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Observability Inequalities for Parabolic Equations over Measurable Sets and Some Applications Related to the Bang-Bang Property for Control Problems

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Abstract

This article presents two observability inequalities for the heat equation over $\Omega \times (0,T)$. In the first one, the observation is from a subset of positive measure in $\Omega \times (0,T)$, while in the second, the observation is from a subset of positive surface measure on $\partial\Omega \times (0,T)$. We will provide some applications for the above-mentioned observability inequalities, the bang-bang property for the minimal time, time optimal and minimal norm control problems, and also establish new open problems related to observability inequalities and the aforementioned applications.

Keywords: Parabolic equations, control theory, controllability, observability inequalities, Bang-Bang properties.
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1 Introduction

This article serves as a review of observability inequalities from measurable sets for solutions to the heat equation. The purpose of trying to obtain the two observability inequalities that we will see and prove in this article, was that in control theory there is a very well known result, the Hilbert Uniqueness Method, that assures that the null controllability of an equation is equivalent to obtain an observability inequality for the adjoint equation. This result is attributed to J.L. Lion. In our previous research we were studying the null controllability of parabolic equations over measurable sets, so, for the Hilbert Uniqueness Method reason, we focused on proving the observability inequalities (Theorems 1 and 2) that we will see in this article.

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In the next lines of the Introduction we will establish the type of problem we will work on, remember some apriori estimates for the parabolic equations and recall some previous results about this kind of work.

Then, in Section 2, we will establish and prove Theorem 1 and 2 which will give us two observability inequalities. We will continue, in Section 3, showing some applications of the obserbavility inequalities, the bang-bang property for the minimal time, optimal time and minimal norm control problems. In Section 4, we will establish some open problems related to observability inequalities and their applications to control theory. Finally, with Section 5, we will finish the article giving some details of a definition and a proof requiered in Section 3.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and T be a fixed positive time. Consider the heat equation:

$$\begin{cases} \partial_t u - \Delta u = 0, \quad \text{in } \Omega \times (0,1), \\ u = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad \text{in } \Omega, \end{cases} \tag{1}$$

with u_0 in $L^2(\Omega)$. The solution of (1) will be treated as either a function from $[0,T]$ to $L^2(\Omega)$ or a function of two variables x and t. Two important apriori estimates for the above equation are as follows:

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathscr{D}) \int_{\mathscr{D}} |u(x,t)|\, dxdt, \tag{2}$$

for all $u_0 \in L^2(\Omega)$, where $\mathscr{D}$ is a subset of $\Omega \times (0,T)$, and

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathscr{J}) \int_{\mathscr{J}} |\frac{\partial}{\partial \nu} u(x,t)|\, d\sigma dt, \tag{3}$$

for all $u_0 \in L^2(\Omega)$, where $\mathscr{J}$ is a subset of $\partial\Omega \times (0,T)$. Such apriori estimates are called observability inequalities.

In the case that $\mathscr{D} = \omega \times (0,T)$ and $\mathscr{J} = \Gamma \times (0,T)$ with ω and Γ accordingly open and nonempty subsets of Ω and $\partial\Omega$, both inequalities (2) and (3) (where $\partial\Omega$ is smooth) were essentially first established, via the Lebeau-Robbiano spectral inequalities in [8]. These two estimates were set up to the linear parabolic equations (where $\partial\Omega$ is of class C^2), based on the Carleman inequality provided in [7]. In the case when $\mathscr{D} = \omega \times (0,T)$ and $\mathscr{J} = \Gamma \times (0,T)$ with ω and Γ accordingly subsets of positive measure and positive surface measure in Ω and $\partial\Omega$, both inequalities (2) and (3) were built up in [1] with the help of a propagation of smallness estimate from measurable sets for real-analytic functions first established in [13]. For $\mathscr{D} = \omega \times E$, with ω and E accordingly an open subset of Ω and a subset of positive measure in $(0,T)$, the inequality (2) (when $\partial\Omega$ is smooth) was proved in [14] with the aid of the Lebeau-Robbiano spectral inequality, and it was then verified for heat equations (when Ω is convex) with lower terms depending on the time variable, through a frequency function method in [11]. When $\mathscr{D} = \omega \times E$, with ω and E accordingly subsets of positive measure in Ω and $(0,T)$, the estimate (2) (when $\partial\Omega$ is real-analytic) was obtained in [15].

In [2], we stablished the inequalities (2) and (3) when $\mathscr{D}$ and $\mathscr{J}$ were arbitrary subsets of positive measure and of positive surface measure in $\Omega \times (0,T)$ and $\partial\Omega \times (0,T)$ respectively. Such inequalities not only are mathematically interesting but also have important applications in the control theory of the heat equation, such as the bang-bang control, the time optimal control, the null controllability over a measurable set and so on.

We will see how we proved the two above-mentioned inequalities. We start assuming that the Lebeau-Robbiano spectral inequality stands on Ω. To introduce it, we write

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$$

for the eigenvalues of $-\Delta$ with the zero Dirichlet boundary condition over $\partial\Omega$, and $\{e_j : j \geq 1\}$ for the set of $L^2(\Omega)$-normalized eigenfunctions, i.e.,

$$\begin{cases} \Delta e_j + \lambda_j e_j = 0, \quad \text{in } \Omega, \\ e_j = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{4}$$

For $\lambda > 0$ we define

$$\mathscr{E}_\lambda f = \sum_{\lambda_j \leq \lambda} (f, e_j) e_j \quad \text{and} \quad \mathscr{E}_\lambda^\perp f = \sum_{\lambda_j > \lambda} (f, e_j) e_j,$$

where

$$(f, e_j) = \int_\Omega f e_j \, dx, \text{ when } f \in L^2(\Omega), \; j \geq 1.$$

Throughout this paper the following notations are used:

$$(f, g) = \int_\Omega f g \, dx \quad \text{and } \|f\|_{L^2(\Omega)} = (f, f)^{\frac{1}{2}}.$$

