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Perturbation analysis of a matrix differential equation $\dot{x} = ABx$

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Abstract

Two complex matrix pairs (A,B) and (A',B') are contragrediently equivalent if there are nonsingular S and R such that $(A',B') = (S^{-1}AR, R^{-1}BS)$. M.I. García-Planas and V.V. Sergeichuk (1999) constructed a miniversal deformation of a canonical pair (A,B) for contragredient equivalence; that is, a simple normal form to which all matrix pairs $(A+\widetilde{A}, B+\widetilde{B})$ close to (A,B) can be reduced by contragredient equivalence transformations that smoothly depend on the entries of $\widetilde{A}$ and $\widetilde{B}$. Each perturbation $(\widetilde{A},\widetilde{B})$ of (A,B) defines the first order induced perturbation $A\widetilde{B}+\widetilde{A}B$ of the matrix AB, which is the first order summand in the product $(A+\widetilde{A})(B+\widetilde{B}) = AB + A\widetilde{B} + \widetilde{A}B + \widetilde{A}\widetilde{B}$. We find all canonical matrix pairs (A,B), for which the first order induced perturbations $A\widetilde{B}+\widetilde{A}B$ are nonzero for all nonzero perturbations in the normal form of García-Planas and Sergeichuk. This problem arises in the theory of matrix differential equations $\dot{x} = Cx$, whose product of two matrices: $C = AB$; using the substitution $x = Sy$, one can reduce C by similarity transformations $S^{-1}CS$ and (A,B) by contragredient equivalence transformations $(S^{-1}AR, R^{-1}BS)$.

Keywords: Contragredient equivalence; Miniversal deformation; Perturbation.
AMS 2010 codes: 15A21; 93D13

1 Introduction

We study a matrix differential equation $\dot{x} = ABx$, whose matrix is a product of an $m\times n$ complex matrix A and an $n\times m$ complex matrix B. It is equivalent to $\dot{y} = S^{-1}ARR^{-1}BSy$, in which S and R are nonsingular matrices and $x = Sy$. Thus, we can reduce (A,B) by *transformations of contragredient equivalence*

$$(A,B) \mapsto (S^{-1}AR, R^{-1}BS), \qquad S \text{ and } R \text{ are nonsingular.} \tag{1}$$

The canonical form of (A,B) with respect to these transformations was obtained by Dobrovol′skaya and Ponomarev [3] and, independently, by Horn and Merino [5]:

$$\text{each pair } (A,B) \text{ is contragrediently equivalent to a direct sum, uniquely determined up to permutation of summands, of pairs of the types } (I_r, J_r(\lambda)),\ (J_r(0), I_r),\ (F_r, G_r),\ (G_r, F_r), \tag{2}$$

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in which $r = 1,2,\dots,$

$$J_r(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \ (\lambda \in \mathbb{C}), \quad F_r := \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix}, \quad G_r := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}$$

are $r \times r$, $r \times (r-1)$, $(r-1) \times r$ matrices, and

$$(A_1, B_1) \oplus (A_2, B_2) := (A_1 \oplus A_2, B_1 \oplus B_2).$$

Note that $(F_1, G_1) = (0_{10}, 0_{10})$; we denote by 0_{mn} the zero matrix of size $m \times n$, where $m,n \in \{0,1,2,\dots\}$. All matrices that we consider are complex matrices. All matrix pairs that we consider are counter pairs: a matrix pair (A,B) is a *counter pair* if A and B^T have the same size.

A notion of miniversal deformation was introduced by Arnold [1, 2]. He constructed a miniversal deformation of a Jordan matrix J; i.e., a simple normal form to which all matrices $J+E$ close to J can be reduced by similarity transformations that smoothly depend on the entries of E. García-Planas and Sergeichuk [4] constructed a miniversal deformation of a canonical pair (2) for contragredient equivalence (1).

For a counter matrix pair (A,B), we consider all matrix pairs $(A+\widetilde{A}, B+\widetilde{B})$ that are sufficiently close to (A,B). The pair $(\widetilde{A},\widetilde{B})$ is called a *perturbation* of (A,B). Each perturbation $(\widetilde{A},\widetilde{B})$ of (A,B) defines the *induced perturbation* $A\widetilde{B}+\widetilde{A}B+\widetilde{A}\widetilde{B}$ of the matrix AB that is obtained as follows:

$$(A+\widetilde{A})(B+\widetilde{B}) = AB + A\widetilde{B} + \widetilde{A}B + \widetilde{A}\widetilde{B}.$$

Since $\widetilde{A}$ and $\widetilde{B}$ are small, their product $\widetilde{A}\widetilde{B}$ is "very small"; we ignore it and consider only *first order induced perturbations* $A\widetilde{B}+\widetilde{A}B$ of AB.

