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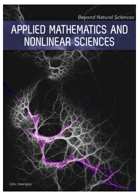

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Applications of the Generalized Kummer's Summation Theorem to Transformation Formulas and Generating Functions

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Abstract

In this paper, we establish two general transformation formulas for Exton's quadruple hypergeometric functions K_5 and K_{12} by application of the generalized Kummer's summation theorem. Further, a number of generating functions for Jacobi polynomials are also derived as an applications of our main results.

Keywords: Transformation formulas, Generating functions, generalized Kummer's theorem, Exton functions, Saran functions, Jacobi polynomials
AMS 2010 codes: 33B15,33C05, 33C45.

1 Introduction

In our present investigation, we begin by recalling the following definitions:
The Exton's quadruple hypergeometric functions K_5 and K_{12} [1]:

$$K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,t) = \sum_{p,q,r,s=0}^{\infty} \frac{(a)_{p+q+r+s}(b_1)_{p+q}(b_2)_{r+s}\, x^p\, y^q\, z^r\, t^s}{(c_1)_p(c_2)_q(c_3)_r(c_4)_s\; p!q!r!s!} \tag{1}$$

and

$$K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;x,y,z,t) = \sum_{p,q,r,s=0}^{\infty} \frac{(a)_{p+q+r+s}(b_1)_p(b_2)_q(b_3)_r(b_4)_s\, x^p\, y^q\, z^r\, t^s}{(c_1)_{p+q}(c_2)_{r+s}\; p!q!r!s!}, \tag{2}$$

where $(a)_n$ denotes the Pochhammer's symbol defined by

$$(a)_n = \begin{cases} 1 & ,if\ n=0 \\ a(a+1)(a+2)\dots(a+n-1) & ,if\ n=1,2,3,\dots \end{cases} \tag{3}$$

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The Exton's triple hypergeometric functions X_4 and X_7 [2]:

$$X_4(a,b;c_1,c_2,c_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b)_{n+p}\, x^m\, y^n\, z^p}{(c_1)_m(c_2)_n(c_3)_p\, m!n!p!} \tag{4}$$

and

$$X_7(a,b_1,b_2;c_1,c_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{2m+n+p}(b_1)_n(b_2)_p\, x^m\, y^n\, z^p}{(c_1)_{n+p}(c_2)_m\, m!n!p!}. \tag{5}$$

The Saran's triple hypergeometric functions F_E and F_G [6]:

$$F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(b_1)_m(b_2)_{n+p}\, x^m\, y^n\, z^p}{(c_1)_m(c_2)_n(c_3)_p\, m!n!p!} \tag{6}$$

and

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p\, x^m\, y^n\, z^p}{(c_1)_m(c_2)_{n+p}\, m!n!p!}. \tag{7}$$

The Exton's double hypergeometric function [3]

$$X^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a) & : & (b)\ ; & (b')\ ; \\ (c) & : & (d)\ ; & (d')\ ;\end{matrix}\ x\ ,\ y\right] = \sum_{m,n=0}^{\infty} \frac{((a))_{2m+n}((b))_m((b'))_n\, x^m\, y^n}{((c))_{2m+n}((d))_m((d'))_n\, m!\, n!}, \tag{8}$$

where the symbol $((a))_m$ denotes the product $\prod_{j=1}^{A}(a_j)_m$.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [5]

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n\ ; \\ 1+\alpha\ ;\end{matrix}\ \frac{1-x}{2}\right]. \tag{9}$$

In order to obtain our main results, we require the following generalization of the classical Kummer's summation theorem for the series ${}_2F_1(-1)$ due to Lavoie *et al* [4]

$$\begin{aligned} {}_2F_1\left[\begin{matrix}a\ ,\ b\ ; \\ 1+a-b+i\ ;\end{matrix}\ -1\right] &= \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))} \\ \times\left\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)}\right. &+ \left.\frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\right\} \end{aligned} \tag{10}$$

for $(i = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5)$.

