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A global solution for a reaction-diffusion equation on bounded domains

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Abstract

In the literature, it has been proved the existence of a pullback global attractor for reaction-diffusion equation on a bounded domain and under some conditions, a uniform bound on the dimension of its sections. Using those results and putting a bound on the diameter of the domain, we proved that the pullback global attractor consists only of one global solution. As an application to this result, a bounded perturbation of Chafee-Infante equation has been studied.

Keywords: non-autonomous reaction-diffusion equation, pullback global attractor, fractal dimension, Chafee-Infante equation
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1 Introduction

Pullback attractors have become a suitable tool for studying non-autonomous dissipative dynamical systems generated by evolution equations arising in physical phenomena. Pullback global attractor is a family of compact subsets of the phase space which is strictly invariant under the evolution process and attracts all bounded subsets in a pullback sense. For pullback exponential attractor, we have a positively invariant family of compact subsets which have a uniformly bounded fractal dimension and pullback attract all bounded subsets at an exponential rate. We recall the definitions from [4]:

Definition 1. The two parameter family $\{U(t,s)|t,s\in\mathbb{R},t\geq s\}$ of continuous operators from the metric space X to itself is called an *evolution process* in X if it satisfies the properties

$$U(t,s)\,o\,U(s,r)=U(t,r),\quad t\geq s\geq r$$
$$U(t,t)=Id,\quad t\in\mathbb{R}\quad and$$
$$\mathscr{T}\times X\ni(t,s,x)\to U(t,s)x\in X\ is\ continuous,$$

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Where $\mathscr{T} = \{(t,s) \in \mathbb{R} \times \mathbb{R} | t \geq s\}$

Definition 2. The family of non-empty subsets $\{\mathscr{A}(t) | t \in \mathbb{R}\}$ of X is called a *pullback global attractor for the process* $\{U(t,s) | t \geq s\}$ if $\mathscr{A}(t)$ is compact for all t, the family $\{\mathscr{A}(t) | t \in \mathbb{R}\}$ is strictly invariant , that is

$$U(t,s)\mathscr{A}(s) = \mathscr{A}(t) \quad for\ all\ t \geq s,$$

it pullback attracts all bounded subsets of X, that is for every bounded $D \subset X$ and $t \in \mathbb{R}$

$$\lim_{s\to\infty} dist_H(U(t,t-s)D, \mathscr{A}(t)) = 0$$

and the family is minimal within the families of closed subsets that pullback attract all bounded subsets of X.

Definition 3. We call the family $\mathscr{M} = \{\mathscr{M}(t) | t \in \mathbb{R}\}$ a *pullback exponential attractor for the process* $\{U(t,s) | t \geq s\}$ in X if

1. the subsets $\mathscr{M}(t) \subset X$ are non-empty and compact in X for all $t \in X$,
2. the family is positively semi-invariant, that is
$$U(t,s)\mathscr{M}(s) \subset \mathscr{M}(t) \quad for\ all\ t \geq s$$
3. the fractal dimension in X of the sections $\mathscr{M}(t)$, $t \in \mathbb{R}$, is uniformly bounded and
4. the family $\{\mathscr{M}(t) | t \in \mathbb{R}\}$ exponentially pullback attracts bounded subsets of X; that is, there exists a positive constant $\omega > 0$ such that for every bounded subset $D \subset X$ and $t \in \mathbb{R}$
$$\lim_{s\to\infty} e^{\omega s} dist_H(U(t,t-s)D, \mathscr{M}(t)) = 0.$$

For some constructions of these attractors refer for example to [1, 4, 6–9]. Reaction-diffusion equation, in this regard, has been studied as an example of dissipative systems.
We consider

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(t,u) = h(t), \\ u|_{\partial\Omega} = 0, \\ u(s) = u_s, \qquad t > s. \end{cases}$$

where the forcing term $h(t)$ is allowed to have an exponential growth in time.
When the phase space is $L^2(\Omega)$, in [1], it was obtained a uniform bound on the dimensions of sections of the pullback global attractor under a Lipschitz condition on the nonlinear term $f(t,u)$.
In [7], under some requirements for the phase space and the process defined on it, it was constructed a uniform bound for the dimensions of sections of the pullback exponential attractor and the pullback global attractor contained in it and as an application, it was applied to the reaction-diffusion equation in the phase space $H_0^1(\Omega)$.
We use the bound obtained in these works for proving that the dimension of sections is zero providing a bound on the diameter of the domain Ω which brings about a unique global solution as the pullback global attractor.
We, moreover, release the Lipschitz condition on f and replace it with a condition on f_u which affords us to deal with such problems with a polynomial nonlinearity as Chafee-Infante equation and obtain its trivial dynamics under a bounded perturbation.