ν is the unit exterior normal vector to Ω. $d\sigma$ is surface measure on $\partial\Omega$. $B_R(x_0)$ stands for the ball centered at x_0 in $\mathbb{R}^n$ of radius R, $\triangle_R(x_0)$ denotes $B_R(x_0) \cap \partial\Omega$, $B_R = B_R(0)$ and $\triangle_R = \triangle_R(0)$. For measurable sets $\omega \subset \mathbb{R}^n$ and $\mathscr{D} \subset \mathbb{R}^n \times (0,T)$, $|\omega|$ and $|\mathscr{D}|$ stand for the Lebesgue measures of the sets. For each measurable set $\mathscr{J}$ in $\partial\Omega \times (0,T)$, $|\mathscr{J}|$ denotes its surface measure on the lateral boundary of $\Omega \times \mathbb{R}$. $\{e^{t\Delta} : t \geq 0\}$ is the semigroup generated by Δ with zero Dirichlet boundary condition over $\partial\Omega$. Consequently, $e^{t\Delta} f$ is the solution of equation (1) with the initial state f in $L^2(\Omega)$. The Lebeau-Robbiano spectral inequality is as follows:

For each $0 < R \leq 1$, there is $N = N(\Omega, R)$, such that the inequality

$$\|\mathscr{E}_\lambda f\|_{L^2(\Omega)} \leq N e^{N\sqrt{\lambda}} \|\mathscr{E}_\lambda f\|_{L^2(B_R(x_0))} \tag{5}$$

holds, when $B_{4R}(x_0) \subset \Omega$, $f \in L^2(\Omega)$ and $\lambda > 0$.

2 Observability inequalities

Our main results related to the observability inequalities are stated as follows, but, first, we will define the real-analyticity of the set $\triangle_{4R}(q_0)$.

Definition 1. Let $q_0 \in \partial\Omega$ and $0 < R \leq 1$. We say that $\triangle_{4R}(q_0)$ is real-analytic with constants ρ and δ if for each $q \in \triangle_{4R}(q_0)$, there are a new rectangular coordinate system where $q = 0$, and a real-analytic function $\phi : B'_\rho \subset \mathbb{R}^{n-1} \to \mathbb{R}$ verifying

$$\begin{cases} \phi(0') = 0, \;\; |\partial^\alpha \phi(x')| \leq |\alpha|! \delta^{-|\alpha|-1}, \\ \text{when } \; x' \in B'_\rho, \; \alpha \in \mathbb{N}^{n-1}, \\ B_\rho \cap \Omega = B_\rho \cap \{(x', x_n) : x' \in B'_\rho, \;\; x_n > \phi(x')\}, \\ B_\rho \cap \partial\Omega = B_\rho \cap \{(x', x_n) : x' \in B'_\rho, \;\; x_n = \phi(x')\}. \end{cases} \tag{6}$$

Here, B'_ρ denotes the open ball of radius ρ and with center at $0'$ in $\mathbb{R}^{n-1}$.

In the next two theorems, we establish two observability inequalities for the heat equation over $\Omega \times (0,T)$. In Theorem 1, the observation is from a subset of positive measure in $\Omega \times (0,T)$, while in Theorem 2, the observation is from a subset of positive surface measure on $\partial\Omega \times (0,T)$.

Theorem 1. *Suppose that a bounded domain Ω verifies the condition (5) and $T>0$. Let $x_0 \in \Omega$ and $R \in (0,1]$ be such that $B_{4R}(x_0) \subset \Omega$. Then, for each measurable set $\mathscr{D} \subset B_R(x_0) \times (0,T)$ with $|\mathscr{D}|>0$, there is a positive constant $B = B(\Omega,T,R,\mathscr{D})$, such that*

$$\|e^{T\Delta}f\|_{L^2(\Omega)} \le e^B \int_{\mathscr{D}} |e^{t\Delta}f(x)|\,dxdt, \tag{7}$$

when $f \in L^2(\Omega)$.

Theorem 2. *Suppose that a bounded Lipschitz domain Ω verifies the condition (5) and $T>0$. Let $q_0 \in \partial\Omega$ and $R \in (0,1]$ be such that $\triangle_{4R}(q_0)$ is real-analytic. Then, for each measurable set $\mathscr{J} \subset \triangle_R(q_0) \times (0,T)$ with $|\mathscr{J}|>0$, there is a positive constant $B = B(\Omega,T,R,\mathscr{J})$, such that*

$$\|e^{T\Delta}f\|_{L^2(\Omega)} \le e^B \int_{\mathscr{J}} |\frac{\partial}{\partial\nu} e^{t\Delta}f(x)|\,d\sigma dt, \tag{8}$$

when $f \in L^2(\Omega)$.

Next, we will see some results that will be necessary in the proof of the previous Theorem 1.

Lemma 3. *Let $B_R(x_0) \subset \Omega$ and $\mathscr{D} \subset B_R(x_0) \times (0,T)$ be a subset of positive measure. Set*

$$\mathscr{D}_t = \{x \in \Omega : (x,t) \in \mathscr{D}\},\ E = \{t \in (0,T) : |\mathscr{D}_t| \ge |\mathscr{D}|/(2T)\},\ t \in (0,T). \tag{9}$$

Then, $\mathscr{D}_t \subset \Omega$ is measurable for a.e. $t \in (0,T)$, E is measurable in $(0,T)$, $|E| \ge |\mathscr{D}|/2|B_R|$ and

$$\chi_E(t)\chi_{\mathscr{D}_t}(x) \le \chi_{\mathscr{D}}(x,t),\ \text{in } \Omega \times (0,T). \tag{10}$$

Proof. From Fubini's theorem,

$$|\mathscr{D}| = \int_0^T |\mathscr{D}_t|\,dt = \int_E |\mathscr{D}_t|\,dt + \int_{[0,T]\setminus E} |\mathscr{D}_t|\,dt \le |B_R||E| + |\mathscr{D}|/2.$$

