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Travelling waves and conservation laws of a (2+1)-dimensional coupling system with Korteweg-de Vries equation

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Abstract

In this paper we study a (2+1)-dimensional coupling system with the Korteweg-de Vries equation, which is associated with non-semisimple matrix Lie algebras. Its Lax-pair and bi-Hamiltonian formulation were obtained and presented in the literature. We utilize Lie symmetry analysis along with the $(G'/G)-$expansion method to obtain travelling wave solutions of this system. Furthermore, conservation laws are constructed using the multiplier method.

Keywords: (2+1)-dimensional coupling system with the Korteweg-de Vries equation, Lie symmetries, $(G'/G)-$expansion method, travelling wave solutions, conservation laws.
AMS 2010 codes: 35C07, 35L65.

1 Introduction

It is well-known that nonlinear partial differential equations (NLPDEs) are extensively used to model many nonlinear physical phenomena of the real world, which can be seen from the number of research papers published in the literature. One such NLPDE is the celebrated Korteweg-de Vries (KdV) equation [1]

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

which has applications in nonlinear dynamics, plasma physics and mathematical physics. It is an important equation in scientific fields and in the theory of integrable systems. It describes the unidirectional propagation of

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long waves of small amplitude and has a lot of applications in a number of physical contexts such as hydromagnetic waves, stratified internal waves, ion-acoustic waves, plasma physics and lattice dynamics [2]. Equation (1) has multiple-soliton solutions and an infinite number of conservation laws and many other physical properties. See for example [3–5] and references therein.

Recently focus has shifted to the study of coupled systems of Korteweg-de Vries equations because of their many applications in scientific fields. See for example [5–9].

However, in this work we study the (2+1)-dimensional coupling system with the Korteweg-de Vries equation [2], namely

$$ckdv11u_t - 6uu_x - u_{xxx} = 0, \tag{2a}$$

$$ckdv12v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx} = 0. \tag{2b}$$

This system is a (2+1)-dimensional integrable coupling with the Korteweg-de Vries equation, which is associated with non-semisimple matrix Lie algebras. In the references [10] and [11], its Lax pair and bi-Hamiltonian formulation were presented respectively. It should be noted that its bi-Hamiltonian structure is the first example of local bi-Hamiltonian structures, which lead to hereditary recursion operators in (2+1)-dimensions.

Several methods have been developed to find exact solutions of the NLPDEs. Some of these are the homogeneous balance method [12], the ansatz method [13], the inverse scattering transform method [14], the Bäcklund transformation [15], the Darboux transformation [16], the Hirota bilinear method [17], the simplest equation method [18], the $(G'/G)-$expansion method [19, 20], the Jacobi elliptic function expansion method [21], the Kudryashov method [22], the Lie symmetry method [23–28].

The outline of the paper is as follows. In Section 2 we determine the travelling wave solutions for the system (2a) using the Lie symmetry method along with the $(G'/G)-$expansion method. Conservation laws for (2a) are constructed in Section 3 by employing the multiplier approach [26, 29–37]. Finally concluding remarks are presented in Section 4.

2 Travelling wave solutions of (2a)

In this section we use Lie symmetry analysis together with the $(G'/G)-$expansion method to obtain travelling wave solutions of (2a).

2.1 Lie point symmetries and symmetry reductions of (2a)

Lie symmetry analysis was introduced by Marius Sophus Lie (1842-1899), a Norwegian mathematician, in the later half of the nineteenth century. He developed the theory of continuous symmetry groups and applied it to the study of geometry and differential equations. This theory contains powerful methods which can be used to obtain exact analytical solutions of differential equations [23–25]. The theory is called symmetry groups theory or the classical Lie method of infinitesimal transformations. The symmetry group of a differential equation is the largest local Lie group of transformations of the independent and dependent variables of the differential equation that transforms solutions of the differential equation to other solutions. The symmetry group associated to a differential equation can be obtained by Lie's infinitesimal criterion of invariance.

The (2+1)-dimensional coupling system with the Korteweg-de Vries equation (2a) is invariant under the symmetry group with the generator

$$\Gamma = \xi^1(t,x,y,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,y,u,v)\frac{\partial}{\partial x} + \xi^3(t,x,y,u,v)\frac{\partial}{\partial y} + \eta^1(t,x,y,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,y,u,v)\frac{\partial}{\partial v} \tag{3}$$

if and only if

$$ckdv21 \qquad \Gamma^{[3]}\left(u_t - 6uu_x - u_{xxx}\right)|_{(2a)} = 0, \tag{4a}$$

$$ckdv22 \qquad \Gamma^{[3]}\left(v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx}\right)|_{(2a)} = 0, \tag{4b}$$

where $\Gamma^{[3]}$ denotes the third prolongation [23] of the generator (3) and the symbol $|_{(2a)}$ means it is evaluated on equations (2a). The third prolongation $\Gamma^{[3]}$ is given by

$$\Gamma^{[3]} = \Gamma + \zeta_t^1 \frac{\partial}{\partial u_t} + \zeta_x^1 \frac{\partial}{\partial u_x} + \zeta_y^1 \frac{\partial}{\partial u_y} + \zeta_t^2 \frac{\partial}{\partial v_t} + \zeta_x^2 \frac{\partial}{\partial v_x} + \zeta_{xxx}^1 \frac{\partial}{\partial u_{xxx}} + \zeta_{xxy}^1 \frac{\partial}{\partial u_{xxy}} + \zeta_{xxx}^2 \frac{\partial}{\partial v_{xxx}}, \tag{5}$$