In this paper, we describe all canonical matrix pairs (A,B) of the form (2), for which the first order induced perturbations $A\widetilde{B}+\widetilde{A}B$ are nonzero for all miniversal perturbations $(\widetilde{A},\widetilde{B}) \neq 0$ in the normal form defined in [4].

Note that $z = ABx$ can be considered as the superposition of the systems $y = Bx$ and $z = Ay$:

$$x \longrightarrow \boxed{B} \xrightarrow{y} \boxed{A} \longrightarrow z \qquad \text{implies} \qquad x \longrightarrow \boxed{AB} \longrightarrow z$$

2 Miniversal deformations of counter matrix pairs

In this section, we recall the miniversal deformations of canonical pairs (2) for contragredient equivalence constructed by García-Planas and Sergeichuk [4].

Let

$$(A,B) = (I,C) \oplus \bigoplus_{j=1}^{t_1} (I_{r_{1j}}, J_{r_{1j}}) \oplus \bigoplus_{j=1}^{t_2} (J_{r_{2j}}, I_{r_{2j}}) \oplus \bigoplus_{j=1}^{t_3} (F_{r_{3j}}, G_{r_{3j}}) \oplus \bigoplus_{j=1}^{t_4} (G_{r_{4j}}, F_{r_{4j}}) \tag{3}$$

be a canonical pair for contragredient equivalence, in which

$$C := \bigoplus_{i=1}^{t} \Phi(\lambda_i), \qquad \Phi(\lambda_i) := J_{m_{i1}}(\lambda_i) \oplus \dots \oplus J_{m_{ik_i}}(\lambda_i) \qquad \text{with } \lambda_i \neq \lambda_j \text{ if } i \neq j,$$

$m_{i1} \geqslant m_{i2} \geqslant \dots \geqslant m_{ik_i}$, and $r_{i1} \geqslant r_{i2} \geqslant \dots \geqslant r_{it_i}$.

For each matrix pair (A,B) of the form (3), we define the matrix pair

$$\Big(I, \bigoplus_i \Phi(\lambda_i) + N\Big) \oplus \left(\left[\begin{array}{c|c|c} \oplus_j I_{r_{1j}} & 0 & 0 \\ \hline 0 & \oplus_j J_{r_{2j}}(0) + N & N \\ \hline 0 & N & \begin{matrix} P_3 & N \\ 0 & Q_4 \end{matrix} \end{array}\right], \left[\begin{array}{c|c|c} \oplus_j J_{r_{1j}}(0) + N & N & N \\ \hline N & \oplus_j I_{r_{2j}} & 0 \\ \hline N & 0 & \begin{matrix} Q_3 & 0 \\ N & P_4 \end{matrix} \end{array}\right] \right), \tag{4}$$

of the same size and of the same partition of the blocks, in which

$$N := [H_{ij}] \tag{5}$$

is a parameter block matrix with $p_i \times q_j$ blocks H_{ij} of the form

$$H_{ij} := \left[\begin{matrix} * & \\ \vdots & 0 \\ * & \end{matrix}\right] \text{ if } p_i \leqslant q_j, \qquad H_{ij:} = \left[\begin{matrix} 0 \\ * \cdots * \end{matrix}\right] \text{ if } p_i > q_j. \tag{6}$$

$$P_l := \begin{bmatrix} F_{r_{l1}}+H & H & \cdots & H \\ & F_{r_{l2}}+H & \ddots & \vdots \\ & & \ddots & H \\ 0 & & & F_{r_{lt_l}}+H \end{bmatrix}, \qquad Q_l := \begin{bmatrix} G_{r_{l1}} & & & 0 \\ H & G_{r_{l2}} & & \\ \vdots & \ddots & \ddots & \\ H & \cdots & H & G_{r_{lt_l}} \end{bmatrix} \qquad (l = 3,4), \tag{7}$$

N and H are matrices of the form (5) and (6), and the stars denote independent parameters.