2 Statement of the problem

Let us consider the initial boundary value problem for the non-autonomous reaction-diffusion equation as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(t,u) = h(t), \\ u|_{\partial\Omega} = 0, \\ u(s) = u_s, \qquad t > s. \end{cases} \tag{1}$$

where $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $h(\cdot) \in L^2_{loc}(\mathbb{R}, L^2(\Omega)), \Omega$ is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and there exists $p \geq 2, c_i > 0, i = 1, ..., 6$, such that

$$c_1|u|^p - c_2 \leq f(t,u)u \leq c_3|u|^p + c_4, \tag{2}$$

$$f_u(t,u) \geq -c_5, \tag{3}$$

$$f(t,0) = 0, \quad \forall t \in \mathbb{R}, \tag{4}$$

$$\|h(t)\|_{L^2(\Omega)} \leq c_6 e^{\alpha|t|}, \tag{5}$$

and $0 \leq \alpha < \lambda_1$ where $\lambda_1 > 0$ is the first eigenvalue of the operator $A = -\Delta$ where Δ is the Laplace operator in $L^2(\Omega)$ with zero Dirichlet boundary condition [7].

When the phase space is $L^2(\Omega)$, in [1], it was proved that the problem (1) has a pullback global attractor in $L^2(\Omega)$ with f satisfying (2),(3),(4) and under the condition of a polynomial bound on the forcing term instead of (5):

$$\|h(t)\|_{L^2(\Omega)} \leq a|t|^r + b, \quad t \in \mathbb{R}, a, b > 0, r \geq 0.$$

and obtained a uniform bound on the fractal dimension of its sections under an additional assumption that there exists a positive and nondecreasing function $\xi : \mathbb{R} \to (0, \infty)$ such that:

$$|f(s,u) - f(s,v)| \leq \xi(t)\,|u - v|, \quad s \leq t,\, u, v \in \mathbb{R}. \tag{6}$$

Here, we briefly state the assumptions on the process to have a pullback exponential attractor [7]:

Let we have a Banach space V and $U(t,s)$ an evolution process on it. We assume that

($\mathscr{A}_1$) There is a positively invariant family of closed bounded subsets of V, $B(t)$.

($\mathscr{A}_2$) $B(t)$ grows exponentially in the past, i.e. $\mathrm{diam}(B(t)) < Me^{\gamma_0 t} \quad t \leq t_0, \gamma_0 \geq 0$.

($\mathscr{A}_3$) The family pullback absorbs all bounded subsets of V.

Next, we assume that the process has the decomposition $U(t,s) = C(t,s) + S(t,s)$, where, $\{C(t,s) : t_0 \geq t \geq s\}$ and $\{S(t,s) : t_0 \geq t \geq s\}$ are families of operators satisfying the following properties:

($\mathscr{H}_1$) There exists $\tilde{t} > 0$ such that $C(t, t - \tilde{t})$ are contractions within the absorbing sets, where the contraction constant λ is independent of time and
$0 \leq \lambda \leq \frac{1}{2} e^{-\gamma_0 \tilde{t}}$ with $\gamma_0 \geq 0$ from $\mathscr{A}_2$,

($\mathscr{H}_2$) There exists an auxiliary normed space $(W, \|\cdot\|_W)$ such that V is compactly embedded into W and $S(t - \tilde{t})$ satisfies the smoothing property within the absorbing sets,

($\mathscr{H}_3$) The process is Lipschitz continuous within the absorbing sets.

In [7],theorem 2.2, it was shown that under these assumptions, there exists a pullback exponential attractor for the process and it was applied to the above problem in $H_0^1(\Omega)$ and obtained a uniform bound on the dimension of sections of its pullback global attractor under the assumption (6). Here, we state a useful corollary of it, to which our results rely upon.

