# Noether's theorems of variable mass systems on time scales

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## Abstract

This paper deals with the Noether's theory for variable mass system on time scales. The calculus on time scales unifies and extends variable mass system continuous model and discrete model into a single theory. Firstly, Hamilton's principle of the variable mass system on time scales is given. Secondly, based on the quasi-invariance of the Hamilton's action under a group of infinitesimal transformations, Noether's theorem and its inverse theorem of the variable mass system on time scales are presented. Finally, two examples are given to illustrate the applications of the results.

**Keywords:** variable mass system, Noether's theorem, time scale.
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## 1 Introduction

The theory of time scales was born in 1988 with the work of Stefen Hilger in order to unify and generalize continuous and discrete analysis [1, 2]. The calculus of variations on time scales has been developing rapidly in the past thirteen years, after the pioneering work of Bohner in 2004 [3]. Cai and Fu established the Noether symmetries of the non-conservative and non-holonomic systems on time scales, and obtained the symmetry theorem for constrained mechanical systems on time scales [4, 5]. More recently, Noether theory for Bikhoffian systems on time scales was established by Song and Zhang [6]. Zhai and Zhang obtain the Noether theorem for non-conservative systems with time delay on time scales [7].The time scales has a tremendous potential for applications and has recently received much attention in other areas such as engineering, biology, economics, and physics [9–12].

In 1918, Noether proposed famous Noether symmetry theorem which deal with the invariance of the Hamilton action under the infinitesimal transformations: when a system exhibits a symmetry, then a conservation law can be obtained [16]. The symmetries and conservation laws can also be studied by using differential variational

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principles [17]. The calculus of variations and control theory are disciplines in which there appears to be many opportunities for application of time scales [13, 14]. The Noether method is making good progress, such as Herglotz variational problems [15]. And in recent years, a series of important results have been obtained on the study of the Noether symmetry and conservation law of classical mechanical systems, such as Torres made use of the Euler-Lagrange equations on time scales to generalize one of the most beautiful results of the calculus of variations-the celebrated Noether's theorem [18, 19].

The problem of variable mass has attracted people's attention as early as the middle of the nineteenth century. With the development of space technology and other industrial technologies, the study of variable mass system dynamics becomes more and more important. There are many studies on variable mass systematics have been done by Mei [20, 21]. A series of new theories and methods have been put forward, and a series of innovative research results have been obtained [22–26].

In this article, we will study the Noether theorems and its inverse problem of variable mass on time scales. In Section 2, we review some basic definitions and properties about the calculus on time scales. In Section 3, we obtain the Lagrange equations of systems by deriving Hamilton's principle for variation mass systems with delta derivative. In Section 4, based on the quasi-invariance of Hamiltonian action of the variation mass systems under the infinitesimal transformations with respect to the time scales and generalized coordinates, the Noether's theorem and the conservation laws for variation mass systems on time scales are obtained. In Section 5, the Noether's inverse theorem of variable mass systems on time scales is given. In the end, two examples are given to illustrate the applications of the results.

## 2 Basics on the time scales calculus

In this section we give basic definitions and facts concerning the calculus on time scales. More can be found elsewhere [27].

A time scales is a nonempty closed subset of real numbers, and we usually denote it by symbol $T$. The two most popular examples are $(T =)$ and $(T =)$. We define the forward and backward jump operators $\sigma, \rho$.

**Definition 2.1** Let $T$ be a time scale. For $t \in T$ we define the forward and backward jump operators $\sigma, \rho : T \to T$ by

$$\sigma(t) := \inf\{s \in T : s > t\} \text{ and } \rho(t) := \sup\{s \in T : s < t\} \text{ for all } t \in T,$$

(supplemented by $\inf \phi = \sup T$ and $\sup \phi = \inf T$) and the graininess function $\mu : T \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for each $t \in T$.

If $T =$, then $\sigma(t) = t = \rho(t)$ and $\mu(t) = 0$ for any $t \in T$. If $T =$, then $\sigma(t) = t+1, \rho(t) = t-1$ and $\mu(t) \equiv 1$ for every $t \in$. A point $t \in T$ is called right scattered, right dense, left scattered and left dense if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, $\rho(t) = t$, respectively. We can consider that $t$ is isolated if $\rho(t) < t < \sigma(t)$, then $t$ is dense if $\rho(t) = t = \sigma(t)$. If $\sup T$ is finite and left-scattered, we set $T^\kappa = T \setminus \{\sup T\}$. Otherwise, .

**Definition 2.2.** Assume $f : T \to$ is a function and let $t \in T^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of (i.e., $U = (t - \delta, t + \delta) \cap T$ for some $\delta > 0$ ) such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U$$

we call $f^\Delta(t)$ the delta (or Hilger) derivative of $f$ at $t$.

