

With the knowledge of the two parameters the equivalent function $a_0 f_{0z}(\propto_0 x)$ can be constructed.

3 Error of the approximations

Let us assume that the composite functions have four components with the following identical normalized parameters for all four component functions:

$a_1 = 5$ $\alpha_1 = 0.75$
$a_2 = 3$ $\alpha_2 = 0.9$
$a_3 = 2$ $\alpha 3 = 1.5$
$a_4 = 0.5$ $\alpha 4 = 3$

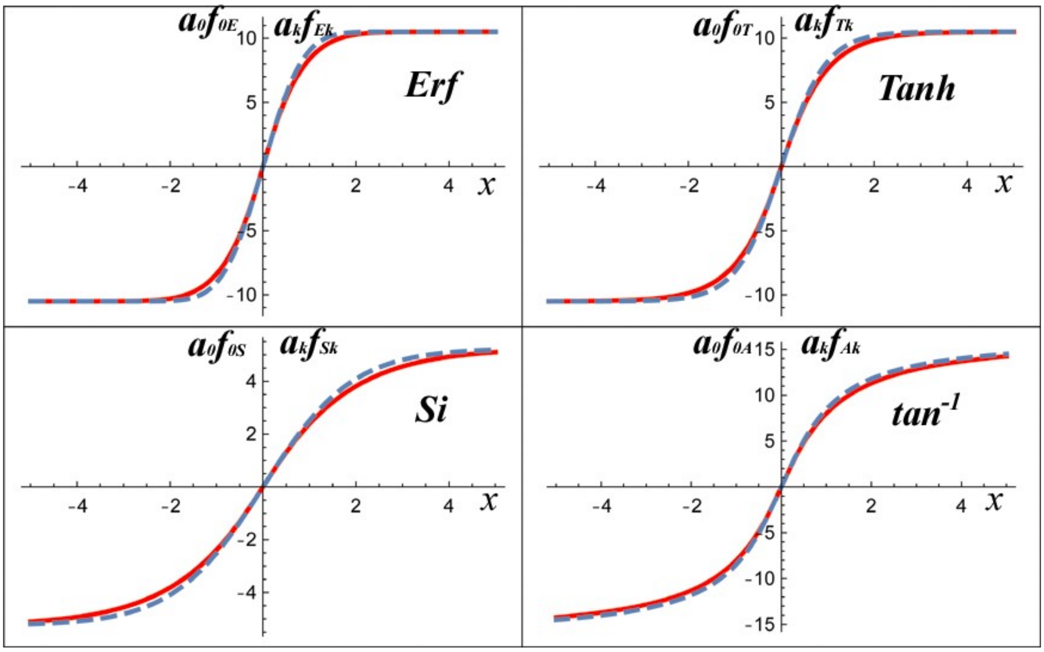

Fig. 1 The numerical and analytical sums of sigmoid functions Solid lines numerical broken lines analitical

By using (8) and (11) the numerical (computer aided) values of the two parameters can be calculated as: $a_0 = 10.5$ and $\alpha_0 = 1.043$.

The numerical and the modeling functions are shown in Fig. 1.

Due to the inherent different shapes of the sampled functions all functions need to be normalized to approximate the shape of the B_{j0}/L_{r0} functions. The following normalization to models (3), (4), (5) and (6) make the closest fit to the Brillouin/Langevin functions:

$$B_{J0} = \sum_{k}^{n} a_k \left(\frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} 2.5 \propto_k x \right) - \frac{1}{2J} \coth\left(\frac{2.5 \propto_k x}{2J} \right) \right) \tag{12}$$

$$L_{r0} = \sum_{k}^{n} a_k \coth(2.5 \propto_k x) - \frac{1}{2.5 \propto_k x} \tag{13}$$

$$Erf_0 = \sum_{k}^{n} \frac{0.88 a_k}{\sqrt{\pi}} \int_{-0.75 \propto_k x}^{+0.75 \propto_k x} e^{t^2} dt \tag{14}$$

$$Tanh_0 = \sum_{k}^{n} 0.936 a_k \tanh(0.955 \propto_k x) \tag{15}$$

$$Si_0 = \sum_{k}^{n} \frac{1.79 a_k}{1 + e^{-1.75 \propto_k x}} - \frac{1.79 a_k}{2} \tag{16}$$

$$At_0 = \sum_{k}^{n} 0.7 tan^{-1}(1.2 \propto_k x) \tag{17}$$

By forming the difference Δ between numerical sum of the *Br/Lr* and $c_k\ a_0 f_{0z}(\ \gamma_k \propto_0 x)$ functions we can assess the accuracy of the each of the modeling functions. Fig. 2 shows that Δ deviation is maximum from the numerical (computer aided) curve of *Br/Lr* in case of the *Erf* function (14) followed by the *Si* (16) and the *Tanh* (15) functions. The smallest difference created by the tan^{-1}(17) function.