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients A_i and B_i are given respectively in [4]. When $i = 0$, (10) reduces immediately to the classical Kummer's theorem [5]

$${}_2F_1\left[\begin{matrix}a\ ,\ b\ ; \\ 1+a-b\ ;\end{matrix}\ -1\right] = \frac{\Gamma(1+a-b)\Gamma(\frac{1}{2})}{2^a\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}a+\frac{1}{2})}. \tag{11}$$

We also require the following identities [8]:

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n \tag{12}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A(n,m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} A(n,m-n) \tag{13}$$

$$(\alpha)_{m-n} = \frac{(-1)^n(\alpha)_m}{(1-\alpha-m)_n},\quad 0\le n\le m \tag{14}$$

$$(m-n)! = \frac{(-1)^n\, m!}{(-m)_n},\quad 0\le n\le m. \tag{15}$$

2 Main Results

Theorem 1. *The following general transformation formulas for K_5 holds true.*

$$\begin{aligned}
&K_5(a,a,a,a;b',b',b,b;c',c,d,d+i;x,y,z,-z)\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+n+2p}(b')_{m+n}(b)_{2p}\,x^m\,y^n\,z^{2p}}{(c')_m(c)_n(d)_{2p}\,m!\,n!\,(2p)!}\\
&\times\Bigg\{A_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p)}{\Gamma(d+2p+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right])\Gamma(p+d+\frac{1}{2}i)}\\
&+B_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p)}{\Gamma(d+2p+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-\left[\frac{i}{2}\right])\Gamma(p+d-\frac{1}{2}+\frac{1}{2}i)}\Bigg\}\\
&+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+n+2p+1}(b')_{m+n}(b)_{2p+1}\,x^m\,y^n\,z^{2p+1}}{(c')_m(c)_n(d)_{2p+1}\,m!\,n!\,(2p+1)!}\\
&\times\Bigg\{A_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p+1)}{\Gamma(d+2p+1+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-\left[\frac{1+i}{2}\right])\Gamma(p+\frac{1}{2}+d+\frac{1}{2}i)}\\
&+B_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p+1)}{\Gamma(d+2p+1+\frac{1}{2}(i+|i|))\Gamma(-p-\frac{1}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right])\Gamma(p+d+\frac{1}{2}i)}\Bigg\}
\end{aligned}\tag{16}$$

for $(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5)$.
The coefficients A_i' and B_i' can be obtained from the tables of A_i and B_i given in [4] by taking $a=-2p,\ b=1-d-2p$ and the coefficients A_i'' and B_i'' can be also obtained from the same tables by taking $a=-2p-1,\ b=-d-2p$.

Theorem 2. *The following general transformation formulas for K_{12} holds true.*

$$\begin{aligned}
&K_{12}(a,a,a,a;b',b,c-i,c;d,d,e,e;x,y,z,-z)\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+n+2p}(b')_m(b)_n(c-i)_{2p}\,x^m\,y^n\,z^{2p}}{(d)_{m+n}(e)_{2p}\,m!\,n!\,(2p)!}\\
&\times\Bigg\{E_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(1+2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right])\Gamma(1-p-c+\frac{1}{2}i)}\\
&+F_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(1-2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-\left[\frac{i}{2}\right])\Gamma(-p+\frac{1}{2}-c+\frac{1}{2}i)}\Bigg\}\\
&+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+n+2p+1}(b')_m(b)_n(c-i)_{2p+1}\,x^m\,y^n\,z^{2p+1}}{(d)_{m+n}(e)_{2p+1}\,m!\,n!\,(2p+1)!}\\
&\times\Bigg\{E_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(-2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-\left[\frac{1+i}{2}\right])\Gamma(-p+\frac{1}{2}-c+\frac{1}{2}i)}\\
&+F_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(-2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p-\frac{1}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right])\Gamma(-p-c+\frac{1}{2}i)}\Bigg\}
\end{aligned}\tag{17}$$

for $(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5)$.
The coefficients E_i' and F_i' can be obtained from the tables of A_i and B_i given in [4] by taking $a=-2p,\ b=c$. Also the coefficients E_i'' and F_i'' can be obtained from the same tables by taking $a=-2p-1,\ b=c$.