For differentiable $f$, the formula

$$f^\sigma(t) = f + \mu f^\Delta \text{ and } f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{1}$$

**Definition 2.3.** A function $f : T \to$ is called re-continuous if it is continuous at the right- dense points in $T$ and its left-sided limits exist at all left-dense points in $T$. A function $f : T \to^n$ is re-continuous if all its components are re-continuous.

The set of all re-continuous is denoted by $C_{rd}$. Similarly, $C_{rd}^1$ will denote the set of functions from $C_{rd}$ whose delta derivative belongs to $C_{rd}$.

**Theorem 2.1.** Let $f$ be regulated. Then there exists a function $F : \mathrm{T} \to$ is called an pre-antiderivative of $f : \mathrm{T} \to$ if it satisfies $F^\Delta(t) = f(t)$, for all $t \in \mathrm{T}^\kappa$.

**Definition 2.4.** Assume $f : \mathrm{T} \to$ is a regulated function. Any function $F$ as in Theorem 2.1. is called a pre-antiderivative of $f$. We define the indefinite integral of a regulated function $f$ by

$$\int f(t)\Delta t = F(t) + C \tag{2}$$

where $C$ is an arbitrary constant and $F$ is a pre-antiderivative of $f$. We define the Cauchy integral by

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \text{ for all } a, b \in \mathrm{T}. \tag{3}$$

We shall need the following properties of delta derivatives and integrals:

$$(fg)^\Delta = f^\Delta g^\sigma + fg^\Delta, \tag{4}$$

$$(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t), \tag{5}$$

$$\int_a^b f(\alpha(t))\alpha^\Delta(t)\,\Delta t = \int_{\alpha(a)}^{\alpha(b)} f(t^*)\Delta t^*, \tag{6}$$

where $\alpha : [a,b] \cap \mathrm{T} \to$ is an increasing $C_{rd}^1$ function and image is a new time scale.

**Lemma 2.1.** (Dubois-Reymond) Let $g \in C_{rd}$, $g : [a,b] \to {}^n$, then

$$\int_a^b g^{\mathrm{T}}(t) \cdot \eta^\Delta(t)\Delta t = 0 \tag{7}$$

for all $\eta \in C_{rd}^1$ with $\eta(a) = \eta(b) = 0$, holds if and only if $g(t) \equiv c$ on $[a,b]^\kappa$ for some $c \in$.

## 3 Hamilton's principle and Lagrange equations for variable mass systems with delta derivatives

Consider a mechanical system consisting of $N$ variable mass particles. Suppose at time $t$, the mass of the particle $i$ is supposed to be $m_i (i = 1, 2, \cdots, N)$. At the moment $t + \Delta t$ the mass of a small particle separated from the particle $i$ or combined with the particle $i$ is supposed to be $\Delta m_i$. The configuration of the system is determined by $n$ generalized coordinates $q_s (s = 1, 2, \cdots, n)$ and the mass of the particle depends on time, generalized coordinates and generalized velocity

$$m_i = m_i\left(t, q_s^\sigma, q_s^\Delta\right).$$

Assuming that the kinetic energy function of the variable mass system on time scales is $T = T(t, q_s^\sigma, q_s^\Delta)$, Hamilton's principle states that the actual pace exists when the Hamiltonian action has determining value. Thus the Hamilton's principle for variable mass systems with delta derivatives can be written in the following form:

$$\int_a^b (\delta T + Q_s \delta q_s^\sigma + P_s \delta q_s^\sigma)\Delta t = 0 \tag{8}$$

where $Q_s \delta q_s^\sigma$ is the virtual work of generalized force, $P_s \delta q_s^\sigma$ is the virtual work of generalized counter thrust, $\delta q_s^\sigma = \varepsilon \left( \xi^\sigma - \tau^\sigma q_s^{\Delta\sigma} \right)$.

$$P_s = \left( R_i + m_i^\Delta r_i^\Delta \right) \frac{\partial r_i}{\partial q_s} - \frac{1}{2} r_i^\Delta \cdot r_i^\Delta \frac{\partial m_i}{\partial q_s} + \frac{\Delta}{\Delta t} \left( \frac{1}{2} r_i^\Delta \cdot r_i^\Delta \frac{\partial m_i}{\partial q_s^\Delta} \right) \tag{9}$$

where $r_i$ and $r_i^\Delta$ are respectively the position vector and the velocity vector of the $i$-th particle and the velocity vector of the $i$-th particle and $R_i = \frac{\Delta m_i}{\Delta t} u_i$, where $u_i$ is the corpuscle's velocity relative to the $i$-th particle.

The exchanging relationships with respect to the derivatives on time scales and isochronous variation on time scales [5]:

$$\delta q_s^\Delta = (\delta q_s)^\Delta, \tag{10}$$

and following eq. (1) we can find $q_s^{\Delta\sigma} = q_s^\Delta + \mu(t) q_s^{\Delta\Delta}$.