Corollary 1. *[7] Assume that the process $\{U(t,s): t \geq s\}$ on a Banach space V satisfies $(\mathscr{A}_1)$-$(\mathscr{A}_3)$, $(\mathscr{H}_3)$ and admits the above decomposition with $(\mathscr{H}_1)$ and let $\nu \in (0, \frac{1}{2}e^{-\gamma_0\tilde{t}} - \lambda)$. Assume further that*

(H_2) *there exists $N = N_\nu \in \mathbb{N}$ such that for any $t \leq t_0$, any $R > 0$ and any $u \in B(t-\tilde{t})$,there exists $v_1, \ldots, v_n \in V$ such that*

$$S(t,t-\tilde{t})(B(t-\tilde{t}) \cap B_R^V(u)) \subset \bigcup_{i=1}^{N} B_{\nu R}^V(v_i). \tag{7}$$

Then there exists a pullback exponential attractor $\{\mathscr{M}(t) = \mathscr{M}^\nu(t) : t \in \mathbb{R}\}$ where the bound for the fractal dimension is as follows:

$$\sup_{t\in\mathbb{R}} \dim_f^V(\mathscr{M}(t)) \leq \frac{-\ln N_\nu}{\ln 2(\nu+\lambda)+\gamma_0\tilde{t}} \quad , \tag{8}$$

For the existence of global solutions $u(t)$ for the problem on both $L^2(\Omega)$ and $H_0^1(\Omega)$ refer to [7], Theorem 4.1. Define $U(t,s)u_s := u(t)$, then we have the evolution process $\{U(t,s): t \geq s\}$ in $L^2(\Omega)$ and let us denote by $\|\cdot\|$ the $L^2(\Omega)$ norm.

3 Main Results

A priori estimate method is widely used to capture the asymptotic behavior of PDE systems. In the following lemma, using some standard estimates and a new trick, we obtain a useful estimate of the solutions suitable for our purpose.

Lemma 2. *Consider the initial boundary value problem (1) with the assumptions (2)-(5) and let $\{U(t,s): t \geq s\}$ be the process of it in $L^2(\Omega)$. let $u_s, v_s \in L^2(\Omega)$ are two initial values and $u(t), v(t)$ are the solutions corresponding to them. we have:*

$$\|u(t)-v(t)\| \leq e^{-(\lambda_1-c_5)(t-s)}\|u_s - v_s\|. \tag{9}$$

Proof. We write (1) for u and v and consider their difference. Let $\omega = u - v$ then we have

$$\partial_t\omega - \Delta\omega + (f(t,u) - f(t,v)) = 0,$$

We multiply it by ω and integrate over Ω. Using the boundary condition and the Green formula we have

$$\frac{1}{2}\partial_t\|\omega(t)\|^2 + \|\nabla\omega\|^2 + \int_\Omega (f(t,u)-f(t,v))\omega\, dx = 0,$$

Let $\Omega_1 = \{x \in \Omega : u(t,x) \neq v(t,x)\}$, since $f \in C^1$ as a function of u, by Mean value theorem $\forall x \in \Omega_1 \quad \exists z := z(t,x) \in \mathbb{R} : \frac{f(t,u)-f(t,v)}{u-v} = f_u(z,t)$. Note that z is between u and v as values in $\mathbb{R}$ and Ω_1 is a Lebesgue measurable set. Note also that $f_u(t,z)$ as a function of x is not necessarily even measurable but $f_u(t,z)|\omega|^2 = (f(t,u) - f(t,v))\omega$ is an integrable function on Ω_1. Hence, we have

$$\frac{1}{2}\partial_t\|\omega(t)\|^2 + \|\nabla\omega\|^2 + \int_{\Omega_1} f_u(t,z)|\omega|^2\, dx = 0,$$

Then by (3)

$$-c_5\int_{\Omega}|\omega|^2 \le -c_5\int_{\Omega_1}|\omega|^2 \le \int_{\Omega_1} f_u(t,z)|\omega|^2$$

and then

$$\frac{1}{2}\partial_t\|\omega(t)\|^2+\|\nabla\omega\|^2-c_5\|\omega\|^2\le 0,$$

Due to the Poincaré inequality

$$\frac{1}{2}\partial_t\|\omega(t)\|^2+(\lambda_1-c_5)\|\omega(t)\|^2\le 0 \quad t\ge s,$$

Thus, using the Gronwall lemma, the relation (9) follows.