with

$$\begin{aligned}
\zeta_t^1 &= D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2) - u_y D_t(\xi^3),\\
\zeta_x^1 &= D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3),\\
\zeta_y^1 &= D_y(\eta^1) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3),\\
\zeta_t^2 &= D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_t(\xi^2) - v_y D_t(\xi^3),\\
\zeta_x^2 &= D_x(\eta^2) - v_t D_x(\xi^1) - v_x D_x(\xi^2) - v_y D_x(\xi^3),\\
\zeta_{xx}^1 &= D_x(\zeta_x^1) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3),\\
\zeta_{xx}^2 &= D_x(\zeta_x^2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2) - v_{xy} D_x(\xi^3),\\
\zeta_{xxx}^1 &= D_x(\zeta_{xx}^1) - u_{txx} D_x(\xi^1) - u_{xxx} D_x(\xi^2) - u_{xxy} D_x(\xi^3),\\
\zeta_{xxy}^1 &= D_x(\zeta_{xx}^1) - u_{txy} D_x(\xi^1) - u_{xxy} D_x(\xi^2) - u_{yxy} D_x(\xi^3),\\
\zeta_{xxx}^2 &= D_x(\zeta_{xx}^2) - v_{txx} D_x(\xi^1) - v_{xxx} D_x(\xi^2) - v_{xxy} D_x(\xi^3),
\end{aligned}$$

and D_t, D_x and D_y are the operators of total differentiation defined as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + \cdots,\\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + \cdots,\\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v} + \cdots,
\end{aligned} \tag{6}$$

respectively. Expanding (4a) and then splitting on the derivatives of u and v, we obtain the following overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi_x^1 = 0,\ \xi_y^1 = 0,\ \xi_u^1 = 0,\ \xi_v^1 = 0,\ \xi_u^2 = 0,\ \xi_v^2 = 0,\ \xi_t^3 = 0,\ \xi_x^3 = 0,\ \xi_y^2 = 0\\
&\xi_v^3 = 0,\ \eta_v^1 = 0,\ \eta_{uu}^1 = 0,\ \eta_{uu}^2 = 0,\ \eta_{vv}^2 = 0,\ \eta_{uv}^2 = 0,\ \eta_{xu}^1 - \xi_{xx}^2 = 0,\\
&2\eta_{xu}^1 - \xi_{xx}^2 = 0,\ \eta_{xv}^2 - \xi_{xx}^2 = 0,\ \eta_{xu}^2 + \eta_{yu}^1 = 0,\ -3\xi_x^2 + \xi_t^1 = 0,\\
&4u\xi_x^2 + 2\eta^1 + \eta_{xxu}^1 = 0, \eta_{xxx}^1 - \eta_t^1 + 6u\eta_x^1 = 0 - \eta_u^1 + \xi_y^3 + \eta_v^2 - \xi_x^2 = 0,\\
&2v\xi_x^2 + 2v\xi_y^3 + 2\eta^2 + \eta_{xxu}^2 + 2\eta_{xyu}^1 = 0, 12u\xi_x^2 + 6\eta^1 - \xi_{xxx}^2 + 3\eta_{xxu}^1 + \xi_t^2 = 0,\\
&\xi_t^1 - 2\xi_x^2 - \xi_y^3 - \eta_v^2 + \eta_u^1 = 0, 6u\eta_y^1 + 6v\eta_x^1 - \eta_t^2 + 3\eta_{yxx}^1 + \eta_{xxx}^2 + 6u\eta_x^2 = 0,\\
&6u\xi_x^2 + 6u\xi_y^3 - 6u\eta_u^1 + 6\eta^1 - \xi_{xxx}^2 + 3\eta_{xxv}^2 + \xi_t^2 + 6u\eta_v^2 = 0.
\end{aligned}$$

Solving the above system of partial differential equations, one obtains

$$\xi^1 = C_1 + 3C_3 t,\ \xi^2 = C_2 + C_3 x,\ \xi^3 = C_4\{F(y) - vF'(y)\},\ \eta^1 = -2C_3 u,\ \eta^2 = -C_3 v,$$

where $C_1,\cdots,C_4$ are arbitrary constants and $F(y)$ is an arbitrary function of y. Thus the Lie algebra of infinitesimal symmetries of the system (2a) is spanned by the four vector fields

$$\begin{aligned} time\,translation\;\; \Gamma_1 &= \frac{\partial}{\partial t}, \\ space\,translation\;\; \Gamma_2 &= \frac{\partial}{\partial x}, \\ scaling\;\; \Gamma_3 &= 3t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-2u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}, \\ \Gamma_4 &= F(y)\frac{\partial}{\partial y}-vF'(y)\frac{\partial}{\partial y}. \end{aligned}$$