Theorem 1 (see [4])**.** *Let (A,B) be the canonical pair (3). Then all matrix pairs $(A+\widetilde{A}, B+\widetilde{B})$ that are sufficiently close to (A,B) are simultaneously reduced by some transformation*

$$(A+\widetilde{A}, B+\widetilde{B}) \mapsto (S^{-1}(A+\widetilde{A})R, R^{-1}(B+\widetilde{B})S),$$

in which S and R are matrix functions that depend holomorphically on the entries of $\widetilde{A}$ and $\widetilde{B}$, $S(0) = I$, and $R(0) = I$, to the form (4), whose stars are replaced by complex numbers that depend holomorphically on the entries of $\widetilde{A}$ and $\widetilde{B}$. The number of stars is minimal that can be achieved by such transformations.

3 Main theorem

Each matrix pair $(A+\widetilde{A}, B+\widetilde{B})$ of the form (4), in which the stars are complex numbers, we call a *miniversal normal pair* and $(\widetilde{A}, \widetilde{B})$ a *miniversal perturbation* of (A,B).

The following theorem is the main result of the paper.

Theorem 2. *Let (A,B) be a canonical pair (2). The following two conditions are equivalent:*

(a) $A\widetilde{B} + \widetilde{A}B \neq 0$ *for all nonzero miniversal perturbations* $(\tilde{A}, \tilde{B})$.

(b) (A,B) *does not contain*

  (i) $(I_r, J_r(0)) \oplus (J_r(0), I_r)$ *for each r,*

  (ii) $(F_1, G_1) \oplus (G_2, F_2)$*, and*

  (iii) $(F_m, G_m) \oplus (G_m, F_m)$ *for each m.*

Proof. (a) $\Longrightarrow$ (b). Let (A,B) be a canonical pair (2). We should prove that if (A,B) contains a pair of type (i), (ii), or (iii), then $A\widetilde{B} + \widetilde{A}B = 0$ for some miniversal perturbation $(\widetilde{A}, \widetilde{B}) \neq (0,0)$. It is sufficient to prove this statement for (A,B) of types (i)–(iii).

Case 1: $(A,B) = (I_r, J_r(0)) \oplus (J_r(0), I_r)$ for some r. We should prove that there exists a nonzero miniversal perturbation $(\widetilde{A}, \widetilde{B})$ such that $A\widetilde{B} + \widetilde{A}B = 0$.

If $r = 1$, then

$$(A,B) = (I_1, J_1(0)) \oplus (J_1(0), I_1) = \left(\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right] \right).$$

Its miniversal deformation (4) has the form

$$\left(\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \varepsilon \end{array}\right], \left[\begin{array}{c|c} \lambda & \mu \\ \hline \delta & 1 \end{array}\right]\right),$$

in which $\varepsilon, \lambda, \mu$ and δ are independent parameters. We have that

$$A\widetilde{B} + \widetilde{A}B = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \varepsilon \end{array}\right] + \left[\begin{array}{c|c} \lambda & \mu \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \lambda & \mu \\ \hline 0 & \varepsilon \end{array}\right].$$

Choosing $\varepsilon = \mu = \lambda = 0$ and $\delta \neq 0$, we get $\widetilde{A}B + \widetilde{B}A = 0$.

If $r = 2$, then $(A,B) = (I_2, J_2(0)) \oplus (J_2(0), I_2)$ and

$$(A+\widetilde{A}, B+\widetilde{B}) = \left(\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & \varepsilon_7 & \varepsilon_8 \end{array}\right], \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \hline \varepsilon_5 & 0 & 1 & 0 \\ \varepsilon_6 & 0 & 0 & 1 \end{array}\right]\right),$$

We get

$$A\widetilde{B} + \widetilde{A}B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_7 & \varepsilon_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ \varepsilon_6 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_7 & \varepsilon_8 \end{bmatrix}.$$

Choosing $\varepsilon_5 \neq 0$ and $\varepsilon_i = 0$ if $i \neq 5$, we get $A\widetilde{B} + \widetilde{A}B = 0$.