Let $V_n=\text{span}\{e_1,\dots,e_n\}$ be the linear space spanned by the first n eigenfunctions of $A=-\Delta$ in $L^2(\Omega)$ and let $P_n: L^2(\Omega)\to V_n$ denotes the orthogonal projection and Q_n its complementary projection. For $u\in L^2(\Omega)$ we write $u=P_n(u)+Q_n(u)=u_1+u_2$. We consider the difference $\omega=u-v$ of two solutions of (1) with $u_s,v_s\in L^2(\Omega)$.Taking the inner product in $L^2(\Omega)$ with $Q_n(\omega)=\omega_2$, we obtain

$$\frac{1}{2}\partial_t\|\omega_2\|^2+\|\nabla\omega_2\|^2+\int_{\Omega}(f(t,u)-f(t,v))\omega_2\,dx=0 \tag{10}$$

As in [7] (section 4), we can obtain the absorbing sets $B(t)$ for the problem so that $diam(B(t))\le Le^{-\frac{\alpha}{2}t} \quad t\le t_0\le 0$.

As refers to the uniform bound on the dimension of sections of pullback global attractor, in [2, 3], the Lipschitz condition on f is as a sufficient condition.

Here, we make our new assumption which replaces the Lipschitz condition on f with an assumption on f_u.

As before in the Lemma 2, set $\Omega_1=\{x\in\Omega: u(t,x)\ne v(t,x)\}$ and $z=z(t,x)$. we have

$$\int_{\Omega}(f(t,u)-f(t,v))\omega_2\,dx=\int_{\Omega_1}\frac{f(t,u)-f(t,v)}{u-v}\omega\omega_2\,dx=\int_{\Omega_1}f_u(t,z)\omega\omega_2\,dx \tag{11}$$

Let $\tilde t>0$. $B(t-\tilde t)$ is absorbing, Hence, for the solutions starting at $t-\tilde t$ in $B(t-\tilde t)$, according to [11] (Theorem 11.6 and lines after its proof), we can get $\tilde t$ so large that $u(t),v(t)\in L^\infty(\Omega)$ and thus $|u(t)|\le Le^{-\frac{\alpha}{2}t}$ and $|v(t)|\le Le^{-\frac{\alpha}{2}t}$ almost everywhere($\tilde t$ needs to be further adjusted in the following lines). Since z is between u and v, then, $|z(t)|\le Le^{-\frac{\alpha}{2}t}$ a.e. Since f_u is continuous w.r.t. u, so let

$$g(t)=\sup_u\{f_u(t,u):|u|\le Le^{-\frac{\alpha}{2}t}\} \qquad t\le t_0,$$

Now, suppose that:

$$g(t) \quad \text{is a bounded function} \quad (t\le t_0) \tag{12}$$

This is our new assumption which takes the place of the Lipschitz condition in dealing with (10). According to (12), there exists $\eta_0\ge 0$ such that $|f_u(t,z)|\le\eta_0$ a.e. for $t\le t_0$.

Let $\Omega_1^+=\{x\in\Omega_1:\omega\omega_2\ge 0\}$ and $\Omega_1^-=\{x\in\Omega_1:\omega\omega_2<0\}$.

In the sequel, we try to separate the third term in (10), that is, (11) into positive and negative parts and drop the positive one to remain the expression negative. We have

$$\frac{1}{2}\partial_t\|\omega_2\|^2+\|\nabla\omega_2\|^2-c_5\int_{\Omega_1^+}\omega\omega_2\,dx+\eta_0\int_{\Omega_1^-}\omega\omega_2\,dx\le 0 \qquad t\le t_0$$

We have $\int\omega\omega_2\,dx=\int|\omega_2|^2\,dx+\int\omega_1\omega_2\,dx$, Hence,

$$\frac{1}{2}\partial_t\|\omega_2\|^2+\|\nabla\omega_2\|^2-c_5\|\omega_2\|^2-c_5\int_{\Omega_1^+}\omega_1\omega_2\,dx+\eta_0\int_{\Omega_1^-}\omega_1\omega_2\,dx\le 0 \quad t\le t_0$$

We have $\int_{\Omega_1^+} \omega_1 \omega_2 \, dx + \int_{\Omega_1^-} \omega_1 \omega_2 \, dx = \int_{\Omega_1} \omega_1 \omega_2 \, dx = -\int_{\Omega - \Omega_1} \omega_1^2 \leq 0$,
As a consequence,

$$\frac{1}{2}\partial_t \|\omega_2\|^2 + \|\nabla \omega_2\|^2 - c_5\|\omega_2\|^2 + (\eta_0 + c_5)\int_{\Omega_1^-} \omega_1 \omega_2 \, dx \leq 0 \qquad t \leq t_0$$

