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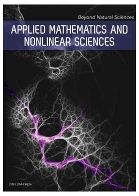

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Numerical investigation on global dynamics for nonlinear stochastic heat conduction via global random attractors theory

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Abstract

In term of the global random attractors theory, global dynamics of nonlinear stochastic heat conduction driven by multiplicative white noise with a variable coefficient are investigated numerically. It is shown that global $\mathscr{D}$-bifurcation, secondary global $\mathscr{D}$-bifurcation and complex dynamical behavior occur in motion of the system with increasing the intensity of linear component in the heat source. Furthermore, the results obtained here indicate that Hasudorff dimension which is relevant to global Lyapunov exponent can be used to describe global dynamics of the associated system qualitatively.

Keywords:nonlinear stochastic heat conduction, global dynamics, random invariant measure, Hausdorff dimension, global Lyapunov exponent

1 Introduction

The definition of global random attractors for random dynamical system (RDS) established by Arnold [1] were proposed by Crauel and Flandoli [2], Schmalfuss [3]. Whereafter, Crauel et al [4] introduced the notion of stochastic dynamical system (SDS) and the global random attractors associated with SDS. Furthermore, the assertion that global random attractors are uniquely determined by attracting deterministic compact sets in phase space was attained by Crauel in [5]. Invoking these theory, the existence of global random attractors for SDS and RDS related to a plenty of mathematical physics problems have been studied by many researchers, for example, Ref [6, 7] and the references therein.

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Once the infinite dynamical system has global attractors, the study on the structure of this invariant set is an interesting problem. Essentially, the Hausdorff dimension is one of the few pieces of information to describe the structure of global attractor in the aspect of analytical analysis. With respect to deterministic situation, the approaches to obtain the finite estimation for Hausdorff dimension of global attractors were due to Temam [8], Chepyzhov and Vishik [9], Caraballo et al [10]. As for the estimation for Hausdorff dimension of global random attractors, according to extend the methods in deterministic case, Crauel and Flandoli [11], Schmalfuss [12] provided the way to estimate the Hausdorff dimension of global random attractors, however, the conditions that assumed in that kind method are too conservative. The Assumptions in treatment given by Debussche in [13] can be satisfied by many RDS and SDS induced from mathematical physics problems, nevertheless, the results on estimation for the Hausdorff dimension obtained by this method do not have the relationship with global Lyapunov exponents clearly. Invoking the property of ergodicity, Debussche [14] established the powerful procedure to estimate the Hausdorff dimension of global random attractors. Furthermore, the estimation derived by the method possesses a strongly relationship with global Lyapunov exponents. By using this method, abundant literatures treated the estimation for Hausdorff dimension of global random attractors for a plenty of mathematical physics problems, for instance [15–19] and the references therein.

There exist two views in the study on behavior of dynamical system, the "static" standpoint and the "dynamical" standpoint, which are not distinguished in the deterministic case, however, the clear connection between the two views in the stochastic status do not exists, for more detail, one can refer to [1, 20]. There exist some results on "dynamical" global dynamics on stochastic dynamical system, for example, Crauel and Flandoli [20] asserted that additive noise destroys pitchfork bifurcation in one dimensional system. The fact that parametric noise (even a multiplicative white noise) destroys Hopf bifurcation was proved by Arnold et al [21]. Wang [22] focused on the bifurcation for stochastic parabolic equations. The investigation on stochastic bifurcation in Duffing system by the theory of random attractors was due to Schenk-Hoppé [23].

Heat conduction process [24], the purpose of which is to determine the distribution of temperature in solid or static fluid, is a very important problem in practical applications, such as civil engineering, high-speed aircraft. Actually, the dynamical govern equation for heat conduction is one kinds of nonlinear stochastic reaction-diffusion equations, the results on global attractor for the system in deterministic case, see Ref [8] and [9]. Debussche [14] obtained the estimation for Hausdorff dimension of global random attractors of reaction-diffusion equations driven by additive white noise. As for the case that the system under multiplicative noise loading with a constant coefficient, the Hausdorff dimension was estimated by Caraballo et al [16]. Afterwards, Caraballo et al [25] studied the stochastic pitchfork bifurcation of the system, the main approach in which is based on finding some invariant manifold to derive the lower bounds on the dimension of global random attractors. It is a very wonderful work in the investigation on global dynamics of stochastic system, however, it is hardly understood by the engineering. Thus, in light of the Hasudorff dimension obtained by Fan and Chen [18], the global dynamics of stochastic heat conduction with multiplicative white noise with a variable coefficient are studied numerically by employing the stochastic subdivision algorithm method proposed by Keller and Ochs [26] to achieve the global random attractors. To our best knowledge, there hardly exist such investigation on this problem.