We now use these Lie point symmetries to find exact solutions of (2a). The linear combination of the three symmetries Γ_1, Γ_2 and Γ_4 with $F(y)=1$ provides us with the three invariants

$$f=t-y,\; g=x-y,\; u=\theta(f,g),\; v=\psi(f,g),$$

the system (2a) is reduced to a system of partial differential equations of two functions θ and ψ in two independent variables f and g;

$$ckdv41\theta_f-6\theta\theta_g-\theta_{ggg}=0, \tag{7a}$$
$$ckdv42\psi_f-6(\theta\psi)_g+6\theta\theta_f+6\theta\theta_g+3\theta_{ggf}+3\theta_{ggg}-\psi_{ggg}=0. \tag{7b}$$

System (7a) has the following symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial f}, \\ X_2 &= \frac{\partial}{\partial g}, \\ X_3 &= 6f\frac{\partial}{\partial f}+(2g+3f)\frac{\partial}{\partial g}-(4\theta+\frac{1}{2})\frac{\partial}{\partial\theta}-(8\psi+2\theta+\frac{1}{2})\frac{\partial}{\partial\psi}, \end{aligned}$$

Considering the symmetry $X=X_1+\alpha X_2$ given by the linear combination of X_1 and X_2 we get the invariants

$$z=g-\alpha f,\; \theta=H(z),\; \psi=J(z).$$

This further reduces (2a) to a system of third-order ordinary differential equations in two functions $H(z)$ and $J(z)$.

$$ckdv51\alpha H'+6HH'+H'''=0, \tag{8a}$$
$$ckdv52\alpha J'+6(HJ)'+6(\alpha-1)HH'+3(\alpha-1)H'''+J'''=0, \tag{8b}$$

where the prime denotes derivative with respect to z.

2.2 Application of the $(G'/G)-$expansion method

In this section we employ the $(G'/G)-$expansion method to construct travelling wave solutions of the system of third order ordinary differential equatons (8a). This method was developed by the authors of [19] and has been extensively used by researchers. It assumes the solutions of the system (8a) to be of the form

$$H(z)=\sum_{i=0}^{M}\mathscr{A}_i\left(\frac{G'(z)}{G(z)}\right)^i, \qquad J(z)=\sum_{j=0}^{N}\mathscr{B}_j\left(\frac{G'(z)}{G(z)}\right)^j, \tag{9}$$

where $G(z)$ satisfies the second-order ODE given by

$$G'' + \lambda G' + \mu G = 0 \tag{10}$$

with λ and μ being arbitrary constants. The homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (8a) determines the values of M and N. The parameters $\mathscr{A}_i$ and $\mathscr{B}_j$, $i = 0, 1, \cdots, M$ and $j = 0, 1, \cdots, N$ need to be determined. In our case the balancing procedure yields $M = 2$ and $N = 2$, so the solutions of the system of ordinary differential equations (8a) are of the form

$$H(z) = \mathscr{A}_0 + \mathscr{A}_1(G'/G) + \mathscr{A}_2(G'/G)^2,\ J(z) = \mathscr{B}_0 + \mathscr{B}_1(G'/G) + \mathscr{B}_2(G'/G)^2. \tag{11}$$

Substituting (11) into (8a) and making use of (10), and then collecting all terms with same powers of (G'/G) and equating each coefficient to zero, yields a system of algebraic equations. Solving this system of algebraic equations, using Mathematica, we obtain the following set of values for the constants $\mathscr{A}_i$ and $\mathscr{B}_j$, $i, j = 0, 1, 2$:

$$\begin{aligned}
&\mathscr{A}_0 = -\frac{1}{6}\left(\alpha + \lambda^2 + 8\mu\right),\ \mathscr{A}_1 = -2\lambda,\ \mathscr{A}_2 = -2,\\
&\mathscr{B}_0 = \frac{1}{6}(\alpha - 1)\left(\alpha - 2\lambda^2 - 16\mu\right),\ \mathscr{B}_1 = 4\lambda(\alpha - 1),\ \mathscr{B}_2 = 4(\alpha - 1).
\end{aligned}$$

Substituting these values of $\mathscr{A}_i$ and $\mathscr{B}_j$ into the corresponding solutions (11) of ordinary differential equations (5), we obtain the following three types of travelling wave solutions of equation (2a):

Case 1: When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$\begin{aligned}
u_1(t,x,y) = &-\frac{1}{6}\left(\alpha + \lambda^2 + 8\mu\right) - 2\lambda\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}\right)\right]\\
&-2\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}\right)\right]^2,
\end{aligned} \tag{12}$$

$$\begin{aligned}
v_1(t,x,y) = &\frac{1}{6}(\alpha - 1)\left(\alpha - 2\lambda^2 - 16\mu\right) + 4\lambda(\alpha - 1)\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}\right)\right]\\
&+4(\alpha - 1)\left[-\frac{\lambda}{2} + \delta_1\left(\frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}\right)\right]^2,
\end{aligned} \tag{13}$$

where $z = x + (\alpha - 1)y - \alpha t$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