Taking total variation for function $T$, we have

$$\delta T = \frac{\partial T}{\partial q_s^\sigma} \delta q_s^\sigma + \frac{\partial T}{\partial q_s^\Delta} \delta q_s^\Delta. \tag{11}$$

substituting eq. (11) into eq. (8), we have

$$\int_a^b \left[ \frac{\partial T}{\partial q_s^\sigma} \delta q_s^\sigma + \frac{\partial T}{\partial q_s^\Delta} \delta q_s^\Delta + Q_s \delta q_s^\sigma + P_s \delta q_s^\sigma \right] \Delta t$$

$$= \int_a^b \left\{ \left( Q_s + P_s + \frac{\partial T}{\partial q_s^\sigma} \right) (\delta q_s)^\sigma + \frac{\partial T}{\partial q_s^\Delta} (\delta q_s)^\Delta \right\} \Delta t$$

$$= \int_a^b \left\{ \left( Q_s + P_s \frac{\partial T}{\partial q_s^\sigma} \right) (\partial q_s)^\sigma + \frac{\partial T}{\partial q_s^\Delta} (\delta q_s)^\Delta \right\} \Delta T$$

$$- \int_a^t \left( Q_s + P_s + \frac{\partial T(\tau, q^\sigma(\tau),) q^\Delta(\tau)}{\partial q_s^\sigma} \right) \Delta \tau \right] (\delta q_s)^\Delta + \frac{\partial T}{\partial q_s^\Delta} (\delta q_s)^\Delta \right\} \Delta t$$

$$= \int_a^b \left\{ \frac{\partial T}{\partial q_s^\Delta} - \int_a^t \left( Q_s + P_s + \frac{\partial T(\tau, q^\sigma(\tau),) q^\Delta(\tau)}{\partial q_s^\sigma} \right) \Delta \tau \right\} (\delta q_s)^\Delta \Delta t = 0.$$

Therefore, by Lemma 2.1, we can derive

$$\frac{\partial T}{\partial q_s^\Delta} - \int_a^b \left( Q_s + P_s + \frac{\partial T(\tau, q^\sigma(\tau),) q^\Delta(\tau)}{\partial q_s^\sigma} \right) \Delta \tau \equiv const, \tag{12}$$

hence

$$\frac{\Delta}{\Delta t} \frac{\partial T}{\partial q_s^\Delta} - \frac{\partial T}{\partial q_s^\sigma} - Q_s - P_s = 0. \tag{13}$$

When contains conservative force and nonconservative force $Q_s''$, and $Q_s'$ satisfies the following conditions: If is potential, that is, there exists a function such that

$$Q_s' = -\frac{\partial V}{\partial q_s^\sigma}. \tag{14}$$

substituting eq.(14) into eq.(13), we have

$$\frac{\Delta}{\Delta t} \frac{\partial T}{\partial q_s^\Delta} - \frac{\partial T}{\partial q_s^\sigma} + \frac{\partial V}{\partial q_s^\sigma} - Q_s'' - P_s = 0. \tag{15}$$

as the function $V = V(q_s^\sigma, t)$ only depends on the generalized coordinates, therefore

$$\frac{\partial V}{\partial q_s{}^\Delta} = 0. \tag{16}$$

Then eq.(13) can be written in the form

$$\frac{\Delta}{\Delta t} \frac{\partial L}{\partial q_s{}^\Delta} - \frac{\partial L}{\partial q_s{}^\sigma} = Q_s'' + P_s. \tag{17}$$

where $L = T - V$.

If $Q_s'$ has generalized potential, that is, there exists a function $U = U(t, q_s^\sigma, q_s^\Delta)$ such that

$$Q_s' = \frac{\partial U}{\partial q_s{}^\sigma} - \frac{\Delta}{\Delta t} \frac{\partial U}{\partial q_s{}^\Delta} \tag{18}$$

then eq.(13) can be written as:

$$\frac{\Delta}{\Delta t} \frac{\partial L}{\partial q_s{}^\Delta} - \frac{\partial L}{\partial q_s{}^\sigma} = Q_s'' + P_s. \tag{19}$$

where $L = T + U = T - V$ is the Lagrangian of the variable mass systems with derivatives on time scales.

## 4 Noether's theorem of variable mass systems on time scales

In order to simplify expressions, we write $L(t, q_s^\sigma, q_s^\Delta)$ instead of $L(t, q_s^\sigma(t), q_s^\Delta(t))$, similarly for the partial derivatives of $L$.