The rest content of this paper is organized as follows. In Section 2, the main preliminaries related to existence and Hausdorff dimension of the stochastic heat conductio are given. Section 3 is intended to provided the main numerical results and conclusion.

2 Preliminaries

This section comprises two components, one of which describe the model nonlinear stochastic heat conduction. Some theoretical results which are invoked in numerical study are listed in the other part.

2.1 Model for nonlinear stochastic heat conduction

Let D be a bounded domain with regular boundary in $\mathbb{R}^n$, $n = 1,2,3$, the govern equation for a heat conduction with nonlinear stochastic heat conduction is as follows

$$\frac{\partial u}{\partial t} = \Delta u + au - u^3 + g(t)u\xi \tag{1}$$

with Dirichlet boundary value

$$x \in \partial : u = 0 \tag{2}$$

where $u = u(t,x)$ represents the distribution of temperature of thermal conductors, $au - u^3 + g(t)u\xi$ is the nonlinear stochastic thermal source, ξ is used to signify the random influences in thermal source, then $u\xi$ denotes the multiplicative random influences, the time-varying coefficient of which is $g(t)$. For more detail about physical background of the model, we refer to [24].

Let $A := -\Delta : (H^2 \bigcap H_0^1)(D) \subset L^2(D) \to L^2(D)$, then A is self-adjoint, positive, unbounded linear operators and $A^{-1} \in \mathscr{L}(L^2(D))$ is compact. Therefore, their eigenvalues $\{\Lambda_i\}_{i\in\mathbb{N}}$ satisfy $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ and the corresponding eigenvalues $\{w_i\}_{i=1}^{\infty}$ forms an orthonormal basis in $L^2(D)$. On the other hand, let $\Omega = \{\omega | \omega(\cdot) \in \mathbb{C}(\mathbb{R},\mathbb{R}), \omega(0) = 0\}$, $\mathscr{F}$ is the $\sigma-$ algebra induced by the compact open topology for this set and $\mathbb{P}$ is the Wiener measure on $\mathscr{F}$, then $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space. In this paper, the white noise which is the derivative of one dimensional two-sided real-valued standard Wiener process $W(t)$ supported by probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is invoked to model the random influences ξ. Moreover, let $\mathscr{F}_t = \sigma(W(s), s \leq t)$, $\forall t \in \mathbb{R}$, the $\{\mathscr{F}_t\}_{t\in\mathbb{R}}$ is called as a filtration in $\mathscr{F}$. For more detail, we refer to Protter [27]. Thus, the system (1)-(2) is equivalent to the following system in Itö form.

$$\begin{cases} du = (-Au + \alpha u - u^3)dt + g(t)udW \\ x \in \partial D : u = 0 \end{cases} \tag{3}$$

Moreover, suppose $g(t)$ satisfies

$$0 \leq \beta_1 \leq |g(t)| \leq \beta_2, \tag{4}$$

in which β_1, β_2 are constants.

2.2 Some existed theoretical results

This subsection is devoted to introduce basic theory related to random attractors theory as well as asymptotic behavior about system (3) which is used in this paper

For the theoretical results associated with asymptotic behavior of system (4) in deterministic case which means $g(t) = 0$, we refer to the part (a) in Section 2 of Caraballo et al [25], the duplicate narration is not made here.

In order to give the notion of global random attractors, the definition of RDS and SDS needed to be given firstly. Set

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(\cdot), \ \forall t \in \mathbb{R}, \tag{5}$$

in which, $\omega \in \Omega$ defined in subsection 2.1, according to Ref [1], we have $\mathbb{P}$ is ergodic with respect to the flow $\{\theta_t\}_{t\in\mathbb{R}}$. Thus, the $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta\}_{t\in\mathbb{R}})$ is the metric dynamical systems [1] employed in this paper.

The ensuring definition of RDS was inaugurated by Arnold [1].