Proof. Denoting the left hand side of (16) by S, expanding K_5 in a power series and using the results (12) – (15), then after simplification, we obtain

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+n+p}(b')_{m+n}(b)_p\ x^m\ y^n\ z^p}{(c')_m(c)_n(d)_p\ m!\ n!\ p!}\times{}_2F_1\left[\begin{matrix}-p\ ,\ 1-d-p\ ; \\ d+i\ ;\end{matrix}\ -1\right]$$

Separating into even and odd powers of z, we have

$$S=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+n+2p}(b')_{m+n}(b)_{2p}\ x^m\ y^n\ z^{2p}}{(c')_m(c)_n(d)_{2p}\ m!\ n!\ (2p)!}{}_2F_1\left[\begin{matrix}-2p\ ,\ 1-d-2p\ ; \\ d+i\ ;\end{matrix}\ -1\right]$$
$$+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{m+n+2p+1}(b')_{m+n}(b)_{2p+1}\ x^m\ y^n\ z^{2p+1}}{(c')_m(c)_n(d)_{2p+1}\ m!\ n!\ (2p+1)!}{}_2F_1\left[\begin{matrix}-2p-1\ ,\ -d-2p\ ; \\ d+i\ ;\end{matrix}\ -1\right]$$

Now,by applying the generalized Kummer's theorem (10) to each ${}_2F_1[-1]$, then after simplification, we arrive at the right hand side of (16). This completes the proof of (16). The proof of (17) is similar to that of (16) and we use here the result (2).

Remark 1. On taking $x=0$ in (16) and (17), we obtain the following transformation formulas for Saran's triple hypergeometric functions F_E and F_G:

Corollary 3.

$$\begin{aligned}F_E(a,a,a;b',b,b;c,d,d+i;y,z,-z)&=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+2p}(b')_n(b)_{2p}\ \ y^n\ z^{2p}}{(c)_n(d)_{2p}\ n!\ (2p)!}\\
&\times\left\{A'_i\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p)}{\Gamma(d+2p+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(p+d+\frac{1}{2}i)}\right.\\
&\left.+B'_i\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p)}{\Gamma(d+2p+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-[\frac{i}{2}])\Gamma(p+d-\frac{1}{2}+\frac{1}{2}i)}\right\}\\
&+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+2p+1}(b')_n(b)_{2p+1}\ y^n\ z^{2p+1}}{(c)_n(d)_{2p+1}\ n!\ (2p+1)!}\\
&\times\left\{A''_i\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p+1)}{\Gamma(d+2p+1+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(p+\frac{1}{2}+d+\frac{1}{2}i)}\right.\\
&\left.+B''_i\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(d+i)\Gamma(d+2p+1)}{\Gamma(d+2p+1+\frac{1}{2}(i+|i|))\Gamma(-p-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(p+d+\frac{1}{2}i)}\right\}\end{aligned}\tag{18}$$

for $(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5)$.

Corollary 4.

$$\begin{aligned}F_G(a,a,a;b,c-i,c;d,e,e;y,z,-z)&=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+2p}(b)_n(c-i)_{2p}\ \ y^n\ z^{2p}}{(d)_n(e)_{2p}\ n!\ (2p)!}\\
&\times\left\{E'_i\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(1-2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1-p-c+\frac{1}{2}i)}\right.\\
&\left.+F'_i\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(1-2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-p+\frac{1}{2}-c+\frac{1}{2}i)}\right\}\\
&+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{n+2p+1}(b)_n(c-i)_{2p+1}\ y^n\ z^{2p+1}}{(d)_n(e)_{2p+1}\ n!\ (2p+1)!}\end{aligned}$$

$$\times\left\{E_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(-2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(-p+\frac{1}{2}-c+\frac{1}{2}i)}\right.$$
$$\left.+F_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(-2p-c+i)\Gamma(1-c)}{\Gamma(1-c+\frac{1}{2}(i+|i|))\Gamma(-p-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(-p-c+\frac{1}{2}i)}\right\} \tag{19}$$

for $(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5)$.