We consider the fundamental problem of the calculus of variations on time scales as defined by Bohner [3,20]

$$S[q_s(\cdot)] = \int_a^b L(t, q_s^\sigma(t), q_s^\Delta(t)) \Delta t \to \min \tag{20}$$

$q_s^\sigma(t) = (q_s \circ \sigma)(t)$, $q_s^\Delta(t)$ is the delta derivative of $q_s$, $t \in \mathrm{T}$, and the Lagrangian $L : \times^n \times^n \to$ is a $C^1$ function with respect to its arguments. By $\partial_i L$ we will denote the partial derivative of $L$ with respects to the $i$th variable, $i = 1, 2, 3$. Admissible functions $q_s(\cdot)$ are assumed to be $C_{rd}^1$.

The relationship between the isochronous variation and the total variation on time scale T:

$$\Delta q_s = \delta q_s + q_s^\Delta \Delta t.$$

Let us consider now the following infinitesimal transformations with respect to the time and the state variable:

$$t^* = H(t, q_s, \varepsilon) = t + \varepsilon \tau(t, q_s) + o(\varepsilon) \quad q_s^* = F(t, q_s, \varepsilon) = q_s + \varepsilon \xi(t, q_s) + o(\varepsilon) \tag{21}$$

Let as before $U$ be a set of $C_{rd}^1$ functions $q_s : [a,b] \to{}^n$ and we assume that the map $t \to \alpha(t) : T(t, q_s, \varepsilon) \in$ is an increasing $C_{rd}^1$ function for every $q_s \in U$, every $\varepsilon$, and any $t \in [a,b]$, and its image is a new time scale with the forward jump operator $\sigma^*$ and the delta derivative $\Delta^*$. We need to employ the following property:

$$\sigma^* \circ \alpha = \alpha \circ \sigma. \tag{22}$$

**Definition 4.1** (Invariance for variable mass systems) Function $I$ is said to be quasi-invariant on $U$ under the infinitesimal transformations (21) if and only if for any subinterval $[t_a, t_b] \in [a,b]$, any $\varepsilon$, any $q_s \in U$:

$$\int_{t_a}^{t_b} L(t, q_s^\sigma, q_s^\Delta) \Delta t = \int_{T(t_a, q_s(t_a))}^{T(t_b, q_s(t_b))} L(t^*, q_s^{*\sigma}(t^*), q_s^{*\Delta}(t^*)) \Delta t^* + \int_{t_a}^{t_b} \left( \frac{\Delta}{\Delta t}(\Delta G) + \left( Q_s'' + P_s \right) \cdot \delta q_s^\sigma \right) \Delta t \tag{23}$$

**Theorem.4.1** If infinitesimal transformations (21) satisfy

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + L\tau^\Delta - \frac{\partial L}{\partial q_s^\Delta}\tau^\Delta q_s^\Delta + \left(Q_s^{''}+P_s\right)\left\{\xi^\sigma - \tau^\sigma\left[q_s^\Delta + \mu\left(t\right)q_s^{\Delta\Delta}\right]\right\} = -\frac{\Delta}{\Delta t}G$$

then transformations (21) is the Noether generalized quasi-symmetric transformation of variable mass system on time scales.

**Proof.** Substituting formula $\delta q_s^\sigma = \varepsilon\left(\xi^\sigma - \tau^\sigma q_s^{\Delta\sigma}\right) = \varepsilon\left\{\xi^\sigma - \tau^\sigma\left[q_s^\Delta + \mu\left(t\right)q_s^{\Delta\Delta}\right]\right\}$ into eq. (19), we obtain

$$\varepsilon\left[-\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta} + \frac{\partial L}{\partial q_s^\sigma} + Q_s^{''} + P_s\right]\left(\xi_s^\sigma - \tau^\sigma q_s^{\Delta\sigma}\right) = 0 \tag{24}$$

adding and subtracting a function $\frac{\partial L}{\partial t}\tau$ from eq. (24), we obtain

$$\varepsilon\left\{\left[\frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \left(Q_s^{''}+P_s\right)\left(\xi_s^\sigma - \tau^\sigma q_s^{\Delta\sigma}\right)\right] - \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_s^\sigma}q_s^\Delta + \frac{\partial L}{\partial q_s^\Delta}q_s^{\Delta\Delta}\right)\tau \right.$$
$$\left. + \frac{\partial L}{\partial t}\tau - \frac{\partial L}{\partial q_s^\Delta}q_s^\Delta\tau^\Delta - \frac{\Delta}{\Delta t}\left[\frac{\partial L}{\partial q_s^\Delta}\right]\left(\xi_s^\Delta - \tau^\sigma q_s^{\Delta\sigma}\right)\right\} = 0 \tag{25}$$

adding and subtracting a gauge function $\frac{\Delta}{\Delta t}G(t,q_s^\sigma,q_s^\Delta)$ from eq. (25), we obtain