Definition 2.1. *[1] A RDS on Polish space (X,d) with Borel $\sigma-$ algebra $\mathscr{B}(X)$ over a metric dynamical system $(\Omega, \mathscr{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a $(\mathscr{B}(\mathbb{R}^+) \times \mathscr{F} \times \mathbb{B}(X), \mathbb{B}(X))$measurable mapping*

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \ \ (t, \omega, x) \mapsto \Phi(t, \omega, x)$$

such that, for $\mathbb{P}-a.s.\omega \in \Omega$,

(i). $\phi(0,\omega) = id$ *on* X.

(ii). $\phi(t+s,\omega,\cdot) = \phi(t,\theta_s\omega,\cdot)\circ\phi(s,\omega,\cdot)$ *for all* $s,t \in \mathbb{R}^+$.

A RDS is continuous or differential if $\phi(t,\omega): X \to X$ is continuous or differential. Furthermore, it is emphasized that RDS $\phi(t,\theta_{-t}\omega)$ can be understood as the solution start from $-t$ to 0.

The following notion of SDS can be founded in Crauel et al [4].

Definition 2.2. *[4] A SDS on Polish space* (X,d) *is a family of mapping*

$$\phi: \mathbb{R}\times\mathbb{R}\times\Omega\times X \to X, \ \ (t,s,\omega,x) \mapsto \Phi(t,s;\omega)x, t \geq s.$$

such that, for $\mathbb{P}-a.s.\omega \in \Omega$,

(i). $\phi(s,s;\omega) = id$ *on* X.

(ii). $\phi(t,s;\omega) = \phi(t,\tau;\omega)\circ\phi(\tau,s;\omega)$ *for all* $-\infty < s \leq \tau \leq t < \infty$.

The SDS is not assumed to be measurable with respect to ω, moreover, this definition is provided for the researcher who are not used to probabilistic language. In most applications, for all $t \in \mathbb{R}, x \in X$, $(s,\omega) \to \phi(s,s,\omega)x$ is measurable, for all $s < t$ and $x \in X$, the $\omega \to \phi(s,s,\omega)x$ is measurable and

$$\phi(t,s,\omega)x = \phi(t-s,0,\theta_s\omega)x, \ \ \mathbb{P}-a.s.,$$

then the SDS and RDS is coincident. However, the stochastic nonlinear heat conduction equation driven by a multiplicative white noise with a vary coefficient can generate a family of mapping satisfies the Definition 2.2, but we are not sure that it makes a cocycle. Therefore, we distinguish the aforementioned definitions here.

The coming definitions related to random attractors for RDS was established by Crauel et.al [2,28] .

Definition 2.3. *Let* $\mathscr{B} \subset 2^X$ *is a collection of subsets of* X*, then a closed random set* $\mathscr{A}(\omega)$ *is called* $\mathscr{B}$*-random attractor associated with the RDS* ϕ*, if* $\mathbb{P}-a.s.$

(i). $\mathscr{A}(\omega)$ *is a random compact set.*

(ii). $\mathscr{A}(\omega)$ *is invariant i.e.* $\phi(t,\omega)\mathscr{A}(\omega) = \mathscr{A}(\theta_t\omega)$ *for all* $t \geq 0$

(iii). For every $B \in \mathscr{B}$,

$$\lim_{t\to\infty} dist(\phi(t,\theta_{-t}\omega)B, \mathscr{A}(\omega)) = 0$$

where dist denotes the Hausdorff semidistance:

$$dis(A,B) = \sup_{x\in A}\inf_{y\in B} d(x,y), \quad A,B \subset X.$$

When $\mathscr{B}$ *is composed of all bounded set of* X*, then* $\mathscr{A}(\omega)$ *is the global random attractors for* ϕ*. If* $\mathscr{B} = \{\{x\}: x \in X\}$*,* $\mathscr{A}(\omega)$ *is said to be global random point attractor.*

Actually, Definition 2.3 is the notion of random attractor for RDS, if omit 0 in $S(0,s,\omega)$ in definition of random attractors for SDS proposed in Ref [4], random attractor for RDS and for SDS are the same.

The follow assertion provides the relationship between random attractors and invariant measures which is important to exploit the numerical results to expound the global dynamics for RDS.