□

Theorem 4. *Let $x_0 \in \Omega$ and $R \in (0,1]$ be such that $B_{4R}(x_0) \subset \Omega$. Let $\mathscr{D} \subset B_R(x_0) \times (0,T)$ be a measurable set with $|\mathscr{D}|>0$. Write E and $\mathscr{D}_t$ for the sets associated to $\mathscr{D}$ in Lemma 3. Then, for each $\eta \in (0,1)$, there are $N = N(\Omega,R,|\mathscr{D}|/(T|B_R|),\eta)$ and $\theta = \theta(\Omega,R,|\mathscr{D}|/(T|B_R|),\eta)$ with $\theta \in (0,1)$, such that*

$$\|e^{t_2\Delta}f\|_{L^2(\Omega)} \le \left(Ne^{N/(t_2-t_1)} \int_{t_1}^{t_2} \chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathscr{D}_s)}\,ds\right)^{\theta} \|e^{t_1\Delta}f\|_{L^2(\Omega)}^{1-\theta}, \tag{11}$$

when $0 \le t_1 < t_2 \le T$, $|E \cap (t_1,t_2)| \ge \eta(t_2-t_1)$ and $f \in L^2(\Omega)$. Moreover,

$$\begin{aligned} &e^{-\frac{N+1-\theta}{t_2-t_1}}\|e^{t_2\Delta}f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{q(t_2-t_1)}}\|e^{t_1\Delta}f\|_{L^2(\Omega)} \\ &\le N\int_{t_1}^{t_2} \chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathscr{D}_s)}\,ds,\ \text{when } q \ge (N+1-\theta)/(N+1). \end{aligned} \tag{12}$$

The reader can find the proof of the following Lemma 2 in either [10, pp. 256-257] or [11, Proposition 2.1].

Lemma 5. *Let E be a subset of positive measure in $(0,T)$. Let l be a density point of E. Then, for each $z>1$, there is $l_1 = l_1(z,E)$ in (l,T) such that, the sequence $\{l_m\}$ defined as*

$$l_{m+1} = l + z^{-m}(l_1 - l),\ m = 1,2,\cdots,$$

verifies

$$|E \cap (l_{m+1},l_m)| \ge \frac{1}{3}(l_m - l_{m+1}),\ \text{when } m \ge 1. \tag{13}$$

Proof. [Theorem 1] Let E and $\mathscr{D}_t$ be the sets associated to $\mathscr{D}$ in Lemma 3 and l be a density point in E. For $z>1$ to be fixed later, $\{l_m\}$ denotes the sequence associated to l and z in Lemma 5. Because (13) holds, we may apply Theorem 4, with $\eta=1/3$, $t_1=l_{m+1}$ and $t_2=l_m$, for each $m\geq 1$, to get that there are $N=N(\Omega,R,|\mathscr{D}|/(T|B_R|))>0$ and $\theta=\theta(\Omega,R,|\mathscr{D}|/(T|B_R|))$, with $\theta\in(0,1)$, such that

$$\begin{aligned}
&e^{-\frac{N+1-\theta}{l_m-l_{m+1}}}\|e^{l_m\Delta}f\|_{L^2(\Omega)}-e^{-\frac{N+1-\theta}{q(l_m-l_{m+1})}}\|e^{l_{m+1}\Delta}f\|_{L^2(\Omega)}\\
&\leq N\int_{l_{m+1}}^{l_m}\chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathscr{D}_s)}\,ds,\ \text{ when }\ q\geq\frac{N+1-\theta}{N+1}\ \text{ and }\ m\geq 1.
\end{aligned}\tag{14}$$

Setting $z=1/q$ in (14) (which leads to $1<z\leq\frac{N+1}{N+1-\theta}$) and

$$\gamma_z(t)=e^{-\frac{N+1-\theta}{(z-1)(l_1-l)t}},\ t>0,$$

recalling that

$$l_m-l_{m+1}=z^{-m}(z-1)(l_1-l),\ \text{for } m\geq 1,$$

we have

$$\begin{aligned}
&\gamma_z(z^{-m})\|e^{l_m\Delta}f\|_{L^2(\Omega)}-\gamma_z(z^{-m-1})\|e^{l_{m+1}\Delta}f\|_{L^2(\Omega)}\\
&\leq N\int_{l_{m+1}}^{l_m}\chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathscr{D}_s)}\,ds,\ \text{ when }\ m\geq 1.
\end{aligned}\tag{15}$$

Choose now

$$z=\frac{1}{2}\left(1+\frac{N+1}{N+1-\theta}\right).$$

The choice of z and Lemma 5 determines l_1 in (l,T) and from (15),

$$\begin{aligned}
&\gamma(z^{-m})\|e^{l_m\Delta}f\|_{L^2(\Omega)}-\gamma(z^{-m-1})\|e^{l_{m+1}\Delta}f\|_{L^2(\Omega)}\\
&\leq N\int_{l_{m+1}}^{l_m}\chi_E(s)\|e^{s\Delta}f\|_{L^1(\mathscr{D}_s)}\,ds,\ \text{ when }\ m\geq 1.
\end{aligned}\tag{16}$$

with

$$\gamma(t)=e^{-A/t}\ \text{and}\ A=A(\Omega,R,E,|\mathscr{D}|/(T|B_R|))=\frac{2(N+1-\theta)^2}{\theta(l_1-l)}.$$