$$\varepsilon\left\{\left[\frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \left(Q_s^{''}+P_s\right)\left(\xi_s^\sigma - \tau^\sigma q_s^{\Delta\sigma}\right)\right] + \left(L - \frac{\partial L}{\partial q_s^\Delta}q_s^\Delta\right)\tau^\Delta + \frac{\partial L}{\partial t}\tau \right.$$
$$\left. + \frac{\Delta}{\Delta t}G\left(t,q_s^\sigma,q_s^\Delta\right) - \frac{\Delta}{\Delta t}\left[\frac{\partial L}{\partial q_s^\Delta}\xi_s^\sigma + \left(L - \frac{\partial L}{\partial q_s^\Delta}q_s^\Delta\right)\tau + G\left(t,q_s^\sigma,q_s^\Delta\right)\right]\right\} = 0 \tag{26}$$

eq. (26) is the condition of infinitesimal transformations of the variable mass system on time scales.

If generators $\tau,\xi_s$ of infinitesimal transformations and gauge function $G(t,q_s^\sigma,q_s^\Delta)$ satisfy

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q_s^\sigma}\xi^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \left(L - \frac{\partial L}{\partial q_s^\Delta}q_s^\Delta\right)\tau^\Delta + \left(Q_s^{''}+P_s\right)\left\{\xi^\sigma - \tau^\sigma\left[q_s^\Delta + \mu\left(t\right)q_s^{\Delta\Delta}\right]\right\} = -\frac{\Delta}{\Delta t}G \tag{27}$$

eq.(27) is called Noether's identity of the variable mass system on time scales.

**Theorem 4.2.** If functional $I$ is quasi-invariant on $U$ under the infinitesimal transformations (21), then

$$I = \frac{\partial L}{\partial q_s^\Delta}\xi_s + \left[L - \frac{\partial L}{\partial q_s^\Delta}\cdot q_s^\Delta - \frac{\partial L}{\partial t}\cdot\mu\left(t\right)\right]\cdot\tau + G \tag{28}$$

is a conservational law for variable mass dynamical systems on time scales.

**Proof.** Let $\tilde{L}(t;s,q_s;r,v) := L(s-\mu(t)r,q_s,\frac{v}{r})\cdot r$ for $q_s,v\in\mathbb{R}^n, t\in[a,b]$ and $s,r,\in\mathbb{R}, r\neq 0$.

It is readily apparent that for $s(t) = t$ and any $q_s : [a,b]\to\mathbb{R}^n$

$$L(t,q_s^\sigma(t),s_s^\Delta(t)) = \tilde{L}(t;s^\sigma(t),q_s^\sigma(t);s^\Delta(t),q^\Delta(t)), \tag{29}$$

so for the functional:

$$S[q_s(\cdot)] = \tilde{S}[s(\cdot),q_s(\cdot)]$$

and

$$\tilde{S}[s(\cdot),q_s(\cdot)] := \int_a^b \tilde{L}(t;s^\sigma(t),q_s^\sigma(t);s^\Delta,q^\Delta(t))\Delta t \tag{30}$$

Consider the infinitesimal transformation eq.(21) given by $(H_\varepsilon, F_\varepsilon)$ and let $q_s \in U$. For $s(t) = t$, making use of eq. (23) we can obtain

$$\tilde{S}[s(\cdot), q_s(\cdot)] = \int_{\alpha(t_a)}^{\alpha(t_b)} L\left(t^*, q_s^* \circ \sigma^*(t^*), q_s^{*\Delta}(t^*)\right) \Delta t^* + \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + \left(Q_s^{''} + P_s\right) \cdot \delta q_s^\sigma\right) \Delta t$$

$$= \int_{t_a}^{t_b} L(\alpha(t), (q_s^* \circ \sigma^* \circ \alpha)(t), q_s^{*\Delta}(\alpha(t))) \alpha^\Delta(t) \Delta t + \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s^{'} + P_s) \cdot \delta q_s^\sigma\right) \Delta t$$

$$\int_{t_a}^{t_b} L\left(\alpha^\sigma(t) - \mu(t)\alpha^\Delta(t), (q_s^* \circ \sigma^* \circ \alpha)(t), \frac{(q_s^* \circ \alpha)^\Delta(t)}{\alpha^\Delta(t)}\right) \alpha^\Delta(t) \Delta t$$

$$+ \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s^{''} + P_s) \cdot \delta q_s^\sigma\right) \Delta t$$

$$= \int_{t_a}^{t_b} \tilde{L}(t; \alpha^\sigma(t), (q_s^* \circ \alpha)^\sigma(t); \alpha^\Delta(t), (q_s^* \circ \alpha)^\Delta(t)) \Delta t + \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s^{'} + P_s) \cdot \delta q_s^\sigma\right) \Delta t$$

$$= \tilde{S}[\alpha(\cdot), (q_s^* \circ \alpha)(\cdot)] + \int_{t_a}^{t_b} \left(\frac{\Delta}{\Delta t}(\Delta G) + (Q_s^{'} + P_s) \cdot \delta q_s^\sigma\right) \Delta t$$