Proposition 2.4. *When the RDS or SDS* φ *possesses global random attractor comply with Definition 2.3, by the Corollary 4.4 in Crauel [5], this attractors supports every invariant measures. The random point attractor of* φ *given by Definition 2.3 always supports at lest one invariant measure which even is a invariant Markov measure (Crauel [28],P423; Arnold [1], Theorem 1.6.13 and Theorem 1.7.5). When* φ *is a white noise RDS or SDS, together with the Theorem 3.6 in Crauel [28] give that every invariant Markov measure is supported by the global point attractor. On the other hand, if the global random attractors for* φ *exists, then* φ *also has the* *global point attractor. For any fixed $\omega \in \Omega$, taking advantage of pullback mechanism [1], follow the proof of Theorem 5.2 in Birnir [29], the global random attractor can be decomposed into two ingredients, one is random basic attractor which supports all stable invariant Markov measures of white noise RDS or SDS, the other is random remainder. For the definition of basic and remainder, we refer to Birnir [29].*

The global dynamics in this paper are understood as the change in the pattern of existing probability invariant measures of SDS. With the assertion that the SDS possesses a global random attractors, based on Proposition 2.4, this investigation can be accomplished by exploit the numerical results on the structure of global random attractor. Essentially, the useful objects are global random basic attractors and global random point attractors.

With respect to the case $\beta_1 = \beta_2$ in (4), system (3) can induces a RDS which possesses finite dimensional global random attractors, for more detail, we refer to Caraballo et al [16]. According to derive a connection between Itö integral and Lebesgue integral and two type stochastic Gronwall inequalities, the following theoretical results about the global random attractors for system (3) were accomplished by Fan and Chen [18].

Theorem 2.5 ([18],Theorem 2.1)**.** *For any $\mathscr{F}_\tau$−measurable $u_0 \in L^2(D)$, there exists a unique adapted (weak) solution u of* (3) *which satisfies*

$$u \in L^2(\tau,T;H_0^1(D)) \cap L^4(\tau,T;L^4(D)), \ \ \forall T > \tau,$$

$$u \in \mathbf{C}(\tau,+\infty;L^2(D)).$$

The mapping $u_0 \mapsto u$ is Lipschitz continuous in $L^2(D)$. If, furthermore, $u_0 \in H_0^1(D)$, then u belongs to

$$\mathbf{C}(\tau,T;H_0^1(D)) \cap L^2(\tau,T;H^2(D)), \ \ \forall T > \tau.$$

The Theorem 2.5 reveals that system (3) can generate a SDS, denoted by $S(t,\tau;\omega)$. The following theorem addresses the existence of global random attractors for $S(t,\tau;\omega)$, which was given by in Ref [18].

Theorem 2.6. *The SDS $S(t,\tau;\omega)$ owns nonempty tempered random attractors $\mathscr{A}(t,\omega)$, $t \in \mathbb{R}$, $\omega \in$ ■, and $\mathscr{A}(t,\omega) \subseteq \mathscr{D}(t,\omega)$. Moreover, if $\alpha < \lambda_1 + \frac{1}{2}\beta_1^2$, the random attractors become $\{0\}$.*

Utilizing Theorem 2.6, there exist global random point attractors and global random basic attractors for the SDS $S(t,\tau;\omega)$. Base on the methods to estimate the Hausdorff dimension of random invariant set and nonautonomous invariant set, which were given in [14] and [9], the coming assertion holds.

Theorem 2.7 ([18],Theorem 5.2)**.** *Let*

$$d = \min\left\{ k \in \mathbb{N}^+ : \alpha < \frac{1}{k}\sum_{j=1}^{k}\lambda_j + \frac{\beta_1^2 k}{2} \right\},$$

then the Hausdorff dimension of the random attractors $\mathscr{A}(t,\omega)$ satisfies

$$dim(\mathscr{A}(t,\omega)) \leq d, \qquad \forall t \in \mathbb{R}.$$

Moreover, if $\alpha < \lambda_1 + \frac{\beta_1^2}{2}$, then the Hausdorff dimension of the random attractors is 0*.*

From the Theorem 2.7, an interesting phenomenon expressed in the following manner can be derived. If $\min|g(t)| = 0$, then no matter what is value of $\max|g(t)|$, the estimation for Hausdorff dimension of global random attractor does not vary, which is equal to the value of status $g(t) = 0$. Furthermore, in the case of $g(t) = \sigma$, here σ is positive constant, the estimation of estimation for Hausdorff dimension of global random attractor is less than or equal to the result given by Caraballo et al [16], which is expounded in Ref [18].

3 Numerical results and conclusion

This Section is devoted to numerically obtain the global random attractor by stochastic subdivision algorithm method proposed in [26], subsequently, along with Proposition 2.4 analysis the global dynamics of the system. Finally, the conclusion is given.