Finally, because of

$$\|e^{T\Delta}f\|_{L^2(\Omega)}\leq\|e^{l_1\Delta}f\|_{L^2(\Omega)},\ \sup_{t\geq 0}\|e^{t\Delta}f\|_{L^2(\Omega)}<+\infty,\ \lim_{t\to 0+}\gamma(t)=0,$$

and (10), the addition of the telescoping series in (16) gives

$$\|e^{T\Delta}f\|_{L^2(\Omega)}\leq Ne^{zA}\int_{\mathscr{D}\cap(\Omega\times[l,l_1])}|e^{t\Delta}f(x)|\,dxdt,\ \text{ for }\ f\in L^2(\Omega),$$

which proves (7) with $B=zA+\log N$. □

Remark 1. *The constant B in Theorem 1 depends on E because the choice of $l_1=l_1(z,E)$ in Lemma 5 depends on the possible complex structure of the measurable set E (See the proof of Lemma 5 in [11, Proposition 2.1]). When $\mathscr{D}=\omega\times(0,T)$, one may take $l=T/2$, $l_1=T$, $z=2$ and then,*

$$B=A(\Omega,R,|\omega|/|B_R|)/T.$$

Remark 2. *The proof of Theorem 1 also implies the following observability estimate:*

$$\sup_{m\geq 0}\sup_{l_{m+1}\leq t\leq l_m} e^{-z^{m+1}A}\|e^{t\Delta}f\|_{L^2(\Omega)} \leq N\int_{\mathscr{D}\cap(\Omega\times[l,l_1])}|e^{t\Delta}f(x)|\,dxdt,$$

for f in $L^2(\Omega)$, and with z, N and A as defined along the proof of Theorem 1. Here, $l_0 = T$.

Next, we will see some results that will be necessary in the proof of the previous Theorem 2.

Lemma 6. *Let $q_0 \in \partial\Omega$ and $\mathscr{J} \subset \triangle_R(q_0)\times(0,T)$ be a subset with $|\mathscr{J}| > 0$. Set*

$$\mathscr{J}_t = \{x\in\partial\Omega : (x,t)\in\mathscr{J}\},\; E = \{t\in(0,T) : |\mathscr{J}_t| \geq |\mathscr{J}|/(2T)\},\; t\in(0,T).$$

Then, $\mathscr{J}_t \subset \triangle_R(q_0)$ is measurable for a.e. $t\in(0,T)$, E is measurable in $(0,T)$, $|E| \geq |\mathscr{J}|/(2|\triangle_R(q_0)|)$ and $\chi_E(t)\chi_{\mathscr{J}_t}(x) \leq \chi_{\mathscr{J}}(x,t)$ over $\partial\Omega\times(0,T)$.

Proof. From Fubini's theorem,

$$|\mathscr{J}| = \int_0^T |\mathscr{J}_t|\,dt = \int_E |\mathscr{J}_t|\,dt + \int_{[0,T]\setminus E}|\mathscr{J}_t|\,dt \leq |\triangle_R(x_0)||E| + |\mathscr{J}|/2.$$

□

Theorem 7. *Suppose that Ω verifies the condition (5). Assume that $q_0\in\partial\Omega$ and $R\in(0,1]$ such that $\triangle_{4R}(q_0)$ is real-analytic. Let $\mathscr{J}$ be a subset in $\triangle_R(q_0)\times(0,T)$ of positive surface measure on $\partial\Omega\times(0,T)$, E and $\mathscr{J}_t$ be the measurable sets associated to $\mathscr{J}$ in Lemma 6. Then, for each $\eta\in(0,1)$, there are $N = N(\Omega,R,|\mathscr{J}|/(T|\triangle_R(q_0)|),\eta)$ and $\theta = \theta(\Omega,R,|\mathscr{J}|/(T|\triangle_R(q_0)|),\eta)$ with $\theta\in(0,1)$, such that the inequality*

$$\|e^{t_2\Delta}f\|_{L^2(\Omega)} \leq \left(Ne^{N/(t_2-t_1)}\int_{t_1}^{t_2}\chi_E(t)\|\tfrac{\partial}{\partial\nu}e^{t\Delta}f\|_{L^1(\mathscr{J}_t)}\,dt\right)^{\theta}\|e^{t_1\Delta}f\|^{1-\theta}_{L^2(\Omega)}, \tag{17}$$

holds, when $0\leq t_1 < t_2 \leq T$ with $t_2 - t_1 < 1$, $|E\cap(t_1,t_2)| \geq \eta(t_2-t_1)$ and $f\in L^2(\Omega)$. Moreover,

$$\begin{aligned} &e^{-\frac{N+1-\theta}{t_2-t_1}}\|e^{t_2\Delta}f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{q(t_2-t_1)}}\|e^{t_1\Delta}f\|_{L^2(\Omega)} \\ &\leq N\int_{t_1}^{t_2}\chi_E(t)\|\tfrac{\partial}{\partial\nu}e^{t\Delta}f\|_{L^1(\mathscr{J}_t)}\,dt,\;\; \text{when}\;\; q\geq \tfrac{N+1-\theta}{N+1}. \end{aligned} \tag{18}$$

Proof. [Theorem 2] Let E and $\mathscr{J}_t$ be the sets associated to $\mathscr{J}$ in Lemma 6 and l be a density point in E. For $z > 1$ to be fixed later, $\{l_m\}$ denotes the sequence associated to l and z in Lemma 5. Because of (13) and from Theorem 7 with $\eta = 1/3$, $t_1 = l_{m+1}$ and $t_2 = l_m$, with $m\geq 1$, there are $N = N(\Omega,R,|\mathscr{J}|/(T|\triangle_R(q_0)|)) > 0$ and $\theta = \theta(\Omega,R,|\mathscr{J}|/(T|\triangle_R(q_0)|))$, with $\theta\in(0,1)$, such that

$$\begin{aligned} &e^{-\frac{N+1-\theta}{l_m-l_{m+1}}}\|e^{l_m\Delta}f\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{q(l_m-l_{m+1})}}\|e^{l_{m+1}\Delta}f\|_{L^2(\Omega)} \\ &\leq N\int_{l_{m+1}}^{l_m}\chi_E(s)\|\tfrac{\partial}{\partial\nu}e^{s\Delta}f\|_{L^1(\mathscr{J}_s)}\,ds,\;\; \text{when}\;\; q\geq\frac{N+1-\theta}{N+1}\;\; \text{and}\;\; m\geq 1. \end{aligned}$$