If r is arbitrary, then $(A,B) = (I_r, J_r(0)) \oplus (J_r(0), I_r)$ and its miniversal deformation has the form

$$\left(\left[\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & 0 & & \\ & & & 1 & & & & \\ \hline & & & & 0 & 1 & & \\ & & & & & \ddots & \ddots & \\ & 0 & & & & & 0 & 1 \\ & & & & \alpha_1 & \alpha_2 & \dots & \alpha_s \end{array}\right], \left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & \\ & \ddots & \ddots & & & 0 & & \\ & & 0 & 1 & & & & \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_r & \varepsilon_{r+1} & \varepsilon_{r+2} & \dots & \varepsilon_{r+s} \\ \hline \beta_1 & & & & 1 & & & \\ \beta_2 & & & & & 1 & & \\ \vdots & & 0 & & & & \ddots & \\ \beta_s & & & & & & & 1 \end{array}\right]\right),$$

in which all α_i, β_i, ε_i are independent parameters. Taking all parameters zero except for $\beta_1 \neq 0$, we get that $A\widetilde{B} + \widetilde{A}B = 0$.

Case 2: $(A,B) = (F_1, G_1) \oplus (G_2, F_2)$. Then

$$(A+\widetilde{A}, B+\widetilde{B}) = \left(\begin{bmatrix} \varepsilon & \delta \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \lambda & \mu \end{bmatrix}\right),$$

in which $\varepsilon, \delta, \lambda$ and μ are independent parameters. We get

$$A\widetilde{B} + \widetilde{A}B = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \lambda & \mu \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \\ \lambda & \mu \end{bmatrix}.$$

Taking all parameters zero except for $\delta \neq 0$, we get that $A\widetilde{B} + \widetilde{A}B = 0$.

Case 3: $(A,B) = (F_m, G_m) \oplus (G_m, F_m)$ *for some m.*

If $m = 1$, then $(A,B) = (F_1,G_1) \oplus (G_1,F_1) = (0,0)$. For each perturbation $(\widetilde{A},\widetilde{B}) \neq (0,0)$, we get $A\widetilde{B} + \widetilde{A}B = 0$.

If $m = 2$, then the miniversal deformation (4) of (A,B) is

$$(A+\widetilde{A}, B+\widetilde{B}) = \left(\left[\begin{array}{c|cc} 1 & \alpha & 0 \\ \varepsilon & \beta & 0 \\ \hline 0 & 0 & 1 \end{array} \right], \left[\begin{array}{cc|c} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \lambda & \mu & \delta \end{array} \right] \right)$$

in which $\varepsilon, \alpha, \beta, \lambda, \mu$ and δ are independent parameters. We obtain

$$A\widetilde{B} + \widetilde{A}B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & \mu & \delta \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & \beta \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & \beta \\ \lambda & \mu & \delta \end{bmatrix}.$$

Choosing all parameters zero except for $\alpha \neq 0$, we get $A\widetilde{B} + \widetilde{A}B = 0$.

If r is arbitrary, then the miniversal deformation (4) of (A,B) has the form

$$\left(\left[\begin{array}{ccc|ccc} 1 & & 0 & \varepsilon_r & & \\ & \ddots & & \vdots & & 0 \\ 0 & & 1 & \varepsilon_{2r-2} & & \\ \varepsilon_1 & \dots & \varepsilon_{r-1} & \varepsilon_{2r-1} & & \\ \hline & & & 0 & 1 & 0 \\ & 0 & & & \ddots & \ddots \\ & & & 0 & & 0 \;\; 1 \end{array} \right], \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ & \ddots & \ddots & & 0 & \\ 0 & & 0 \;\; 1 & & & \\ \hline & & & 1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & 1 \\ \alpha_1 & \alpha_2 & \dots \; \alpha_r & \alpha_{r+1} & \dots & \alpha_{2r-1} \end{array} \right] \right)$$

in which all α_i and ε_j are independent parameters. Since the rth row of B is zero, a parameter ε_{2r-2} does not appear in $\widetilde{A}B$, and so in $A\widetilde{B} + \widetilde{A}B$ too. Choosing all parameters zeros except for $\varepsilon_{2r-2} \neq 0$, we get $A\widetilde{B} + \widetilde{A}B = 0$.

(b) $\Longrightarrow$ (a). Let us prove that if there exists a nonzero miniversal perturbation $(\tilde{A},\tilde{B})$ such that $A\widetilde{B} + \widetilde{A}B = 0$, then (A,B) contains $(I_r,J_r(0)) \oplus (J_r(0),I_r)$ for some r, or $(F_1,G_1) \oplus (G_2,F_2)$, or $(F_m,G_m) \oplus (G_m,F_m)$ for some m.