Special cases of (16) and (17)

Here we mention some special cases of our results (16) and (17) and we will use in each case the following results [8]:

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=(\alpha)_n,\ \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \tag{20}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1+\alpha) = 2^{\alpha}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha\right) \tag{21}$$

$$(\alpha)_{2n}=2^{2n}\left(\frac{1}{2}\alpha\right)_n\left(\frac{1}{2}\alpha+\frac{1}{2}\right)_n \tag{22}$$

$$(2n)! = 2^{2n}\left(\frac{1}{2}\right)_n n! \ \textit{and}\ (2n+1)! = 2^{2n}\left(\frac{3}{2}\right)_n n!. \tag{23}$$

1. Taking $i=0$ and $d=b$ in (16), we get

$$K_5(a,a,a,a;b',b',b,b;c',c,b,b;x,y,z,-z)=X_4(a,b';b,c',c;-z^2,x,y). \tag{24}$$

2. Taking $i=1$ and $d=b-1$ in (16), we get

$$K_5(a,a,a,a;b',b',b,b;c',c,b-1,b;x,y,z,-z)$$
$$=X_4(a,b';b-1,c',c;-z^2,x,y)+\frac{az}{b-1}X_4(a+1,b';b,c',c;-z^2,x,y). \tag{25}$$

3. Taking $i=-1$ and $d=b$ in (16), we get

$$K_5(a,a,a,a;b',b',b,b;c',c,b,b-1;x,y,z,-z)$$
$$=X_4(a,b';b-1,c',c;-z^2,x,y)-\frac{az}{2}X_4(a+1,b';b,c',c;-z^2,x,y). \tag{26}$$

4. Taking $i=0$ and $e=2c$ in (17), we get

$$K_{12}(a,a,a,a;b',b,c,c;d,d,2c,2c;x,y,z,-z)=X_7(a,b',b;d,c+\frac{1}{2};z^2/4,x,y). \tag{27}$$

5. Taking $i=1$ and $e=2c-1$ in (17), we get

$$K_{12}(a,a,a,a;b',b,c-1,c;d,d,2c-1,2c-1;x,y,z,-z)$$
$$=X_7(a,b',b;d,c-\frac{1}{2};z^2/4,x,y)-\frac{az}{2c-1}X_7(a+1,b',b;d,c+\frac{1}{2};z^2/4,x,y). \tag{28}$$

6. Taking $i=-1$ and $e=2c+1$ in (17), we get

$$K_{12}(a,a,a,a;b',b,c+1,c;d,d,2c+1,2c+1;x,y,z,-z)$$
$$=X_7(a,b',b;d,c+\frac{1}{2};z^2/4,x,y)-\frac{az}{2c+1}X_7(a+1,b',b;d,c+\frac{3}{2};z^2/4,x,y). \tag{29}$$

3 Applications to Generating Functions

Two interesting generating functions for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are given by Sharma and Mittal [7]