so for $s(t) = t$ we can obtain

$$(\alpha(\cdot), (q_s^* \circ \alpha)(t)) = (H_\varepsilon(t, q_s(t)), F_\varepsilon(t, q_s(t))) = (H_\varepsilon(s(t), q_s(t)), F_\varepsilon(s(t), q_s(t)))$$

we observe that $\tilde{S}$ is an invariant on $\tilde{U} = \{(s, q_s) | s(t) = t, q_s \in U\}$ under the infinitesimal transformations:

$$(s^*, q_s^*) = (H_\varepsilon(s, q_s), F_\varepsilon(s, q_s)) \tag{31}$$

then

$$\frac{\Delta}{\Delta t}\left\{\frac{\partial L}{\partial q_s^\Delta}\xi_s + \left[L - \frac{\partial L}{\partial q_s^\Delta} \cdot q_s^\Delta - \frac{\partial L}{\partial t} \cdot \mu(t)\right] \cdot \tau + G\right\}$$

$$= \frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \frac{\Delta}{\Delta t}\left[L - \frac{\partial L}{\partial q_s^\Delta} \cdot q_s^\Delta - \frac{\partial L}{\partial t} \cdot \mu(t)\right]\tau^\sigma$$

$$+ \left[L - \frac{\partial L}{\partial q_s^\Delta} \cdot q_s^\Delta - \frac{\partial L}{\partial t} \cdot \mu(t)\right]\tau^\Delta + \frac{\Delta}{\Delta t}G$$

$$= \frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \left[\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_s^\sigma}q_s^{\Delta\sigma} + \frac{\partial L}{\partial q_s^\Delta}q_s^{\Delta\Delta} - \frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}q_s^\Delta - \frac{\partial L}{\partial q_s^\Delta}q_s^{\Delta\Delta} - \frac{\Delta}{\Delta t}\frac{\partial L}{\partial t} \cdot \mu(t)\right]\tau^\sigma$$

$$+ \left[L - \frac{\partial L}{\partial q_s^\Delta} \cdot q_s^\Delta - \frac{\partial L}{\partial t} \cdot \mu(t)\right]\tau^\Delta + \frac{\Delta}{\Delta t}G$$

$$= \left[\frac{\partial L}{\partial q_s^\sigma} + Q_s^{''} + P_s\right]\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \left[\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_s^\sigma}q_s^{\Delta\sigma} - \frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta}q_s^\Delta\right]\tau^\sigma$$

$$+ \left[L - \frac{\partial L}{\partial q_s^\Delta} \cdot q_s^\Delta - \frac{\partial L}{\partial t} \cdot \mu(t)\right]\tau^\Delta + \frac{\Delta}{\Delta t}G$$

$$= \left[\frac{\partial L}{\partial q_s^\sigma} + Q_s^{''} + P_s\right]\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + \left[\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_s^\sigma}q_s^{\Delta\sigma} - \left(\frac{\partial L}{\partial q_s^\sigma} + Q_s^{''} + P\right)q_s^\Delta\right]\tau^\sigma$$

$$+ \left[L - \frac{\partial L}{\partial q_s^\Delta} \cdot q_s^\Delta - \frac{\partial L}{\partial t} \cdot \mu(t)\right]\tau^\Delta + \frac{\Delta}{\Delta t}G$$

$$= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q_s^\sigma}\xi_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\xi_s^\Delta + L \cdot \tau^\Delta - \frac{\partial L}{\partial q_s^\Delta} \cdot \tau^\Delta \cdot q_s^\Delta + \left(Q_s^{''} + P\right)(\xi_s^\sigma - \tau^\sigma q_s^{\Delta\sigma}) + \frac{\Delta}{\Delta t}G$$

$$= 0.$$

The calculus on time scales unifies and extends variable mass system continuous model and discrete model into a single theory.

**Remark 1.** If $T = \mathbb{R}$, then $\sigma(t) = t, \mu(t) = 0$, therefore eq(27) give classical variable mass system Noether equation:

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \left(L - \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\dot{\tau} + \left(Q_s'' + P_s\right)\left(\xi_s - \tau\dot{q}_s\right) = -\frac{\Delta}{\Delta t}G \tag{32}$$

and the conservational law become the classical variable mass system Noether conservational law

$$I = \frac{\partial L}{\partial \dot{q}_s}\xi_s + \left[L - \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right] \cdot \tau(t, q_s) + G \tag{33}$$