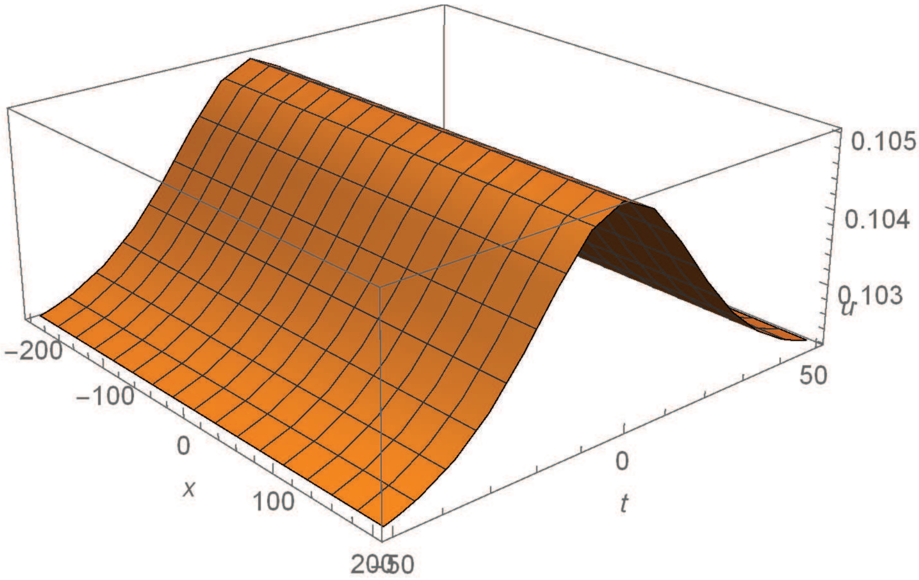

Fig. 1 Profile of solution (12)

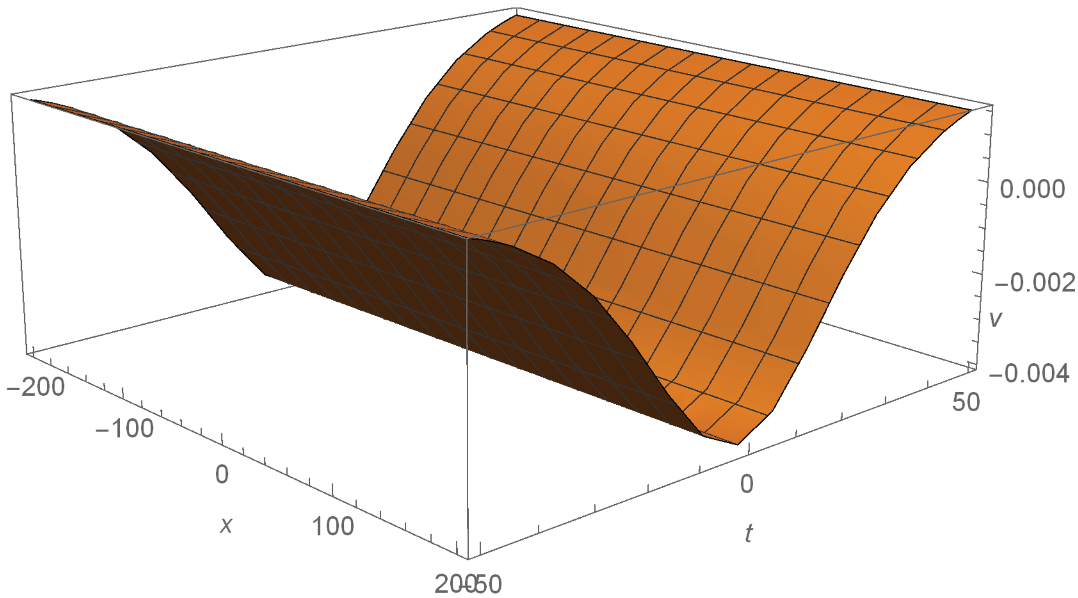

Fig. 2 Profile of solution (13)

Case 2: When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$
\begin{aligned}
u_2(t,x,y) &= -\frac{1}{6}\left(\alpha+\lambda^2+8\mu\right) - 2\lambda\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin\left(\delta_2 z\right)+C_2\cos\left(\delta_2 z\right)}{C_1\cos\left(\delta_2 z\right)+C_2\sin\left(\delta_2 z\right)}\right) \\
&\quad -2\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin\left(\delta_2 z\right)+C_2\cos\left(\delta_2 z\right)}{C_1\cos\left(\delta_2 z\right)+C_2\sin\left(\delta_2 z\right)}\right)^2, \qquad (14) \\
v_2(t,x,y) &= \frac{1}{6}(\alpha-1)\left(\alpha-2\lambda^2-16\mu\right)+4\lambda(\alpha-1)\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin\left(\delta_2 z\right)+C_2\cos\left(\delta_2 z\right)}{C_1\cos\left(\delta_2 z\right)+C_2\sin\left(\delta_2 z\right)}\right)
\end{aligned}
$$

$$+4(\alpha-1)\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2 z)+C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z)+C_2\sin(\delta_2 z)}\right)^2, \tag{15}$$

where $z=x+(\alpha-1)y-\alpha t$, $\delta_2=\frac{1}{2}\sqrt{4\mu-\lambda^2}$, C_1 and C_2 are arbitrary constants.