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_4(\lambda+n,\gamma;\delta,\rho;y,z)t^n=\left[1-\frac{(1-x)t}{2}\right]^{-\lambda}$$

$$\times F_E\left(\lambda,\lambda,\lambda,-\alpha,\gamma,\gamma;-\alpha-\beta,\delta,\rho;\frac{-2t}{2-(1-x)t},\frac{2y}{2-(1-x)t},\frac{2z}{2-(1-x)t}\right) \tag{30}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_1(\lambda+n,\gamma,\rho;\delta;y,z)t^n=\left[1-\frac{(1-x)t}{2}\right]^{-\lambda}$$

$$\times F_G\left(\lambda,\lambda,\lambda,-\alpha,\gamma,\rho;-\alpha-\beta,\delta,\delta;\frac{-2t}{2-(1-x)t},\frac{2y}{2-(1-x)t},\frac{2z}{2-(1-x)t}\right), \tag{31}$$

where F_1 and F_4 are Appell's double hypergeometric functions [8].
Now, in (30), replacing ρ by $\delta+i$ and z by $-y$ and using (18), we get the following families of generating functions for Jacobi polynomials :

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_4(\lambda+n,\gamma;\delta,\delta+i;y,-y)t^n$$

$$=\left[1-\frac{(1-x)t}{2}\right]^{-\lambda}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\lambda)_{n+2p}(-\alpha)_n(\gamma)_{2p}}{(-\alpha-\beta)_n(\delta)_{2p}\,n!\,(2p)!}\left[\frac{-2t}{2-(1-x)t}\right]^n\left[\frac{2y}{2-(1-x)t}\right]^{2p}$$

$$\times\left\{A_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(\delta+i)\Gamma(\delta+2p)}{\Gamma(\delta+2p+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(p+\delta+\frac{1}{2}i)}\right.$$

$$\left.+B_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(\delta+i)\Gamma(\delta+2p)}{\Gamma(\delta+2p+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-[\frac{i}{2}])\Gamma(p+\delta-\frac{1}{2}+\frac{1}{2}i)}\right\}$$

$$+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\lambda)_{n+2p+1}(-\alpha)_n(\gamma)_{2p+1}}{(-\alpha-\beta)_n(\delta)_{2p+1}\,n!\,(2p+1)!}\left[\frac{-2t}{2-(1-x)t}\right]^n\left[\frac{2y}{2-(1-x)t}\right]^{2p+1}$$

$$\times\left\{A_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(\delta+i)\Gamma(\delta+2p+1)}{\Gamma(\delta+2p+1+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-[\frac{1+i}{2}])\Gamma(p+\frac{1}{2}+\delta+\frac{1}{2}i)}\right.$$

$$\left.+B_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(\delta+i)\Gamma(\delta+2p+1)}{\Gamma(\delta+2p+1+\frac{1}{2}(i+|i|))\Gamma(-p-\frac{1}{2}+\frac{1}{2}i-[\frac{i}{2}])\Gamma(p+\delta+\frac{1}{2}i)}\right\} \tag{32}$$

for $(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5)$.
Next, in (31), replacing γ by $\rho-i$ and z by $-y$ and using (19), we get the following families of generating functions for Jacobi polynomials:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_1(\lambda+n,\rho-i,\rho;\delta;y,-y)t^n$$

$$=\left[1-\frac{(1-x)t}{2}\right]^{-\lambda}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\lambda)_{n+2p}(-\alpha)_n(\rho-i)_{2p}}{(-\alpha-\beta)_n(\delta)_{2p}\,n!\,(2p)!}\left[\frac{-2t}{2-(1-x)t}\right]^n\left[\frac{2y}{2-(1-x)t}\right]^{2p}$$