Let

$$z = \frac{1}{2}\left(1+\frac{N+1}{N+1-\theta}\right).$$

Then, we can use the same arguments as those in the proof of Theorem 1 to verify Theorem 2. □

Remark 3. *The proof of Theorem 2 also implies the following observability estimate:*

$$\sup_{m\geq 0}\sup_{l_{m+1}\leq t\leq l_m} e^{-z^{m+1}A}\|e^{t\Delta}f\|_{L^2(\Omega)} \leq N\int_{\mathscr{J}\cap(\partial\Omega\times[l,l_1])}\left|\frac{\partial}{\partial\nu}e^{t\Delta}f(x)\right| d\sigma dt,$$

for f in $L^2(\Omega)$, with $A = 2(N+1-\theta)^2/[\theta(l_1-l)]$ and with z, N and θ as given along the proof of Theorem 2. Here, $l_0 = T$.

Remark 4. *When $\mathscr{J} = \Gamma\times(0,T)$, $\Gamma\subset\triangle_R(q_0)$ is a measurable set, one may take $l = T/2$, $l_1 = T$, $z = 2$ and the constant B in Theorem 2 becomes*

$$B = A(\Omega, R, |\Gamma|/|\triangle_R(q_0)|)/T.$$

3 Applications of observability inequalities

We will now show some applications of the Theorems 1 and 2 in the control theory of the heat equation. Specifically, we will focus on the uniqueness and bang-bang properties of the minimal time, time optimal and minimal L^∞-norm control problems.

In this section we assume that $T>0$ and that Ω is a bounded Lipschitz domain verifying the condition (5).

First of all, we will show that Theorems 1 and 2 imply the null controllability with controls restricted over measurable subsets in $\Omega\times(0,T)$ and $\partial\Omega\times(0,T)$ respectively. Let $\mathscr{D}$ be a measurable subset with positive measure in $B_R(x_0)\times(0,T)$ with $B_{4R}(x_0)\subset\Omega$. Let $\mathscr{J}$ be a measurable subset with positive surface measure in $\triangle_R(q_0)\times(0,T)$, where $q_0\in\partial\Omega$, $R\in(0,1]$ and $\triangle_{4R}(q_0)$ is real-analytic. Consider the following controlled heat equations:

$$\begin{cases}\partial_t u - \Delta u = \chi_{\mathscr{D}} v, \text{in } \Omega\times(0,T],\\ u = 0, \text{on } \partial\Omega\times[0,T],\\ u(0) = u_0, \text{ in } \Omega,\end{cases}\tag{19}$$

and

$$\begin{cases}\partial_t u - \Delta u = 0, \text{ in } \Omega\times(0,T],\\ u = g\chi_{\mathscr{J}}, \text{ on } \partial\Omega\times[0,T],\\ u(0) = u_0, \text{ in } \Omega,\end{cases}\tag{20}$$

where $u_0\in L^2(\Omega)$, $v\in L^\infty(\Omega\times(0,T))$ and $g\in L^\infty(\partial\Omega\times(0,T))$ are controls. *We say that u is the solution to 20 if $v\equiv u-e^{t\Delta}u_0$ is the unique solution defined in [6, Theorem 3.2] to*

$$\begin{cases}\partial_t v - \Delta v = 0, \ in\ \Omega\times(0,T),\\ v = g\chi_{\mathscr{J}}, \ on\ \partial\Omega\times(0,T),\\ v(0) = 0, \ in\ \Omega,\end{cases}\tag{21}$$

with g in $L^p(\partial\Omega\times(0,T))$ for some $2\leq p\leq\infty$.

From now on, we always denote by $u(\cdot\,;u_0,v)$ and $u(\cdot\,;u_0,g)$ the solutions of equations (19) and (20) corresponding to v and g respectively.

Corollary 8. *For each $u_0\in L^2(\Omega)$, there are bounded control functions v and g with*

$$\|v\|_{L^\infty(\Omega\times(0,T))}\leq C_1\|u_0\|_{L^2(\Omega)},$$

$$\|g\|_{L^\infty(\partial\Omega\times(0,T))} \leq C_2\|u_0\|_{L^2(\Omega)},$$

such that $u(T;u_0,v)=0$ *and* $u(T;u_0,g)=0$. *Here* $C_1=C(\Omega,T,R,\mathscr{D})$ *and* $C_2=C(\Omega,T,R,\mathscr{J})$.

Proof. We only prove the boundary controllability. Let E be the measurable set associated to $\mathscr{J}$ in Lemma 6. Write

$$\widetilde{\mathscr{J}}=\{(x,t):(x,T-t)\in\mathscr{J}\} \text{ and } \widetilde{E}=\{t:T-t\in E\}.$$

Let $l>0$ be a density point of $\widetilde{E}$ (Hence, $T-l$ is a density point of E). We choose z, l_1 and the sequence $\{l_m\}$ as in the proof of Theorem 2 but with $\mathscr{J}$ and E accordingly replaced by $\widetilde{\mathscr{J}}$ and $\widetilde{E}$. It is clear that

$$0<l<\cdots<l_{m+1}<l_m\cdots<l_1<l_0=T,\ \lim_{m\to+\infty} l_m=l.$$

We set

$$\mathscr{M}=\mathscr{J}\cap(\partial\Omega\times[T-l_1,T-l])\subset\mathscr{J}.$$