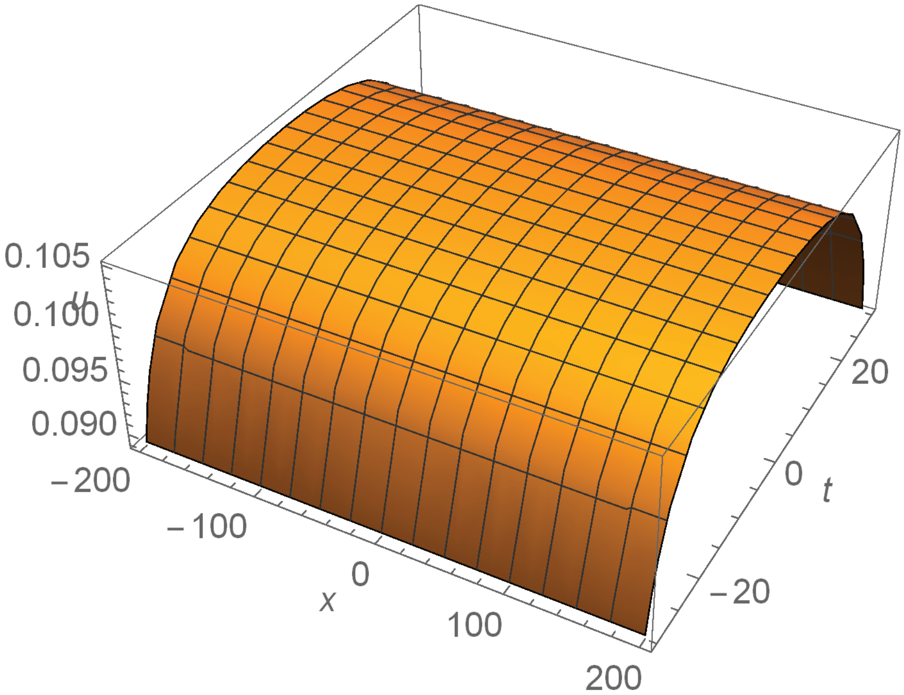

Fig. 3 Profile of solution (14)

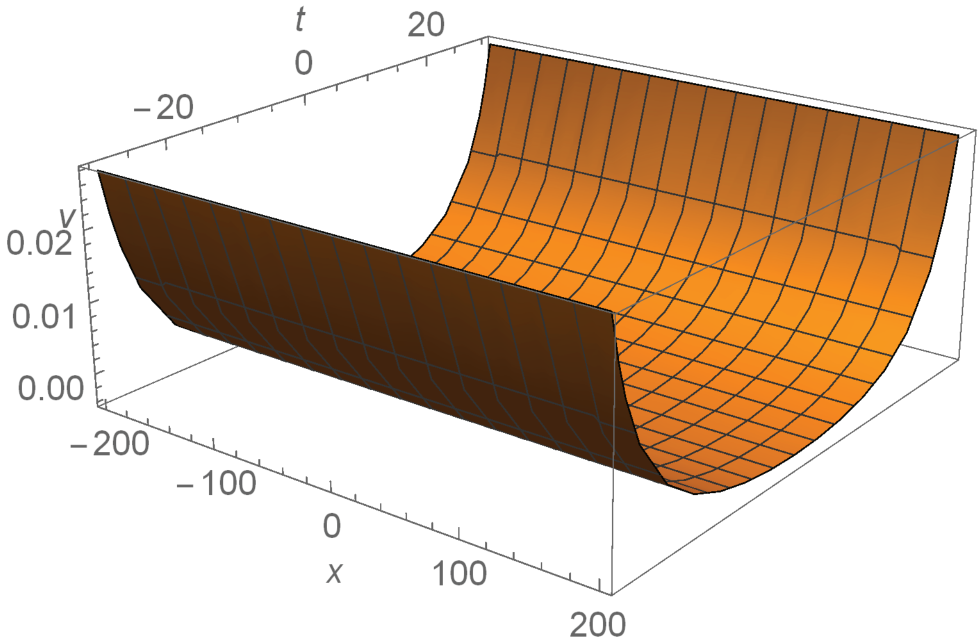

Fig. 4 Profile of solution (15)

Case 3: When $\lambda^2 - 4\mu = 0$, we obtain the rational solutions

$$
\begin{aligned}
u_3(t,x,y) &= -\frac{1}{6}\left(\alpha + \lambda^2 + 8\mu\right) - 2\lambda\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) - 2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \\
v_3(t,x,y) &= \frac{1}{6}(\alpha - 1)\left(\alpha - 2\lambda^2 - 16\mu\right) + 4\lambda(\alpha - 1)\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) \\
&\quad + 4(\alpha - 1)\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2.
\end{aligned}
$$

where $z = x + (\alpha - 1)y - \alpha t$, C_1 and C_2 are arbitrary constants.

3 Conservation laws for (2a)

In this section we construct conservation laws for our (2+1)-dimensional coupling system with the Korteweg-de Vries equation (2a). Conservations laws are physical quantities such as mass, momentum, angular momentum, energy, electrical charge, that do not change in the course of time within a physical system. They play a vital role in the solution process of differential equations. They are significant for exploring integrability and for establishing existence, uniqueness and stability of solutions of differential equations. Also conservation laws play an essential role in the numerical integration of partial differential equations, for example, to control numerical errors and they can be used to construct solutions of partial differential equations.

Several methods have been developed by researchers for constructing conservation laws. These include the Noether's theorem for variational problems, the Laplace's direct method, the characteristic form method by Stuedel, the multiplier approach, Kara and Mahomed partial Noether approach. The computer software packages for computing conservation laws have also been developed over the past few decades.