**Remark 2.** If $T = \hbar\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h, \mu(t) = h$, therefore eq(27) give

$$\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q_s}\xi_s(t+h) + \frac{\partial L}{\partial \frac{q_s(t+h)-q(t)}{h}}\frac{\xi_s(t+h) - \xi_s(t)}{h}$$

$$+ \left(L - \frac{\partial L}{\partial \frac{q_s(t+h)-q(t)}{h}}\frac{q_s(t+h) - q(t)}{h}\right)\frac{\tau(t+h) - \tau(t)}{h} \tag{34}$$

$$+ \left(Q_s'' + P_s\right)(\xi_s - \tau q_s(t+h)) = -\frac{\Delta}{\Delta t}G$$

and the conservational law give

$$I = \frac{\partial L}{\partial \dot{q}_s}\xi_s + \left[L - \frac{\partial L}{\partial \dot{q}_s}\dot{q}_s - h\frac{\partial L}{\partial t}\right] \cdot \tau(t, q_s) + G \tag{35}$$

eq.(34)and eq.(35) are the discrete variable mass system Noether identity and Noether conservational law.

## 5 Noether's inverse theorem of variable mass system on time scales

Suppose that a first integral of the variable mass system on time scales has been given as

$$I = I(t, q_s^\sigma, q_s^\Delta) = \text{const.} \tag{36}$$

then we have

$$\frac{\Delta I}{\Delta t} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial q_s^\sigma}q_s^\Delta + \frac{\partial I}{\partial q_s^\Delta}q_s^{\Delta\Delta} = 0 \tag{37}$$

multiply $\tilde{\xi}_s^\sigma = \xi^\sigma - \tau^\sigma q_s^{\Delta\sigma}$ both sides of eq.(19), we obtain

$$\tilde{\xi}_s^\sigma\left(\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q_s^\Delta} - \frac{\partial L}{\partial q_s^\sigma} - Q_s'' - P_s\right) = 0 \tag{38}$$

according to eq. (9), $P_s$ is generally a linear function of $q_s^{\Delta\Delta}$ and can be written as

$$P_s = W_{sk}(t, q_s^\sigma, q_s^\Delta)q_{sk}^{\Delta\Delta} + W_s(t, q_s^\sigma, q_s^\Delta), \tag{39}$$

where $W_{sk} = \frac{\partial m_i}{\partial q_k^\Delta}(u_i + r_i^\Delta)\frac{\partial r_i}{\partial q_s} + \frac{1}{2}r_i^\Delta \cdot r_i^\Delta \frac{\partial^2 m_i}{\partial q_s^\Delta \partial q_k^\Delta} + \frac{\partial m_i}{\partial q_s^\Delta}r_i^\Delta \frac{\partial r_i}{\partial q_k}$.

Adding eq.(38) to eq.(39), and putting the coefficients of $q_s^{\Delta\Delta}$ equal to zero, we obtain

$$\left(\frac{\partial^2 L}{\partial q_s^\Delta \partial q_k^\Delta} - W_{sk}\right)\tilde{\xi}_s^\sigma - \frac{\partial I}{\partial q_k^\Delta} = 0 \tag{40}$$

suppose

$$\det(h_{sk}) = \det\left(\frac{\partial^2 L}{\partial q_s^\Delta \partial q_k^\Delta} - W_{sk}\right) \neq 0 \tag{41}$$

then from eq.(41) we obtain

$$\tilde{\xi}_s^\sigma = \tilde{h}_{sk}\frac{\partial I}{\partial q_k^\Delta} \tag{42}$$

in which

$$\tilde{h}_{sk}h_{kr} = \delta_{sr}, h_{sk} = \frac{\partial^2 L}{\partial q_s^\Delta \partial q_{sk}^\Delta} - W_{sk} \tag{43}$$

Now let the integral (37) equal conserved quantity (28), i.e.

$$I = \frac{\partial L}{\partial q_s^\Delta}\xi_s + \left[L - \frac{\partial L}{\partial q_s^\Delta}q_s^\Delta - \frac{\partial L}{\partial t}\mu(t)\right]\tau(t, q_s) + G \tag{44}$$

Thus, from eq.(42) and (43), generators $\tau, \xi$ of infinitesimal transformation can be found.

**Theorem 5.1.** If the integral of the variable mass holonomic system has been given, then the infinitesimal transformations determined by eq.(21), (43) and (44) are the system's transformation satisfying Noetheris identity (27).

Theorem 5.1 is called the generalized Noether's inverse theorem of the variable mass holonomic system.