It is clear that $|\mathscr{M}|>0$. The proof of Theorem 2, the change of variables $t=T-\tau$ and Remark 3 show that the observability inequality

$$\|\varphi(0)\|_{L^2(\Omega)} \leq e^B\int_{\mathscr{M}}|\tfrac{\partial\varphi}{\partial\nu}(p,t)|\,d\sigma dt, \tag{22}$$

holds, when φ is the unique solution in $L^\infty([0,T],L^2(\Omega))\cap L^2([0,T],H_0^1(\Omega))$ to

$$\begin{cases}\partial_t\varphi+\Delta\varphi=0, & \text{in } \Omega\times[0,T),\\ \varphi=0, & \text{on } \partial\Omega\times[0,T),\\ \varphi(T)=\varphi_T, & \text{in } \partial\Omega,\end{cases} \tag{23}$$

for some φ_T in $L^2(\Omega)$. Set

$$X=\{\tfrac{\partial\varphi}{\partial\nu}|_{\mathscr{M}}:\varphi(t)=e^{(T-t)\Delta}\varphi_T,\ \text{for } 0\leq t\leq T,\ \text{for some } \varphi_T\in L^2(\Omega)\}.$$

Since $\mathscr{M}\subset\partial\Omega\times[T-l_1,T-l]$, X is a subspace of $L^1(\mathscr{M})$ and from (22), the linear mapping $\Lambda:X\longrightarrow\mathbb{R}$, defined by

$$\Lambda(\tfrac{\partial\varphi}{\partial\nu}|_{\mathscr{M}})=(u_0,\varphi(0)),$$

verifies

$$\left|\Lambda(\tfrac{\partial\varphi}{\partial\nu}|_{\mathscr{M}})\right|\leq e^B\|u_0\|_{L^2(\Omega)}\int_{\mathscr{M}}|\tfrac{\partial\varphi}{\partial\nu}(p,t)|\,d\sigma dt,\ \text{when } \tfrac{\partial\varphi}{\partial\nu}|_{\mathscr{M}}\in X.$$

From the Hahn-Banach theorem, there is a linear extension $T:L^1(\mathscr{M})\longrightarrow\mathbb{R}$ of Λ, with

$$\begin{aligned}&T(\tfrac{\partial\varphi}{\partial\nu}|_{\mathscr{M}})=(u_0,\varphi(0)),\ \text{when } \tfrac{\partial\varphi}{\partial\nu}|_{\mathscr{M}}\in X,\\ &|T(f)|\leq e^B\|u_0\|\|f\|_{L^1(\mathscr{M})},\ \text{for all } f\in L^1(\mathscr{M}).\end{aligned}$$

Thus, T is in $L^1(\mathscr{M})^*=L^\infty(\mathscr{M})$ and there is g in $L^\infty(\mathscr{M})$ verifying

$$T(f)=\int_{\mathscr{M}} fg\,d\sigma dt,\ \text{for all } f\in L^1(\mathscr{M}) \text{ and } \|g\|_{L^\infty(\mathscr{M})}\leq e^B\|u_0\|.$$

We extend g over $\partial\Omega\times(0,T)$ by setting it to be zero outside $\mathscr{M}$ and denote the extended function by g again. Then it holds that $u(T;u_0,g)=0$ provided that we know that

$$\int_\Omega u(T;u_0,g)\varphi_T\,dx=\int_\Omega u_0\varphi(0)\,dx-\int_{\mathscr{M}} g\tfrac{\partial\varphi}{\partial\nu}\,d\sigma dt,\ \text{for all } \varphi_T\in L^2(\Omega). \tag{24}$$

To prove (24), we first use the unique solvability for the problem

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,T], \\ u = \gamma, & \text{on } \partial\Omega \times [0,T], \\ u(0) = 0 & \text{in } \Omega, \end{cases}$$

with lateral Dirichlet data γ in $L^p(\partial\Omega \times (0,T))$, $2 \le p \le \infty$, stablished in [6, Theorem 3.2] (See also [3, Theorems 8.1 and 8.3]). Then, because $g\chi_{\mathscr{M}}$ is bounded and supported in $\partial\Omega \times [T-l_1, T-l] \subset \partial\Omega \times (2\eta, T-2\eta)$ for some $\eta > 0$, the calculations leading to (24) can be justified via the regularization of $g\chi_{\mathscr{M}}$ and the approximation of Ω by smooth domains $\{\Omega_j; j \ge 1\}$ as in [3, Lemma 2.2]. For the sake of completeness we provide the detailed proof of this identity in the Appendix in Section 5.

□

3.1 Definition of the Minimal Time Control Problems and Main Results

In this section, we apply Theorems 1 and 2 to get the bang-bang property for the minimal time control problems usually called the first type of time optimal control problems; they are stated as follows. Let ω be a measurable subset with positive measure in $B_R(x_0)$ and $B_{4R}(x_0) \subset \Omega$. Suppose that $\triangle_{4R}(q_0)$ is real-analytic for some $q_0 \in \partial\Omega$ and $R \in (0,1]$ and let Γ be a measurable subset with positive surface measure of $\triangle_R(x_0)$. For each $M > 0$, we define the following control constraint set:

$$\mathscr{U}_M^1 = \{v \;\text{ measurable on }\; \Omega \times \mathbb{R}^+ : \; |v(x,t)| \le M \;\text{ for a.e. }\; (x,t) \in \Omega \times \mathbb{R}^+\}.$$

$$\mathscr{U}_M^2 = \{g \;\text{ measurable on }\; \partial\Omega \times \mathbb{R}^+ : \; |g(x,t)| \le M \;\text{ for a.e. }\; (x,t) \in \partial\Omega \times \mathbb{R}^+\}.$$

Let $u_0 \in L^2(\Omega) \setminus \{0\}$. Consider the minimal time control problems:

$$(TP)_M^1 : \;\; T_M^1 \equiv \min_{v \in \mathscr{U}_M^1} \left\{ t > 0 : \; e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}(\chi_\omega v)\, ds = 0 \right\}$$

and

$$(TP)_M^2 : \;\; T_M^2 \equiv \min_{g \in \mathscr{U}_M^2} \{ t > 0 : \; u(x,t;g) = 0 \;\text{ for a.e. }\; x \in \Omega \},$$

where $u(\cdot,\cdot;g)$ is the solution to

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = g\chi_\Gamma, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0, & \text{in } \Omega. \end{cases} \tag{25}$$

Any solution of $(TP)_M^i$, $i = 1,2$, is called a minimal time control to this problem. According to Theorem 1 and Theorem 3.3 in [12], problem $(TP)_M^1$ has solutions. By Theorem 2, using the same arguments as those in the proof of Theorem 3.3 in [12], we can verify that there is $g \in \mathscr{U}_M^2$ such that for some $t > 0$, $u(x,t;g) = 0$ for a.e. $x \in \Omega$.