4 Numerical results and conclusion

The modal equations associated with EBS which is not display here (See Equations (A1) in 5) can be got by employing inertial manifold with delay [30] and nonlinear gakerlin method [31].

Let $D=(0,3.1), k_1=2, k_2=1$ in Appendix 5, the eigenvalues $\{\lambda_i\}_{i=1}^{3}$ (see Table 1) and eigenvectors $\{w_i\}_{i=1}^{3}$ of operator A and integration with respect to space variable in (A2) and (A3) listed in Appendix 5 can be performed by COMSOL with Matlab [32]. Thus, the first three modes related to system (3), solution of which

Table 1 The one to for order eigenvalues for A

λ_1	λ_2	λ_3	λ_4
1.0270	4.1081	9.2431	16.4329

can be achieved by stochastic Runge-Kutta method [33], can be attained. The SDS induced by this model equations is denoted by φ_{NGM3}. Suppose $g(t)=1+0.5sin(t)$, then $\beta_1=0.5$, $\beta_2=1.5$. Fixed any sample ω of one dimensional two-sided real-valued standard Wiener process. We mainly consider the global dynamics for φ_{NGM3} in the cases presented in Table 2. For the sake of brevity, let $\underline{l}_1=p_1, \underline{l}_2=q_1, \bar{l}_1=p_2$ and $\{\mu_i^g\}_{i\in\mathbb{N}}$ satisfies $\mu_i^g \le \mu_{i+1}^g, i\in\mathbb{N}$ denotes the global Lyapunov exponent.

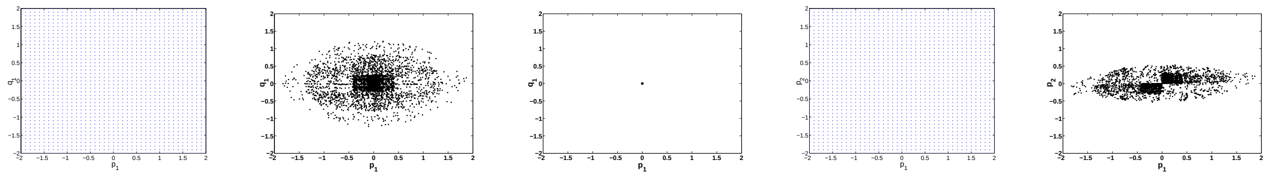
Table 2 The values of α and corresponding Hasudorff dimension

	Case I	Case II	Case III	Case IV
α	1	2.4	4	6
Hausdorff dimenison	0	1	2	3

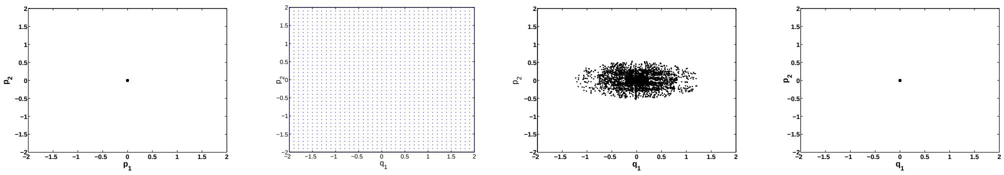
Case I. In this situation $\alpha=1$, together with Theorem 2.7 gives that the estimation of Hausdorff dimension is 0, which means $\mu_1^g<0$. The numerical results on random attractors of φ_{NGM3} is shown by Figure 1. Obviously, φ_{NGM3} possesses a.s. stable invariant Markov measure $\delta_0(\omega)$ which is supported by a random fix point described by Figure 1b, Figure 1e and Figure 1h.

Case II. In this circumstance $\alpha=2.4$, the estimation of Hausdorff dimension is 1, which interprets $\mu_1^g+\mu_2^g<0$ and $\mu_1^g\ge 0$. From the numerical results on global random point attractor (see Figure 3), there exist three invariant Markov measures for φ_{NGM3}, which together with global random basic attractor described by Figure 2 give that two of them is stability, denoted by $\delta_{-x_1(\omega)}, \delta_{x_1(\omega)}$, the rest of invariant Markov measure is instability, represented by δ_0. Correspondingly, the section of global point/basic attractors for $S(\tau,\tau-t,\omega)$ is shown by Figure 5, from which the dynamical behavior obtained is the same as dynamical behavior of φ_{NGM3}. Thus, the Figures that describe dynamical behavior of $S(\tau,\tau-t,\omega)$ is not listed in the sequel.