$$\times\left\{E_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(1-2p-\rho+i)\Gamma(1-\rho)}{\Gamma(1-\rho+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right])\Gamma(1-p-\rho+\frac{1}{2}i)}\right.$$
$$\left.+F_i'\frac{2^{2p}\Gamma(\frac{1}{2})\Gamma(1-2p-\rho+i)\Gamma(1-\rho)}{\Gamma(1-\rho+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-\left[\frac{i}{2}\right])\Gamma(-p+\frac{1}{2}-\rho+\frac{1}{2}i)}\right\}$$
$$+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\lambda)_{n+2p+1}(-\alpha)_n(\rho-i)_{2p+1}}{(-\alpha-\beta)_n(\delta)_{2p+1}\, n!\,(2p+1)!}\left[\frac{-2t}{2-(1-x)t}\right]^n\left[\frac{2y}{2-(1-x)t}\right]^{2p+1}$$
$$\times\left\{E_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(-2p-\rho+i)\Gamma(1-\rho)}{\Gamma(1-\rho+\frac{1}{2}(i+|i|))\Gamma(-p+\frac{1}{2}i-\left[\frac{1+i}{2}\right])\Gamma(-p+\frac{1}{2}-\rho+\frac{1}{2}i)}\right.$$
$$\left.+F_i''\frac{2^{2p+1}\Gamma(\frac{1}{2})\Gamma(-2p-\rho+i)\Gamma(1-\rho)}{\Gamma(1-\rho+\frac{1}{2}(i+|i|))\Gamma(-p-\frac{1}{2}+\frac{1}{2}i-\left[\frac{i}{2}\right])\Gamma(-p-\rho+\frac{1}{2}i)}\right\} \tag{33}$$

for $(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5)$.

Now, we mention some interesting special cases of the results (32) and (33) and we using in each case the results (20)–(23).

1. Taking $i=0$ in (32) and (33), we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_4(\lambda+n,\gamma;\delta,\delta;y,-y)t^n$$
$$=\left(1-\frac{(1-x)t}{2}\right)^{-\lambda}X_{0:3;1}^{1:2;1}\left[\begin{matrix}\lambda: & \frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2} & ; & -\alpha & ;\\ -: & \delta,\frac{1}{2}\delta,\frac{1}{2}\delta+\frac{1}{2} & ; & -\alpha-\beta & ;\end{matrix}\ -\left(\frac{2y}{2-(1-x)t}\right)^2,\frac{-2t}{2-(1-x)t}\right] \tag{34}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_1(\lambda+n,\rho,\rho;\delta;y,-y)t^n$$
$$=\left(1-\frac{(1-x)t}{2}\right)^{-\lambda}X_{0:2;1}^{1:1;1}\left[\begin{matrix}\lambda: & \rho & ; & -\alpha & ;\\ -: & \frac{1}{2}\delta,\frac{1}{2}\delta+\frac{1}{2} & ; & -\alpha-\beta & ;\end{matrix}\ \left(\frac{y}{2-(1-x)t}\right)^2,\frac{-2t}{2-(1-x)t}\right]. \tag{35}$$

2. Taking $i=1$ in (32) and (33), we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_4(\lambda+n,\gamma;\delta,\delta+1;y,-y)t^n$$
$$=\left(1-\frac{(1-x)t}{2}\right)^{-\lambda}\left\{X_{0:3;1}^{1:2;1}\left[\begin{matrix}\lambda: & \frac{1}{2}\gamma,\frac{1}{2}\gamma+\frac{1}{2} & ; & -\alpha & ;\\ -: & \delta,\frac{1}{2}\delta+\frac{1}{2},\frac{1}{2}\delta+1 & ; & -\alpha-\beta & ;\end{matrix}\ -\left(\frac{2y}{2-(1-x)t}\right)^2,\frac{-2t}{2-(1-x)t}\right]\right.$$
$$+\frac{2\lambda\gamma y}{\delta(\delta+1)(2-t+xt)}$$
$$\left.\times X_{0:3;1}^{1:2;1}\left[\begin{matrix}\lambda+1: & \frac{1}{2}\gamma+\frac{1}{2},\frac{1}{2}\gamma+1 & ; & -\alpha & ;\\ - : & \delta+1,\frac{1}{2}\delta+1,\frac{1}{2}\delta+\frac{3}{2} & ; & -\alpha-\beta & ;\end{matrix}\ -\left(\frac{2y}{2-(1-x)t}\right)^2,\frac{-2t}{2-(1-x)t}\right]\right\} \tag{36}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)F_1(\lambda+n,\rho-1,\rho;\delta;y,-y)t^n$$