Since the deformation (4) is the direct sum of

$$\Big(I, \bigoplus_i (\Phi(\lambda_i) + N)\Big) \quad \text{and} \quad \left(\left[\begin{array}{c|c|c} \oplus_j I_{r_{1j}} & 0 & 0 \\ \hline 0 & \oplus_j J_{r_{2j}}(0) + N & N \\ \hline 0 & N & \begin{array}{cc} P_3 & N \\ 0 & Q_4 \end{array} \end{array} \right], \left[\begin{array}{c|c|c} \oplus_j J_{r_{1j}}(0) + N & N & N \\ \hline N & \oplus_j I_{r_{2j}} & 0 \\ \hline N & 0 & \begin{array}{cc} Q_3 & 0 \\ N & P_4 \end{array} \end{array} \right] \right),$$

it is sufficient to consider (A,B) equals

$$\Big(I, \bigoplus_i (\Phi(\lambda_i))\Big) \quad \text{or} \quad \bigoplus_{j=1}^{t_1} (I_{r_{1j}}, J_{r_{1j}}) \oplus \bigoplus_{j=1}^{t_2} (J_{r_{2j}}, I_{r_{2j}}) \oplus \bigoplus_{j=1}^{t_3} (F_{r_{3j}}, G_{r_{3j}}) \oplus \bigoplus_{j=1}^{t_4} (G_{r_{4j}}, F_{r_{4j}}). \tag{8}$$

Let first $(A,B) = \big(I, \bigoplus_i (\Phi(\lambda_i))\big)$. Then

$$(A+\widetilde{A}, B+\widetilde{B}) = \left(\left[\begin{array}{c|c|c} \oplus_j I_{r_{1j}} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \oplus_j I_{r_{lj}} \end{array} \right], \left[\begin{array}{c|c|c} \oplus_j J_{r_{1j}}(\lambda_1) + N & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \oplus_j J_{r_{lj}}(\lambda_l) + N \end{array} \right] \right).$$

If

$$\widetilde{A}B+A\widetilde{B}=\left[\begin{array}{c|c|c} N & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & N \end{array}\right]=0,$$

in which all N have independent parameters, then all N are zero, and so $(\widetilde{A},\widetilde{B})=(0,0)$.

It remains to consider (A,B) equaling the second pair in (8). Write the matrices (7) as follows:

$$P_l=\overline{P}_l+\underline{P}_l, \qquad Q_l=\overline{Q}_l+\underline{Q}_l, \qquad \text{in which } \; l=3,4,$$

$$\overline{P}_l=\begin{bmatrix} F_{r_{l1}} & 0 & \cdots & 0 \\ & F_{r_{l2}} & \ddots & \vdots \\ & & \ddots & 0 \\ 0 & & & F_{r_{lt_l}} \end{bmatrix}, \qquad \underline{P}_l=\begin{bmatrix} H_{r_{l1}} & H & \cdots & H \\ & H_{r_{l2}} & \ddots & \vdots \\ & & \ddots & H \\ 0 & & & H_{r_{lt_l}} \end{bmatrix},$$

$$\overline{Q}_l=\begin{bmatrix} G_{r_{l1}} & & & 0 \\ 0 & G_{r_{l2}} & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & G_{r_{lt_l}} \end{bmatrix}, \qquad \underline{Q}_l=\begin{bmatrix} 0_{r_{l1}} & & & 0 \\ H & 0_{r_{l2}} & & \\ \vdots & \ddots & \ddots & \\ H & \cdots & H & 0_{r_{lt_l}} \end{bmatrix},$$

N and H are matrices of the form (5) and (6), and the stars denote independent parameters.