Here we use the multiplier method to find conservation laws of the system (2a), namely

$$ckdv110E_1 \equiv u_t - 6uu_x - u_{xxx} = 0, \tag{16a}$$

$$ckdv120E_2 \equiv v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx} = 0. \tag{16b}$$

A conservation law of the system (2a) is a space-time divergence such that

$$D_t T + D_x X + D_y Y = 0 \tag{17}$$

holds for all solutions $(u(t,x,y); v(t,x,y))$ of the system (2a). The vector (T, X, Y) is called the conserved vector of the system (2a).

We look for second-order multipliers Q_1 and Q_2, that is, Q_1 and Q_2 depend on t, x, y, u, v and first and second derivatives of u and v. The multipliers Q_1 and Q_2 of the system (2a) have the property that

$$Q_1 E_1 + Q_2 E_2 = D_t T + D_x X + D_y Y, \tag{18}$$

for all functions $u(t,x,y)$ and $v(t,x,y)$. The determining equations for the multipliers are obtained by solving the system

$$ckdv71\frac{\delta}{\delta u}[Q_1(u_t - 6uu_x - u_{xxx}) + Q_2(v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx})] = 0, \tag{19a}$$

$$ckdv72\frac{\delta}{\delta v}[Q_1(u_t - 6uu_x - u_{xxx}) + Q_2(v_t - 6(uv)_x - 6uu_y - 3u_{xxy} - v_{xxx})] = 0, \tag{19b}$$

where $\delta/\delta u$ and $\delta/\delta v$ are the standard Euler-Lagrange operators given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} - D_x^3\frac{\partial}{\partial u_{xxx}} - D_x^2 D_y\frac{\partial}{\partial u_{xxy}} + \cdots \tag{20}$$

and

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} - D_y \frac{\partial}{\partial v_y} - D_x^3 \frac{\partial}{\partial v_{xxx}} + \cdots, \tag{21}$$

respectively. Expanding system (19a) using (20) and (21) yields an overdetermined system of partial differential equations, which after solving with the help of Maple [38], we obtain

$$\begin{aligned}
Q_1 = &-x\left(\frac{5}{2}u^3 + uu_{xx} - \frac{1}{4}(u_x^2 - u_{tx})\right)F_2'(y) - \left(6tux + x^2\right)F_1'(y) + vF_4(y) \\
&-x\left(3u^2 + u_{xx}\right)F_3'(y) - xuF_4'(y) + 6tuF_5'(y) + \frac{15}{2}vu^2 + 6tvF_1(y) \\
&+\frac{1}{4}\left((8u_{xy} + 4v_{xx})u + 4u_{xx}v + (-2u_y - 2v_x)u_x + u_{ty} + v_{tx}\right)F_2(y) \\
&+\frac{1}{4}\left(10u^3 + 4uu_{xx} - {u_x}^2 + u_{tx}\right)F_6(y) + \left(6uv + 2u_{xy} + v_{xx}\right)F_3(y) \\
&+\left(3u^2 + u_{xx}\right)F_7(y) + (6tu + x)F_8(y) + uF_9(y) + F_{10}(y),
\end{aligned} \tag{22}$$

$$\begin{aligned}
Q_2 = &\frac{1}{4}\left(10u^3 + 4uu_{xx} - {u_x}^2 + u_{tx}\right)F_2(y) + (6tu + x)F_1(y) \\
&+\left(3u^2 + u_{xx}\right)F_3(y) + F_4(y)u + F_5(y),
\end{aligned} \tag{23}$$

where F_i, $i = 1, \cdots, 10$ are arbitrary functions of y. As a result the ten conserved vectors are calculated via a homotopy formula [38] and are given by

$$\begin{aligned}
T_1 = &6uvF(y)t - 3xF'(y)tu^2 - ux^2F'(y) + vF(y)x, \\
X_1 = &3u^2x^2F'(y) - xu_xF'(y) + u_{xx}x^2F'(y) - 2u_{xy}F(y)x - v_{xx}F(y)x \\
&-12uu_{xy}F(y)t - 6uv_{xx}F(y)t - 6vu_{xx}F(y)t + 6v_xu_xF(y)t - 6vuF(y)x \\
&-3u_x^2xF'(y)t + 6u_xu_yF(y)t + 6uu_{xx}xF'(y)t + F(y)v_x + u_yF(y) \\
&-36vF(y)u^2t + 12xF'(y)tu^3, \\
Y_1 = &F(y)u_x - 6uu_{xx}F(y)t - u_{xx}F(y)x - 12u^3F(y)t + 3{u_x}^2F(y)t \\
&-3u^2F(y)x;
\end{aligned}$$