$$= \left(1 - \frac{(1-x)t}{2}\right)^{-\lambda} \left\{ X^{1:1;1}_{0:2;1} \left[\begin{matrix} \lambda : & \rho & ; & -\alpha & ; \\ - : & \frac{1}{2}\delta, \frac{1}{2}\delta + \frac{1}{2} & ; & -\alpha - \beta & ; \end{matrix} \left(\frac{y}{2-(1-x)t}\right)^2, \frac{-2t}{2-(1-x)t} \right] \right.$$

$$\left. + \frac{2\lambda y}{\delta(2-t+xt)} X^{1:1;1}_{0:2;1} \left[\begin{matrix} \lambda+1 : & \rho & ; & -\alpha & ; \\ - : & \frac{1}{2}\delta + \frac{1}{2}, \frac{1}{2}\delta + 1 & ; & -\alpha - \beta & ; \end{matrix} \left(\frac{y}{2-(1-x)t}\right)^2, \frac{-2t}{2-(1-x)t} \right] \right\}. \tag{37}$$

3. Taking $i = -1$ in (32) and (33), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x) F_4(\lambda+n, \gamma; \delta, \delta-1; y, -y) t^n$$

$$= \left(1 - \frac{(1-x)t}{2}\right)^{-\lambda} \left\{ X^{1:2;1}_{0:3;1} \left[\begin{matrix} \lambda : & \frac{1}{2}\gamma, \frac{1}{2}\gamma + \frac{1}{2} & ; & -\alpha & ; \\ - : & \delta - 1, \frac{1}{2}\delta, \frac{1}{2}\delta + \frac{1}{2} & ; & -\alpha - \beta & ; \end{matrix} -\left(\frac{2y}{2-(1-x)t}\right)^2, \frac{-2t}{2-(1-x)t} \right] \right.$$

$$\left. - \frac{\lambda \gamma y}{\delta(2-t+xt)} X^{1:2;1}_{0:3;1} \left[\begin{matrix} \lambda+1 : & \frac{1}{2}\gamma + \frac{1}{2}, \frac{1}{2}\gamma + 1 & ; & -\alpha & ; \\ - : & \delta, \frac{1}{2}\delta + \frac{1}{2}, \delta + 1 & ; & -\alpha - \beta & ; \end{matrix} -\left(\frac{2y}{2-(1-x)t}\right)^2, \frac{-2t}{2-(1-x)t} \right] \right\} \tag{38}$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\beta-n)}(x) F_1(\lambda+n, \rho+1, \rho; \delta; y, -y) t^n$$

$$= \left(1 - \frac{(1-x)t}{2}\right)^{-\lambda} \left\{ X^{1:1;1}_{0:2;1} \left[\begin{matrix} \lambda : & \rho + 1 & ; & -\alpha & ; \\ - : & \frac{1}{2}\delta, \frac{1}{2}\delta + \frac{1}{2} & ; & -\alpha - \beta & ; \end{matrix} \left(\frac{y}{2-(1-x)t}\right)^2, \frac{-2t}{2-(1-x)t} \right] \right.$$

$$\left. + \frac{2\lambda y}{\delta(2-t+xt)} X^{1:1;1}_{0:2;1} \left[\begin{matrix} \lambda+1 : & \rho + 1 & ; & -\alpha & ; \\ - : & \frac{1}{2}\delta + \frac{1}{2}, \frac{1}{2}\delta + 1 & ; & -\alpha - \beta & ; \end{matrix} \left(\frac{y}{2-(1-x)t}\right)^2, \frac{-2t}{2-(1-x)t} \right] \right\}. \tag{39}$$

The other special cases of (32) and (33) can also be obtained in the similar manner.

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