## 6 Examples

**Example 1.** The time scale and the Lagrangian of the variable mass system are given as:

$$L = \frac{1}{2}m\left[(q_1^\Delta)^2 + (q_2^\Delta)^2\right] - \frac{1}{2}m\left[(q_1^\sigma)^2 + (q_2^\sigma)^2\right]. \tag{45}$$

The generalized force is $Q_1 = Q_2 = 0$, and the generalized counter thrust is $P_1 = 0, P_s = m^\Delta q_s^\Delta$ following the eq.(19), we find

$$\frac{\Delta}{\Delta t}(mq_1^\Delta) + mq_1^\sigma = 0, \frac{\Delta}{\Delta t}(mq_2^\Delta) + mq_2^\sigma - m^\Delta q_2^\Delta = 0, \tag{46}$$

then the Noether identity (27) becomes

$$\left\{\frac{1}{2}m^\Delta\left[(q_1^\Delta)^2 + (q_2^\Delta)^2\right] - \frac{1}{2}m^\Delta\left[(q_1^\sigma)^2 + (q_2^\sigma)^2\right]\right\}\tau - m(q_1^\sigma\xi_1^\sigma + q_2^\sigma\xi_2^\sigma) + m\left(q_1^\Delta\xi_1^\Delta + q_2^\Delta\xi_2^\Delta\right) \tag{47}$$

we can find solution of eq.(47) as follows

$$\tau = 0, \xi_1 = -1, \xi_2 = -1, \tag{48}$$

substituting the generator (48) into the structure eq. (47) yields

$$G = -mq_1^\Delta - mq_2^\Delta \tag{49}$$

According to Theorem 4.2 , substituting the generator (47) and the gauge function (48) into the formula (28 ), we get the following conserved quantity

$$I = -mq_1^\Delta - mq_2^\Delta. \tag{50}$$

**Example 2.** The time scale and the Lagrangian of the variable mass system are given as:

$$L = \frac{1}{2}m(t)\left[(q_1^\Delta)^2 + (q_2^\Delta)^2\right]. \tag{51}$$

The generalized force is $Q_1 = 0$, $Q_s = q_2^\Delta + q_1 q_1^\Delta$, the generalized counter thrust is $P_1 = P_2 = 0$.
Firstly, following the eq. (19), we find

$$\frac{\Delta}{\Delta t}[m(t)q_1^\Delta] = 0, \frac{\Delta}{\Delta t}[m(t)q_2^\Delta] = q_2^\Delta + q_1 q_1^\Delta \tag{52}$$

then the Noether identity (27) becomes

$$\frac{1}{2}m^\Delta\left[(q_1^\Delta)^2 + (q_2^\Delta)^2\right]\tau + m(q_1^\Delta\xi_1^\Delta + q_2^\Delta\xi_2^\Delta) - \frac{1}{2}m\left[(q_1^\Delta)^2 + (q_2^\Delta)^2\right]\tau^\Delta + (q_2^\Delta + q_1 q_1^\Delta)(\xi_1^\sigma - \tau q_1^{\Delta\sigma}) = -\frac{\Delta}{\Delta t}G. \tag{53}$$

We can find solution of eq. (53) as follows

$$\tau = 0, \xi_1 = 0, \xi_2 = 1, \tag{54}$$

substituting the generator (54) into the structure eq(53) yields

$$G = -q_2 - \frac{1}{2}q_1^2 \tag{55}$$

According to Theorem 4.2 , substituting the generator (53) and the gauge function (54) into the formula (28), we get the following conserved quantity

$$I = mq_2^\Delta - q_2 - \frac{1}{2}q_1^2 \tag{56}$$

Secondly, let us find the corresponding infinitesimal transformations from a known integral. Suppose there is an integral in the form

$$I = mq_2^\Delta - q_2 - \frac{1}{2}q_1^2 \tag{57}$$

According to eq.(42) and eq.(44) we can find

$$\tilde{\xi}_1^\sigma = 0 \tilde{\xi}_2^\sigma = 1 \tag{58}$$

$$L\tau + mq_1^\Delta\delta q_1^\sigma + mq_2^\Delta\delta q_2^\sigma + G = mq_2^\Delta - q_2 - \frac{1}{2}q_1^2 \tag{59}$$

following the $\tilde{\xi}_s^\sigma = \xi^\sigma - \tau^\sigma q_s^{\Delta\sigma}$ we obtain

$$\tau = -\frac{1}{L}\left(G + q_2 + \frac{1}{2}q_1^2\right), \xi_1 = q_1^\Delta, \xi_2 = 1 + q_2^\Delta\tau \tag{60}$$

we have $G = -q_2 - \frac{1}{2}q_1^2$, then $\tau = 0, \xi_1 = 0, \xi_2 = 1$.

## 7 Summary

In this manuscript, the Noether's theorems of variable mass systems on time scales have been studied. We established the Hamilton principle and derived the Lagrange equations for the variable mass system on time scales. Under the kind of infinitesimal transformations, we gave the definitions and criteria of Noether symmetries. And the Noether theorems and its inverse theorem of variable mass system on time scales are established. This paper considered the continuous case and the discrete case, so the results of this paper are of universal meaning. Besides, further study could include Lie symmetry. The approach of this paper can be furthermore generalized to other systems such as relative motion system; Birkhoffian systems and electromectro mechanial coupling system are equally worth studying on time scales.

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