Case III. In this situation $\alpha=4$, the estimation of Hausdorff dimension is 2 gives that $\sum_i^3 \mu_i^g<0$ and $\sum_i^2 \mu_i^g\ge 0$. The assertion that φ_{NGM3} possesses two stability invariant Markov measures can be derived by the section of global random point attractors illustrated by Figure 6. The global random point attractors (see Figure 7) reveal that there exist other three instability of invariant Markov measures. Figure 8 illustrates the boundary of global random attractors.

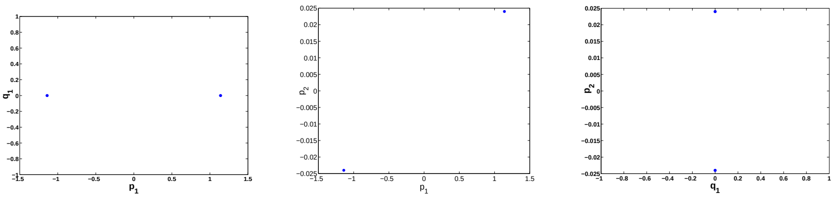


(a) t=0 (b) t=0.5 (c) t=1 (d) t=0 (e) t=0.5



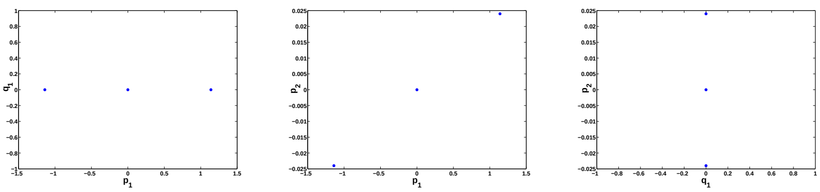
(f) t=1 (g) t=0 (h) t=0.5 (i) t=1

Fig. 1 Supports of invariant measure and random attractors for φ_{NGM3} with case I



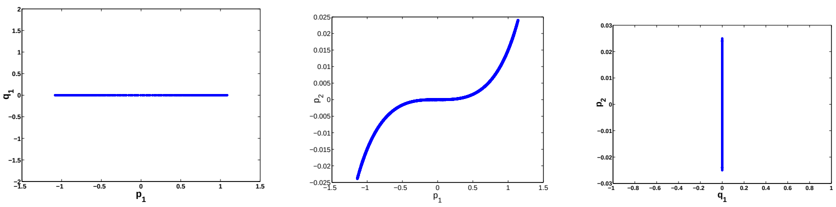
(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

Fig. 2 Section of global random basic attractors for φ_{NGM3} with case II



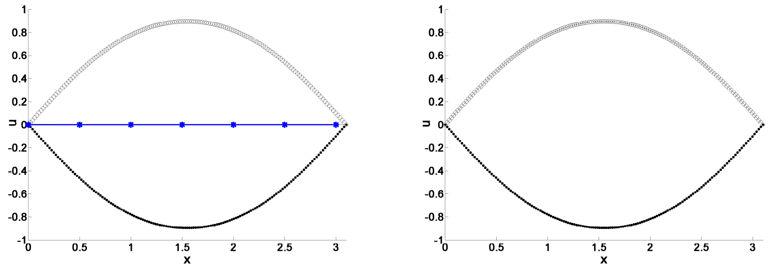
(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

Fig. 3 Section of global random point attractors for φ_{NGM3} with case II



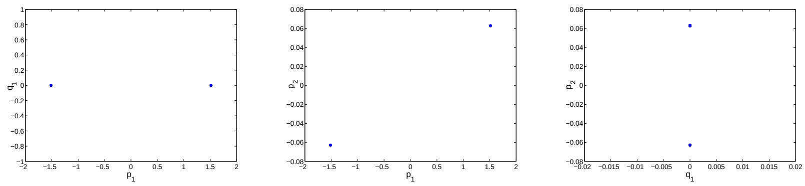
(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

Fig. 4 Section of global random attractors for φ_{NGM3} with case II



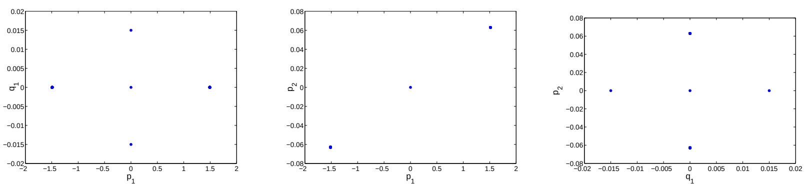
(a) Section of global point attractors (b) Section of global basic attractors