$$\begin{aligned}
T_2 = &\frac{1}{16}F(y)vu_{xxxx} + \frac{1}{16}F(y)uu_{ty} + \frac{7}{6}F(y)u^2u_{xy} + \frac{1}{4}F(y)uu_{xxxy} \\
&+\frac{1}{16}F(y)uv_{tx} + \frac{7}{12}F(y)u^2v_{xx} + \frac{1}{16}F(y)uv_{xxxx} - \frac{1}{12}F(y)u_x^2v + \frac{1}{16}F(y)vu_{tx} \\
&-\frac{5}{8}xF'(y)u^4 + \frac{5}{2}vF(y)u^3 - \frac{1}{16}F(y)u_{xxx}v_x + \frac{1}{16}u_yF(y)u_t \\
&-\frac{1}{16}u_yF(y)u_{xxx} - \frac{3}{16}u_xF(y)u_{xxy} + \frac{1}{16}u_xF(y)v_t - \frac{1}{16}u_xF(y)v_{xxx} \\
&+\frac{1}{16}F(y)u_tv_x - \frac{7}{12}F'(y)u^2u_{xx}x + \frac{1}{12}u{u_x}^2F'(y)x - \frac{1}{6}uF(y)u_xu_y \\
&-\frac{1}{16}u_xF'(y)u_tx + \frac{1}{16}u_xF'(y)u_{xxx}x - \frac{1}{16}uu_{tx}F'(y)x \\
&-\frac{1}{16}uu_{xxxx}F'(y)x + \frac{7}{6}vF(y)uu_{xx} - \frac{1}{6}uF(y)u_xv_x, \\
X_2 = &-\frac{3}{16}F(y)uu_{txxy} - \frac{1}{2}vu_{xx}^2F(y) - \frac{1}{8}u_{xx}F(y)u_{ty} - \frac{1}{8}u_{xx}F(y)v_{tx}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8}v_{xx}F(y)u_{tx}-\frac{1}{4}u_{xy}F(y)u_{tx}-\frac{1}{16}F(y)u_{xxx}v_t+\frac{1}{8}u_yF(y)u_{txx}\\
&-\frac{1}{16}F'(y)u_t^2x-\frac{3}{16}u_tF(y)u_{xxy}+\frac{1}{8}u_tF(y)v_t-\frac{1}{16}u_tF(y)v_{xxx}\\
&-\frac{1}{16}F(y)vu_{txxx}-\frac{1}{16}F(y)vu_{tt}-\frac{7}{12}F(y)u^2u_{ty}-\frac{1}{16}F(y)uv_{tt}-\frac{7}{12}F(y)u^2v_{tx}\\
&-\frac{1}{16}F(y)uv_{txxx}+\frac{1}{6}u_xF'(y)uu_tx-5u^3u_{xy}F(y)+\frac{1}{8}F(y)u_xv_{txx}\\
&-15vF(y)u^4+\frac{1}{4}F(y)u_xu_{txy}+\frac{1}{4}v_{xx}F(y)u_x^2-\frac{5}{2}u^3v_{xx}F(y)+\frac{1}{8}v_xF(y)u_{txx}\\
&+3xF'(y)u^5+\frac{1}{2}u_{xy}F(y)u_x^2+\frac{1}{16}u_tF'(y)u_{xxx}x-\frac{1}{6}uF(y)u_xv_t\\
&-\frac{1}{6}F(y)u_xvu_t-\frac{7}{6}vF(y)uu_{tx}+\frac{5}{2}u^3u_{xx}F'(y)x-\frac{3}{4}u_x^2F'(y)xu^2\\
&+\frac{3}{2}u_xu_yF(y)u^2+\frac{3}{2}v_xu_xF(y)u^2-\frac{7}{2}u^2u_{xx}vF(y)+\frac{3}{2}vuF(y)u_x^2\\
&-\frac{1}{4}u_{xx}u_x^2F'(y)x+\frac{1}{2}u_{xx}u_xF(y)u_y+\frac{7}{12}u^2u_{tx}F'(y)x-2uu_{xx}F(y)u_{xy}\\
&+\frac{1}{2}uu_{xx}^2F'(y)x-uu_{xx}F(y)v_{xx}+\frac{1}{2}v_xF(y)u_xu_{xx}-\frac{1}{6}u_yF(y)uu_t\\
&-\frac{1}{6}u_tF(y)uv_x+\frac{1}{8}u_{xx}F'(y)xu_{tx}-\frac{1}{8}u_{txx}F'(y)u_xx\\
&+\frac{1}{16}uu_{tt}F'(y)x+\frac{1}{16}uu_{txxx}F'(y)x,\\
Y_2=&-\frac{7}{12}F(y)u^2u_{tx}-\frac{1}{16}F(y)uu_{tt}+\frac{1}{8}F(y)u_xu_{txx}-\frac{1}{16}F(y)uu_{txxx}\\
&-\frac{5}{2}u^3F(y)u_{xx}+\frac{3}{4}u^2F(y)u_x^2+\frac{1}{4}u_x^2F(y)u_{xx}-\frac{1}{2}uF(y)u_{xx}^2\\
&-\frac{1}{16}u_tF(y)u_{xxx}-\frac{1}{8}u_{xx}F(y)u_{tx}-3u^5F(y)+\frac{1}{16}u_t^2F(y)\\
&-\frac{1}{6}uF(y)u_xu_t;
\end{aligned}
$$