Using Cauchy-Schwartz inequality, (9) and $\|\omega_1\|, \|\omega_2\| \leq \|\omega\|$, we have

$$\frac{1}{2}\partial_t \|\omega_2\|^2 + \|\nabla \omega_2\|^2 - c_5\|\omega_2\|^2 \leq (\eta_0 + c_5)e^{-(\lambda_1 - c_5)(t-s)}\|\omega(s)\|^2 \qquad s < t \leq t_0$$

By Poincaré inequality and Gronwall lemma and for n large enough, we have:

$$\|\omega_2(t)\|^2 \leq (e^{-2(\lambda_{n+1} - c_5)(t-s)} + (\eta_0 + c_5)e^{-(\lambda_1 - c_5)(t-s)}(\lambda_{n+1} - c_5)^{-1})\|\omega(s)\|^2$$

Setting $s = t - \tilde{t}$, we obtain for $u, v \in B(t - \tilde{t})$ and $t \leq t_0$

$$\begin{aligned} &\|Q_n U(t, t-\tilde{t})u - Q_n U(t, t-\tilde{t})v\| \ \leq \\ &(e^{-2(\lambda_{n+1} - c_5)\tilde{t}} + (\eta_0 + c_5)e^{-(\lambda_1 - c_5)\tilde{t}}(\lambda_{n+1} - c_5)^{-1})^{\frac{1}{2}}\|u - v\|, \quad t \leq t_0, u, v \in B(t - \tilde{t}) \end{aligned}$$

We can choose $n \in \mathbb{N}$ so large that

$$\lambda := (e^{-2(\lambda_{n+1} - c_5)\tilde{t}} + (\eta_0 + c_5)e^{-(\lambda_1 - c_5)\tilde{t}}(\lambda_{n+1} - c_5)^{-1})^{\frac{1}{2}} < \frac{1}{2}e^{-\frac{\alpha}{2}\tilde{t}}.$$

Then $(\mathscr{H}_1)$ is satisfied with $C(t, t-\tilde{t}) = Q_n U(t, t-\tilde{t})$.
Hence, in fulfilling the requirements of Corollary 1 to obtain a pullback exponential attractor, for meeting $(\mathscr{H}_1)$, we can replace the assumption (6) with the assumption (12).

Now, we state the main result in $L^2(\Omega)$:

Theorem 3. *Let $\{U(t,s) : t \geq s\}$ be the process specified in the Lemma 2 and $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ in $L^2(\Omega)$ be the pullback global attractor related to it with a uniform bound on the dimension of its sections. if*

$$\lambda_1 > c_5 + \frac{\alpha}{2}, \tag{13}$$

then, for any $t \in \mathbb{R}$ the section $\mathscr{A}(t)$ has a zero fractal dimension.

Proof. In [7], by checking the requirements of Corollary 1,it was established a pullback global attractor with a uniform bound on its sections. We again check (H_2) using Lemma 2. we proceed as follows: From (9) we have:

$$\|\omega_1(t)\| \leq e^{-(\lambda_1 - c_5)(t-s)}\|\omega(s)\|$$

If $\lambda_1 - c_5 > \frac{\alpha}{2}$, taking s so negative that, fixing $\tilde{t} = t - s$, we have
$\nu = e^{-(\lambda_1 - c_5)\tilde{t}} < \frac{1}{2}e^{-\frac{\alpha}{2}\tilde{t}} - \lambda$. Now, for this ν, any $R > 0$ and $u \in B(t - \tilde{t})$

$$P_n U(t, t-\tilde{t})(B(t-\tilde{t}) \cap B_R^{L^2(\Omega)}(u)) \subset B_{\nu R}^{V_n}(P_n U(t, t-\tilde{t})u)$$

Thus, (H_2) is satisfied with $N_\nu = 1$. Moreover, by (8), the fractal dimension of its sections is 0. Hence, the pullback global attractor has zero dimensional sections.

We know that attractors are connected sets and zero dimensional connected sets are singletons. Then, we have the following corollary.

Corollary 4. *The pullback global attractor under the assumption (13) consists only of one global solution.*

The assumption (13), is interpreted as a restriction on the bounded domain Ω, since λ_1 is only depended on the geometry of the bounded domain. We know from the classic work [10] that if Ω is a bounded, convex, Lipschitz domain with diameter d, then the Poincaré constant is at most $\frac{d}{\pi}$ for $p = 2$ (in $W_0^{1,p}(\Omega)$). It is also well-known that for a smooth, bounded domain Ω, since the Rayleigh quotient for the Laplace operator in $H_0^1(\Omega)$ is minimized by the eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta$, we have $\|u\| \leq \lambda_1^{-1}\|\nabla u\|$. Hence, for our case, If Ω is a bounded, smooth and convex domain, we have $\lambda_1^{-1} \leq \frac{d}{\pi}$ or $\lambda_1 \geq \frac{\pi}{d}$.