Write

$$J_1:=\oplus_j J_{r_{1j}}(0), \qquad J_2:=\oplus_j J_{r_{2j}}(0). \tag{9}$$

Then

$$A=\left[\begin{array}{c|c|cc} I & 0 & 0 & 0 \\ \hline 0 & J_2 & 0 & 0 \\ \hline 0 & 0 & \overline{P}_3 & 0 \\ 0 & 0 & 0 & \overline{Q}_4 \end{array}\right], \qquad \widetilde{A}=\left[\begin{array}{c|c|cc} 0 & 0 & 0 & 0 \\ \hline 0 & N & N & N \\ \hline 0 & N & \underline{P}_3 & N \\ 0 & N & 0 & \underline{Q}_4 \end{array}\right],$$

$$B=\left[\begin{array}{c|c|cc} J_1 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 \\ \hline 0 & 0 & \overline{Q}_3 & 0 \\ 0 & 0 & 0 & \overline{P}_4 \end{array}\right], \qquad \widetilde{B}=\left[\begin{array}{c|c|cc} N & N & N & N \\ \hline N & 0 & 0 & 0 \\ \hline N & 0 & \underline{Q}_3 & 0 \\ N & 0 & N & \underline{P}_4 \end{array}\right],$$

$$A\widetilde{B}=\left[\begin{array}{c|c|cc} N & N & N & N \\ \hline J_2N & 0 & 0 & 0 \\ \hline \overline{P}_3N & 0 & \overline{P}_3\underline{Q}_3 & 0 \\ \overline{Q}_4N & 0 & \overline{Q}_4N & \overline{Q}_4\underline{P}_4 \end{array}\right], \qquad \widetilde{A}B=\left[\begin{array}{c|c|cc} 0 & 0 & 0 & 0 \\ \hline 0 & N & N\overline{Q}_3 & N\overline{P}_4 \\ \hline 0 & N & \underline{P}_3\overline{Q}_3 & N\overline{P}_4 \\ 0 & N & 0 & \underline{Q}_4\overline{P}_4 \end{array}\right],$$

in which we denote by N blocks of the form (5). All blocks denoted by N have distinct sets of independent parameters and may have distinct sizes.

Since $\widetilde{A}B$ and $A\widetilde{B}$ have independent parameters for each (A,B), we should prove that $\widetilde{A}B\neq 0$ for all $\widetilde{A}\neq 0$ and $\widetilde{B}A\neq 0$ for all $\widetilde{B}\neq 0$. Thus, we should prove that

$$J_2N, \quad N\overline{P}_4, \quad \overline{P}_3N, \quad N\overline{Q}_3, \quad \overline{Q}_4N \tag{10}$$

are nonzero if the corresponding parameter blocks N are nonzero.

Let us consider the first matrix in (10):

$$J_2N = \begin{bmatrix} J_{r_1} & & & 0 \\ & J_{r_2} & & \\ & & \ddots & \\ 0 & & & J_{r_n} \end{bmatrix} \begin{bmatrix} H_{r_1} & & & 0 \\ & H_{r_2} & & \\ & & \ddots & \\ 0 & & & H_{r_n} \end{bmatrix} = \begin{bmatrix} & 0 & \\ \varepsilon_{11} & \dots & \varepsilon_{1m_1} \\ 0 & \dots & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} & 0 & \\ \varepsilon_{n1} & \dots & \varepsilon_{nm_n} \\ 0 & \dots & 0 \end{bmatrix},$$

in which all ε_{ij} are independent parameters and $r_1 \leqslant r_2 \leqslant \dots \leqslant r_n$. Clearly, $J_2N \neq 0$ if at least one $\varepsilon_{ij} \neq 0$.

Let us consider the second matrix in (10):

$$N\overline{P}_4 = \begin{bmatrix} H_{r_1} & & & 0 \\ & H_{r_2} & & \\ & & \ddots & \\ 0 & & & H_{r_n} \end{bmatrix} \begin{bmatrix} F_{r_1} & & & 0 \\ & F_{r_2} & & \\ & & \ddots & \\ 0 & & & F_{r_n} \end{bmatrix} = \begin{bmatrix} & 0 & \\ \varepsilon_{11} & \dots & \varepsilon_{1m_1} \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} & 0 & \\ \varepsilon_{n1} & \dots & \varepsilon_{nm_n} \end{bmatrix}.$$

in which all ε_j are independent parameters and $r_1 \geqslant r_2 \geqslant \dots \geqslant r_n$. Clearly, $N\overline{P}_4 \neq 0$ if at least one $\varepsilon_{ij} \neq 0$.

The matrices $\overline{P}_3N$, $\overline{Q}_4N$, $N\overline{Q}_3$, and $\overline{Q}_4N$ in (10) are considered analogously.

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