Lemma 9. *Problem $(TP)_M^2$ has solutions.*

Proof. Let $\{t_n\}_{n \ge 1}$, with $t_n \searrow T_M^2$, and $g_n \in \mathscr{U}_M^2$ be such that $u(x,t_n;g_n) = 0$ over Ω. Hence, on a subsequence,

$$g_n \longrightarrow g^* \;\text{ weakly star in }\; L^\infty(\partial\Omega \times (0,t_1)). \tag{26}$$

It suffices to show that

$$u_n(x,t_n) \equiv u(x,t_n;g_n) \longrightarrow u^*(x,T_M^2) \equiv u(x,T_M^2;g^*), \text{ for all } x \in \Omega. \tag{27}$$

For this purpose, let $G(x,y,t)$ be the Green's function for $\triangle-\partial_t$ in $\Omega\times\mathbb{R}$ with zero lateral Dirichlet boundary condition. [6, Theorems 1.3 and 1.4] and [6, p. 643] show that for $g\in\mathscr{U}_M^2$ and $(x,t)\in\Omega\times(0,T)$,

$$u(x,t;g)=e^{t\triangle}u_0-\int_0^t\int_{\partial\Omega}\frac{\partial G}{\partial\nu_q}(x,q,t-s)\,\chi_\Gamma(q,s)g(q,s)\,d\sigma_q ds \tag{28}$$

and

$$\int_0^T\int_{\partial\Omega}|\tfrac{\partial G}{\partial\nu_q}(x,q,\tau)|^2\,d\sigma_q d\tau<+\infty,\text{ when } x\in\Omega,\ T>0. \tag{29}$$

Also, by standard interior parabolic regularity there is $N=N(n,\varepsilon)$ with

$$|u(x,t;g)-u(x,s;g)|\le N|t-s|\left(\|g\|_{L^\infty(\partial\Omega\times(0,T))}+\|u_0\|_{L^2(\Omega)}\right) \tag{30}$$

when $d(x,\partial\Omega)>\sqrt{\varepsilon}$ and $t>s\ge\varepsilon$. Now, when $x\in\Omega$ with $d(x,\partial\Omega)>\sqrt{\varepsilon}$, it holds that

$$|u_n(x,t_n)-u^*(x,T_M^2)|\le|u_n(x,t_n)-u_n(x,T_M^2)|+|u_n(x,T_M^2)-u^*(x,T_M^2)|.$$

This, along with (26), (28), (29) and (30) indicates that (27) holds for all $x\in\Omega$ with $d(x,\partial\Omega)>\sqrt{\varepsilon}$. Since $\varepsilon>0$ is arbitrary, (27) follows at once. □

Now, we can use the same methods as those in [14], as well as in Lemma 9, to get the following consequences of Theorems 1 and 2 respectively.

Corollary 10. *Problem $(TP)_M^1$ has the bang-bang property: any minimal time control v satisfies that $|v(x,t)|=M$ for a.e. $(x,t)\in\omega\times(0,T_M^1)$. Consequently, this problem has a unique minimal time control.*

Corollary 11. *The problem $(TP)_M^2$ has the bang-bang property: any minimal time boundary control g satisfies that $|g(x,t)|=M$ for a.e. $(x,t)\in\Gamma\times(0,T_M^2)$. Consequently, this problem has a unique minimal time control.*

3.2 Definition of the Time Optimal Control Problems and Main Results

Next, we make use of Theorems 1 and 2 to study the bang-bang property for the time optimal control problems where the interest is on retarding the initial time of the action of a control with bounded L^∞-norm. These problems are usually called the second type of time optimal control problems and are stated as follows: Let $T>0$ and $M>0$. Write ω and Γ for the sets given in Problems $(TP)_M^1$ and $(TP)_M^2$ respectively. Consider the controlled heat equations:

$$\begin{cases}\partial_t u-\Delta u=\chi_\omega\chi_{(\tau,T)}v, & \text{in } \Omega\times(0,T],\\ u=0, & \text{on } \partial\Omega\times[0,T],\\ u(0)=u_0, & \text{in } \Omega\end{cases} \tag{31}$$

and

$$\begin{cases}\partial_t u-\Delta u=0, & \text{in } \Omega\times(0,T],\\ u=\chi_\Gamma\chi_{(\tau,T)}g, & \text{on } \partial\Omega\times[0,T],\\ u(0)=u_0, & \text{in } \Omega,\end{cases} \tag{32}$$

where $u_0\in L^2(\Omega)$. Write accordingly $u(\cdot;\chi_{(\tau,T)}v)$ and $u(\cdot;\chi_{(\tau,T)}g)$ for the solutions to equation (31) corresponding to $\chi_{(\tau,T)}v$, and to equation (32) corresponding to $\chi_{(\tau,T)}g$. Define the following control constraint sets:

$$\mathscr{U}_{M,T}^1=\{v\text{ measurable on }\Omega\times(0,T):|v(x,t)|\le M\text{ for a.e. }(x,t)\in\Omega\times(0,T)\}.$$

$$\mathscr{U}_{M,T}^2=\{g\text{ measurable on }\partial\Omega\times(0,T):|g(x,t)|\le M\text{ for a.e. }(x,t)\in\partial\Omega\times(0,T)\}.$$

Consider the time optimal control problems:

$$(TP)^1_{M,T}: \ \tau^1_{M,T} \equiv \sup_{v\in\mathscr{U}^1_{M,T}} \left\{\tau \in [0,T): \ u(T;\chi_{(\tau,T)}v) = 0\right\}$$

and

$$(TP)^2_{M,T}: \ \tau^2_{M,T} \equiv \sup_{g\in\mathscr{U}^2_{M,T}} \left\{\tau \in [0,T): \ u(T;\chi_{(\tau,T)}g) = 0\right\}.$$

Any solution of $(TP)^i_{T,M}$, $i = 1,2$, is called an optimal control to the corresponding problem.