The maximum linear and the integrated error *I*Δ, (see equ. (18)) between the Br_0/Lr_0 numerical (computer aided) and equivalent analytical functions, in the first quadrant is a good indicator for the most accurate and usable function for mathematical operations.

$$I\triangle = \int_{0}^{\infty} [a_0 c_0 f_{00}(\gamma_0 \propto_0 x) - \sum_{k}^{n} a_k c_k f_z(\gamma_k \propto_k x)] dx \tag{18}$$

Here c_k and γ_κ represent the factors needed to normalize the amplitudes and inclination for the various functions respectively and f_{00} is the sum of the four Brillouin functions of the same parameters as the modeling functions.

The *I*Δ integral function in (18) shown in Fig. 3 for each modeling function, depicts a slightly different picture, that the tan^{-1} function has the smallest error, followed by *tanh* and *Si* functions. The smallest error makes the tan^{-1} the most accurate; however both the *Si* function and the tan^{-1} function are hard functions, not easy to apply them in mathematical operations. Considering accuracy and convenience the *Tanh* function looks the best selection to use for modeling composite hysteretic materials.

4 Conclusion

In modeling practice often necessary to use sums of transcendental functions. One of mathematics unsolved problems however is adding up analytically transcendental functions. There is a need in applied mathematics, used in physics and engineering for an analytical approximation of the process which can be incorporated into

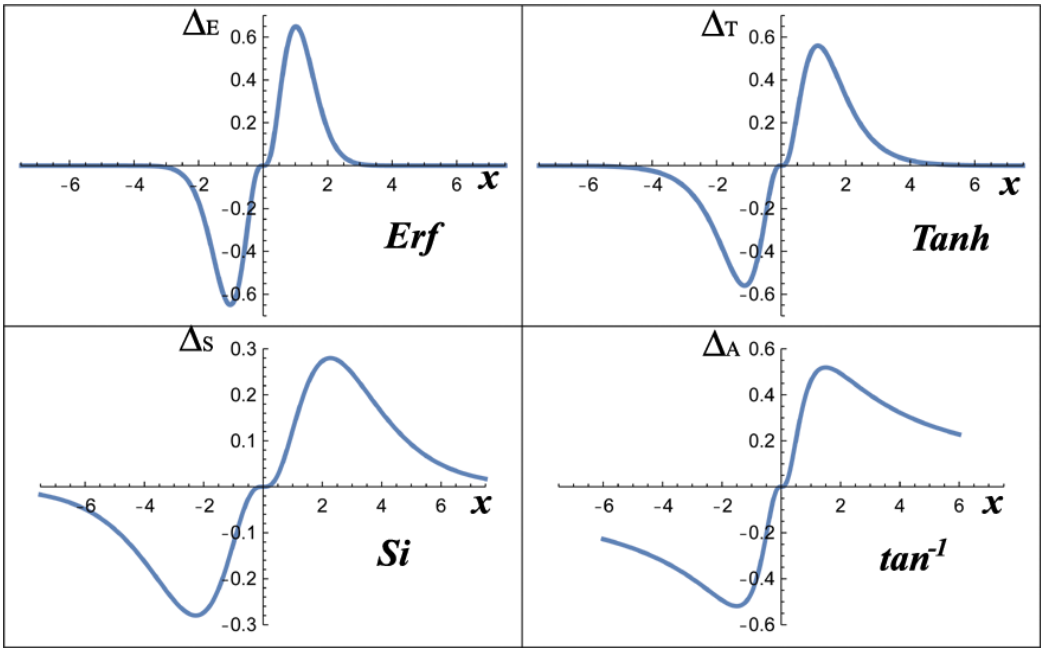

Fig. 2 The difference Δ between the normalized numerical *Br/Lr* and the curves of the equivalent functions shown in Fig. 1.