Now, (13) says $\lambda_1 > c_5 + \frac{\alpha}{2}$, Hence if $c_5 + \frac{\alpha}{2} < \frac{\pi}{d}$, the assumption (13) automatically holds and then the zero dimensionality of $\mathscr{M}(t)$ holds true for the convex domains with $d < \pi(c_5 + \frac{\alpha}{2})^{-1}$ and for a convex Ω with larger diameters, we must check (13) to be held for assuring zero dimensionality of sections of pullback global attractor of reaction-diffusion equation defined on it.

Corollary 5. *Consider the problem* (1) *on a bounded, smooth and convex domain* Ω *with the assumptions* (2)-(5) *and* (12) *and let* $\{\mathscr{A}(t) : t \in \mathbb{R}\}$ *be the pullback global attractor in* $L^2(\Omega)$ *associated to it. If diam*$(\Omega) < \pi(c_5 + \frac{\alpha}{2})^{-1}$*, then* $\mathscr{A}(t) = \{u(t)\}$ *where u is a global solution of the problem.*

In what follows, As an application to Theorem 3, we examine the asymptotic behavior of a new version of non-autonomous Chafee-Infante equation respecting the new assumption (12). Some other versions of this equation in stronger spaces have arisen in [5], [3], [2] and [7]. In [3], the unperturbed version of the following equation has been studied and the trivial asymptotic dynamics has been shown when $\lambda < 1$. We show that under a bounded perturbation, this trivial dynamics remains valid.

Example 6. *Consider the non-autonomous Chafee-Infante equation on the domain* $(0,\pi)$ *as follows:*

$$u_t - u_{xx} = \lambda u - \beta(t)u^3 + h(t), \quad u(t,0) = u(t,\pi) = 0,$$

with the initial condition $u(s,x) = u_0(x)$, $x \in (0,\pi)$ *and* $0 < b_0 \leq \beta(t) \leq B_0$ *is* C^1 *[3] and* $\|h(t)\| \leq M$*. Obviously, this equation meets the conditions* (2)-(5) *with* $f(t,u) = \beta(t)u^3 - \lambda u$, $\alpha = 0$ *and* $c_5 = \lambda$*. For checking* (12)*, We have* $f_u(t,u) = 3\beta(t)u^2 - \lambda$ *and so obviously,* $g(t)$ *is bounded in the past. Hence, Corollary 5 shows that its pullback global attractor in* $L^2(0,\pi)$ *consists only of one global solution if* $\lambda < 1$*. Note that* f *does not meet the Lipschitz condition* (6) *and as a result, we could not prove the uniform bound on the dimensions of the sections of its pullback global attractor by the results in [1] or [7].*
Also, [3] has obtained trivial attractor $\{0\}$ *for the unperturbed equation when* $\lambda < 1$*, while here the perturbed version has a global solution* $\{u(t)\}$ *as its pullback global attractor when* $\lambda < 1$ *which generalizes that result.*
Moreover, this result is consistent with the result about this problem in [12], Section 4, that is, if $h(t) = \varepsilon h_1(t)r(x)$ *where* $h_1 : \mathbb{R} \to \mathbb{R}$ *is an almost periodic function and* $r(x) \in L^2(0,\pi)$ *then* $\lambda \in (0,1)$ *implies* $n = 0$*, and all solutions converge to its unique almost periodic solution* $u(t)$*, so we have the trivial pullback global attractor as* $\{u(t)\}$ *which its sections are one-point sets [[12], Theorem 4.1]. In fact, our result shows that not only for almost periodic perturbations but also for all bounded perturbations, the pullback global attractor only contains a global solution which, in some sense, improves the result of [12] for* $\lambda \in (0,1)$.

Remark 1. When the phase space is $H_0^1(\Omega)$, according to [7], Theorem 4.1, the assumption (6) needs for the existence of solutions and hence, for defining $U(t,s)$ on $H_0^1(\Omega)$. Thus, here we could not replace it with the new assumption (12).

Remark 2. In the case of $H_0^1(\Omega)$, the results of theorem 3 and its corollaries are valid with some modifications for the expressions (9) and (13) and a new bound on the diameter of the domain.

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