Fig. 5 Section of global point/basic attractors for $S(\tau, \tau - t, \omega)$ with case II



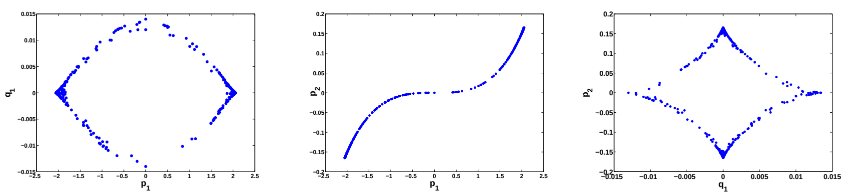
(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

Fig. 6 Section of global random basic attractors for φ_{NGM3} with case III



(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

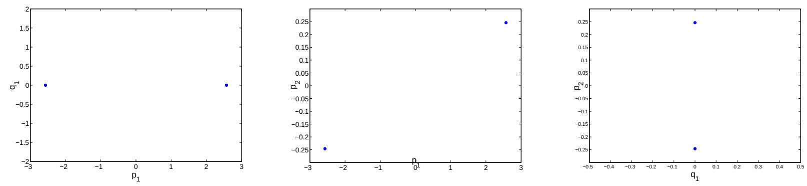
Fig. 7 Section of global random point attractors for φ_{NGM3} with case III



(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

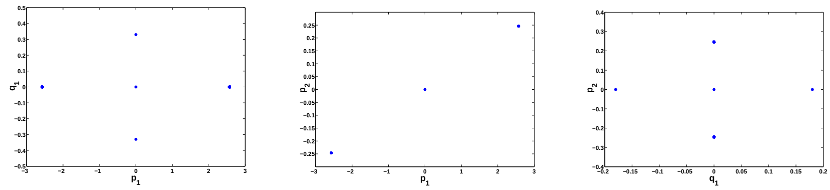
Fig. 8 Section of boundary of global random attractors for φ_{NGM3} with case III

Case IV. In this status $\alpha = 4$, the estimation of Hausdorff dimension that is 3 asserts that $\sum_i^4 \mu_i^g < 0$ and $\sum_i^3 \mu_i^g \geq 0$. The numerical results on global random point attractors (see Figure 10) and global random basic attractors expressed by Figure 9 indicate that φ_{NGM3} possesses two stable invariant Markov measure and three instability invariant Markov measures. Moreover, complex dynamical behavior is demonstrated by boundary of global random attractors (see Figure 11) appears in the dynamics of φ_{NGM3}.



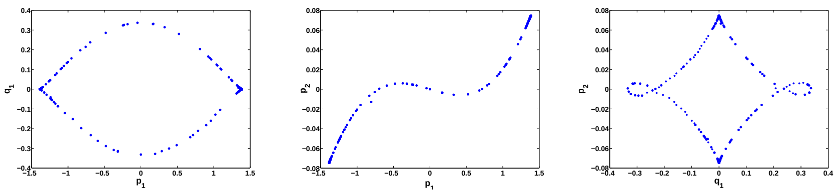
(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

Fig. 9 Section of global random basic attractors for φ_{NGM3} with case IV



(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

Fig. 10 Section of global random point attractors for φ_{NGM3} with case IV



(a) $p_1 - q_1$ (b) $p_1 - p_2$ (c) $q_1 - p_2$

Fig. 11 Section of boundary of global random attractors for φ_{NGM3} with case IV

By means of the aforementioned numerical results, the following affirmation on global dynamics for system (3) can be derived. When α varies from 1 to 2.4, the global $\mathscr{D}$-bifurcation which actually is a pitchfork bifurcation [1] occurs in motion of the nonlinear stochastic heat conduction. Subsequently, let value of α increase to 4, the secondary global $\mathscr{D}$-bifurcation appears. If the value of α becomes even more big (for instance, $\alpha = 6$ in Case IV), the global dynamics of for system (3) is complex. Furthermore, together with Table 2, we have that with varying α from 1 to 6, the estimation of Hausdorff dimension of global random attractors for system (3) increases from 0 to 3, which indicates that the global Lyapunov exponent maybe change from negative to positive. Therefore, the Hausdorff dimension of global random attractors can be used to demonstrate the complex of the system.