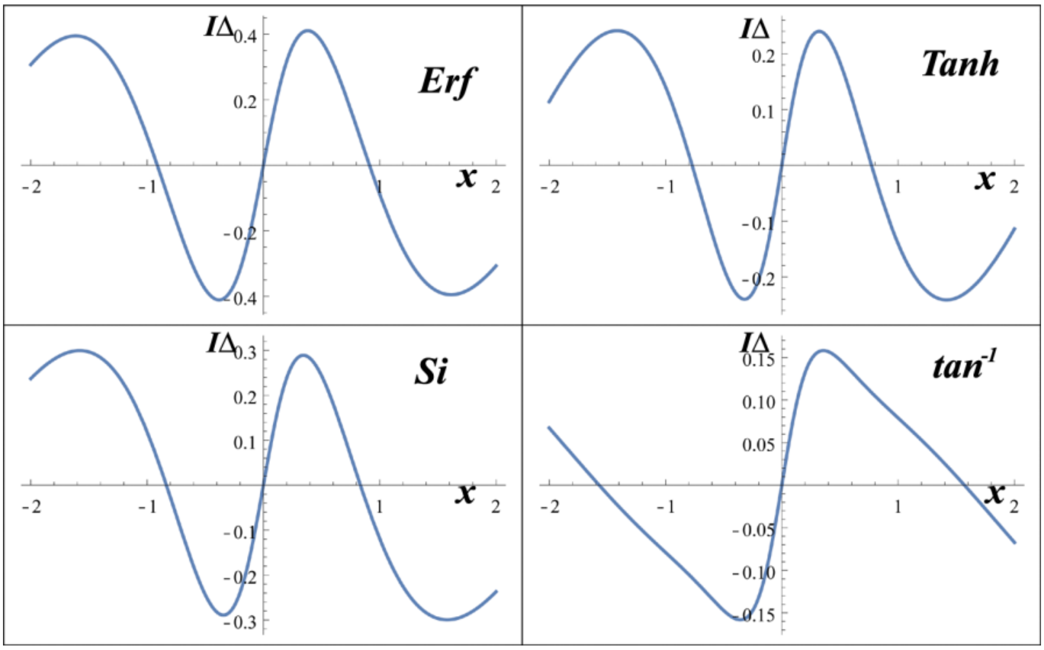

Fig. 3 The difference *I*Δ between the numerical *Br/Lr* and the equivalent curves, integrated for the first quadrant.

mathematical operations. Presently only numerical calculation by computer is available for this. In this paper an analytical process is proposed for sigmoid like functions, mostly applied for calculations in Magnetics. The error between the computer generated numerical result and the proposed analytical equivalents are illustrated graphically, indicating a good accuracy of the proposed analytical approximations. Considering the accuracy and the mathematical usefulness, the *Tanh* function has the most practical advantage over the other three functions, the *Erf, Si* and *tan*$^{-1}$ functions, for inclusion in mathematical calculations.

References

[1] Brillouin M.L. (1927) Les moments de rotation et le magnetisme dans la mechanique ondolatoire. J.de Phys. Radium Vol.VIII. p.74
[2] Langevin M.P. (1905) Magnetisme et theorie des electrons. Ann. Chim. Et Phys. Vol. 94. p. 277.
[3] Takacs J. The Everett integral and its analytical approximation in: L. Malinski (Ed.) pp: 203- 230 in Advanced Magnetic Materials, Intech Publication, (2012).
[4] Finocchio G., Carpentieri M., Cardelli E., Azzerboni B. Analytical solution of Everett integral using Lorentzian Preisach function approximation. JMMM Vol. 300 No.2 (2006) pp. 451-470.
[5] Preisach F., Uber die magnetische Nachwirkung. (1935) Zeitschrift fur Physik, pp. 227-302
[6] Della Torre E., (1999) Magnetic Hysteresis IEEE Press N.Y.
[7] J. Takács. Mathematics of Hysteretic Phenomena (2003) Wiley, Weinheim.
[8] J. Takács and A. Nemcsics (2014) Hysteretic behaviour in some natural processes pp: 89-106, In Hysteresis: Types, Applications and Behaviour Patterns in complex systems, Edited by Jose Carlos Dias Nova S. P. N.Y.
[9] Petrescu L., Cazacu E., Petrescu K. Sigmoid functions used in hysteresis phenomenon modeling. IEEE Advanced Topics in Electrical Engineering (ATEE), 9th International Symposium, (2015) Location: Bucharest, Romania INSPEC Accession Number: 15240830.
[10] Perez-Lojas C. Fitting saturation and hysteresis via arctangent function. IEEE Power Engineering Review, Vol. 20. No. 11 (2000) pp. 55-57.
[11] J. Takács, Analytical way of transforming the hyperbolic hysteresis model into the T(x) model. COMPEL, Vol. 37 No. 3 (2018) pp: 1131-1138