$$
\begin{aligned}
T_3=&-xF'(y)u^3+3F(y)u^2v+F(y)uu_{xy}+\frac{1}{2}F(y)uv_{xx}\\
&+\frac{1}{2}F(y)u_{xx}v-\frac{1}{2}x\left(F'(y)\right)uu_{xx},\\
X_3=&\frac{1}{2}\left(F'(y)\right)u_{xx}^2x-2u_{xx}F(y)u_{xy}-u_{xx}F(y)v_{xx}-\frac{1}{2}F(y)uu_{ty}\\
&-6F(y)u^2u_{xy}-\frac{1}{2}F(y)uv_{tx}-3F(y)u^2v_{xx}-\frac{1}{2}F(y)vu_{tx}+\frac{9}{2}xF'(y)u^4\\
&-18vF(y)u^3+\frac{1}{2}u_yF(y)u_t+\frac{1}{2}u_xF(y)v_t+\frac{1}{2}F(y)u_tv_x\\
&+3F'(y)u^2u_{xx}x-\frac{1}{2}u_xF'(y)u_tx+\frac{1}{2}uu_{tx}F'(y)x\\
&-6vF(y)uu_{xx},\\
Y_3=&-3u^2F(y)u_{xx}+\frac{1}{2}u_xF(y)u_t-\frac{1}{2}uF(y)u_{tx}-\frac{9}{2}u^4F(y)\\
&-\frac{1}{2}u_{xx}^2F(y);
\end{aligned}
$$

$$
\begin{aligned}
T_4 =& -\frac{1}{2}u^2 xF'(y) - uvF(y),\\
X_4 =& 2xF'(y)u^3 - 6F(y)u^2v - \frac{1}{2}xF'(y)u_x^2 - 2F(y)uu_{xy}\\
&- F(y)uv_{xx} + F(y)u_xu_y + F(y)u_xv_x - F(y)u_{xx}v + xF'(y)uu_{xx},\\
Y_4 =& -2u^3F(y) + \frac{1}{2}u_x^2F(y) - uu_{xx}F(y);
\end{aligned}
$$

$$
\begin{aligned}
T_5 =& (y)v + 3tu^2F'(y),\\
X_5 =& -6uu_{xx}tF'(y) + u_xF'(y) - 2F(y)u_{xy} - F(y)v_{xx} - 6F(y)uv\\
&- 12tu^3F'(y) + 3u_x^2tF'(y),\\
Y_5 =& -3u^2F(y) - u_{xx}F(y);
\end{aligned}
$$

$$
\begin{aligned}
T_6 =& -\frac{1}{12}uF(y)u_x^2 + \frac{1}{16}uu_{xxxx}F(y) - \frac{1}{16}u_xF(y)u_{xxx} + \frac{7}{12}u^2F(y)u_{xx}\\
&+ \frac{1}{16}u_xF(y)u_t + \frac{1}{16}uF(y)u_{tx} + \frac{5}{8}u^4F(y),\\
X_6 =& -\frac{7}{12}F(y)u^2u_{tx} - \frac{1}{16}F(y)uu_{tt} + \frac{1}{8}F(y)u_xu_{txx} - \frac{1}{16}F(y)uu_{txxx}\\
&- \frac{5}{2}u^3F(y)u_{xx} + \frac{3}{4}u^2F(y)u_x^2 + \frac{1}{4}u_x^2F(y)u_{xx} - \frac{1}{2}uF(y)u_{xx}^2\\
&- \frac{1}{16}u_tF(y)u_{xxx} - \frac{1}{8}u_{xx}F(y)u_{tx} - 3u^5F(y) + \frac{1}{16}u_t^2F(y)\\
&- \frac{1}{6}uF(y)u_xu_t,\\
Y_6 =& 0;\\
T_7 =& u^3F(y) + \frac{1}{2}uu_{xx}F(y),\\
X_7 =& -3u^2F(y)u_{xx} + \frac{1}{2}u_xF(y)u_t - \frac{1}{2}uF(y)u_{tx} - \frac{9}{2}u^4F(y)\\
&- \frac{1}{2}u_{xx}^2F(y),\\
Y_7 =& 0;
\end{aligned}
$$

$$
\begin{aligned}
T_8 =& 3tu^2F(y) + u_xF(y),\\
X_8 =& F(y)u_x - 6uu_{xx}F(y)t - u_{xx}F(y)x - 12u^3F(y)t + 3u_x^2F(y)t\\
&- 3u^2F(y)x,\\
Y_8 =& 0;
\end{aligned}
$$

$$
\begin{aligned}
T_9 =& \frac{1}{2}u^2F(y),\\
X_9 =& -2u^3F(y) + \frac{1}{2}u_x^2F(y) - uu_{xx}F(y),\\
Y_9 =& 0;
\end{aligned}
$$

$$\begin{aligned}
T_{10} &= F(y)\,u,\\
X_{10} &= -3u^2F(y) - u_{xx}F(y),\\
Y_{10} &= 0.
\end{aligned}$$

Remark: Due to the arbitrary functions in the multipliers Q_1 and Q_2, infinitely many conserved vectors are obtained for the system (2a).

4 Conclusion

In this paper we studied a (2+1)-dimensional coupling system with the Korteweg-de Vries equation (2a). Lie point symmetries of (2a) were computed and used to reduce the system to a system of ordinary differential equations. This ordinary differential equations system was then solved by employing the $(G'/G)-$expansion method and as a result travelling wave solutions of (2a) were obtained. The solutions obtained were expressed in the form of hyperbolic functions, trigonometric functions and rational functions. Some of these solutions were plotted. Furthermore, conservation laws for the system (2a) were derived by using the multiplier approach. The significance of conservation laws was explained in the beginning of Section 3.

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