5 Conclusion

Based on the global random attractors theory and Ref [18], with varying of parameter α, the global dynamics of nonlinear stochastic heat conduction driven by multiplicative white noise are studied numerically, the global $\mathscr{D}$-bifurcation, secondary global $\mathscr{D}$-bifurcation and complex dynamical behavior in motion of the system are founded. Moreover, from the numerical results, it can be concluded that when the estimation of Hausdorff dimension of global random attractors for the system, which possesses a strongly relationship with global Lyapunov exponent, can be calculated precisely, then it can be used to describe the global dynamics of the problem which generates a dynamical system. In practical applications, the parameter α represents the intensity of linear ingredient of the stochastic source, results accomplished by this paper reveal that the global dynamics of nonlinear stochastic becomes complicated with increasing the intensity of linear part in the stochastic heat source.

When complex dynamics occurs, this paper expounded phenomenon simply, such as the dynamical phenomenon in Case IV. Detailed analysis should be considered by more tools. On the other hand, the global Lyapunov exponent can be invoked to describe the global dynamics of the dynamical system, which can be derived by the fact that estimation of Hausdorff dimension of global random attractors can illustrate the global dynamics of the system. However, the results on estimation of Hausdorff dimension is too conservative, therefore, we hope to study global dynamics of other mathematical physics problems by attaining the accurate global Lyapunov exponent numerically in the further.

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APPENDIX

The model equation

Suppose $u = \sum_{i=1}^{k} l_i(t) w_i, k = k_1 + k_2, k_1, k_2 \in \mathbb{N}$, the following model equations associated with nonlinear stochastic heat conduction with the following form

$$\begin{gathered} d\underline{\mathbf{l}} = (-\mathbf{A}_1\underline{\mathbf{l}} + \mathbf{F}_1(\underline{\mathbf{l}}, \bar{\mathbf{l}}_{\mathbf{i}}))dt + \mathbf{B}_1\underline{\mathbf{l}}dW, \\ \bar{\mathbf{l}}_{i+\tau_1 h} - \bar{\mathbf{l}}_i + \mathbf{A}_2\bar{\mathbf{l}}_{i+\tau h} = \mathbf{F}_2(\underline{\mathbf{l}}, \bar{\mathbf{l}}_{\mathbf{i}}))h + \mathbf{B}_2\bar{\mathbf{l}}_i dW, t \in (\tau h - h, \tau h]; \\ \bar{\mathbf{l}} = \bar{\mathbf{l}}_{\tau h - h}, t \notin (\tau h - h, \tau h] \end{gathered} \tag{A1}$$

where $\underline{\mathbf{l}} = (\underline{l}_1, \cdots, \underline{l}_{k_1})^T = (l_1, \cdots, l_{k_1})^T$ which represent the low-frequency modal and the other is high-frequency modal. $\bar{\mathbf{l}} = (\bar{l}_1, \cdots, \bar{l}_{k_2})^T = (l_{k_1}, \cdots, l_{k_1+k_2})^T$, $\mathbf{l}_i$ is the value of $\mathbf{l}$ at time i. h is step size of numerical integration, $\tau \in \mathbb{N}$ is an undetermined constant.

$$\begin{gathered} \mathbf{A}_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{k_1} \end{bmatrix}, \qquad \mathbf{B}_1 = g(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}_{k_1 \times k_1}, \\ \mathbf{F}_1(\underline{\mathbf{l}}, \bar{\mathbf{l}}) = \begin{bmatrix} \alpha - ((\sum_{i=1}^{k} l_i w_i)^3, w_1) \\ \cdots \\ \alpha - ((\sum_{i=1}^{k} l_i w_i)^3, w_{k_1}) \end{bmatrix} \end{gathered} \tag{A2}$$

and

$$\mathbf{A}_2 = \begin{bmatrix} \lambda_{k_1+1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{k_1+k_2} \end{bmatrix}, \qquad \mathbf{B}_2 = g(t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}_{k_2\times k_2},$$

$$\mathbf{F}_2(\underline{\mathbf{l}},\bar{\mathbf{l}}) = \begin{bmatrix} \alpha - ((\sum_{i=1}^{k} l_i w_i)^3, w_{k_1+1}) \\ \cdots \\ \alpha - ((\sum_{i=1}^{k} l_i w_i)^3, w_{k_1+k_2}) \end{bmatrix} \tag{A3}$$

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