Now, we can use the same arguments as those in the proof of Theorem 3.4 in [11] to get the following consequences of Theorem 1 and Theorem 2 respectively:

Corollary 12. *Any optimal control v^* to Problem $(TP)^1_{M,T}$, if it exists, satisfies the bang-bang property: $|v^*(x,t)| = M$ for a.e. $(x,t) \in \omega \times (\tau^1_{M,T}, T)$.*

Corollary 13. *Any optimal control g^* to Problem $(TP)^2_{M,T}$, if it exists, satisfies the bang-bang property: $|g^*(x,t)| = M$ for a.e. $(x,t) \in \Gamma \times (\tau^2_{M,T}, T)$.*

Remark 5. By Theorem 1 (See also Remark 1) and the energy decay property for the heat equation, one can easily prove the following: for a fixed $M > 0$, there is $v \in \mathscr{U}^1_{M,T}$ such that $u(T;\chi_{(0,T)}v) = 0$, when T is large enough (such a control v is called an admissible control); while for a fixed $T > 0$, the same holds when M is large enough. The same can be said about Problem $(TP)^2_{M,T}$ because of Theorem 2 (See also Remark 4). In the case where Problem $(TP)^1_{M,T}$ has admissible controls, one can easily prove the existence of time optimal controls to this problem. In the case when Problem $(TP)^2_{M,T}$ has admissible controls, one can make use of the similar method in the proof of Lemma 9 to verify the existence of time optimal controls for this problem.

3.3 Definition of the Minimal Norm Control Problems and Main Results

In this section, we apply Theorems 1 and 2 to get the bang-bang property for the minimal norm control problems; they are stated as follows. Let $\mathscr{D}$ and $\mathscr{J}$ be the subsets given at the beginning of this section. Let $u_0 \in L^2(\Omega)$, we define two control constraint sets as follows:

$$\mathscr{V}_{\mathscr{D}} = \{v \in L^\infty(\Omega \times (0,T)) : u(T;u_0,v) = 0\}$$

and

$$\mathscr{V}_{\mathscr{J}} = \{g \in L^\infty(\partial\Omega \times (0,T)) : \ u(T;u_0,g) = 0\}.$$

Consider the minimal norm control problems:

$$(NP)_{\mathscr{D}}: \quad M_{\mathscr{D}} \equiv \min\left\{\|v\|_{L^\infty(\Omega\times(0,T))}: \ v \in \mathscr{V}_{\mathscr{D}}\right\}$$

and

$$(NP)_{\mathscr{J}}: \quad M_{\mathscr{J}} \equiv \min\left\{\|g\|_{L^\infty(\partial\Omega\times(0,T))}: \ g \in \mathscr{V}_{\mathscr{J}}\right\}.$$

Any solution of $(NP)_{\mathscr{D}}$ (or $(NP)_{\mathscr{J}}$) is called a minimal norm control to this problem. According to Corollary 8, the sets $\mathscr{V}_{\mathscr{D}}$ and $\mathscr{V}_{\mathscr{J}}$ are not empty. Since $\mathscr{V}_{\mathscr{D}}$ is not empty, it follows from the standard arguments that Problem $(NP)_{\mathscr{D}}$ has solutions. Because $\mathscr{V}_{\mathscr{J}}$ is not empty, by using the similar arguments as those in the proof of Lemma 9, we can justify that Problem $(NP)_{\mathscr{J}}$ has solutions.

We can use the same methods as those in [11] to get the following consequences of Theorem 1 and Theorem 2 respectively:

Corollary 14. *Problem $(NP)_{\mathscr{D}}$ has the bang-bang property: any minimal norm control v satisfies that $|v(x,t)| = M_{\mathscr{D}}$ for a.e. $(x,t) \in \mathscr{D}$. Consequently, this problem has a unique minimal norm control.*

Corollary 15. *The problem $(NP)_{\mathscr{J}}$ has the bang-bang property: any minimal norm boundary-control g satisfies that $|g(x,t)| = M_{\mathscr{J}}$ for a.e. $(x,t) \in \mathscr{J}$. Consequently, this problem has a unique minimal norm control.*

4 Open problems

In tis section we will establish the heat equation with similar conditions to what we studied before, but in this case we will require it to verify other type of boundary conditions instead of Dirichlet boundary conditions.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and consider the following heat equation,

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,1), \\ \frac{\partial}{\partial \nu} u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{33}$$

with Neumann boundary condition and

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,1), \\ \frac{\partial}{\partial \nu} u + \alpha u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{34}$$

with Robin boundary condition, where $\alpha \in \mathbb{R}$ and u_0 in $L^2(\Omega)$.

We proved two obserbavility inequalities (Theorems 1 and 2) for these kind of equations over measurable sets with Dirichlet boundary conditions, but if we change that condition to now use Neumann or Robin conditions, would we be able to prove some similar observability inequalities? And, if that's the case, could we apply them to prove some bang-bang properties?

The idea of facing these questions is to spread our mathematical knowledge about this kind of problems and also to discover new interesting ways or limitations in the techniques we are used to working with. It could also be physically interesting because of the physical meaning of these new boundary conditions, as we will see now.

The Dirichlet boundary condition states that we have a constant temperature at the boundary. This can be considered as a model of an ideal cooler in a good contact having infinitely large thermal conductivity.

With the Neumann boundary condition case for the heat flow, we can say that we have a constant heat flux at the boundary or that it corresponds to a perfectly insulated boundary. If the flux is equal to zero, the boundary condition describes the ideal heat insulator with the heat diffusion. For the Laplace equation and drum modes, we could think this corresponds to allowing the boundary to flap up and down but not move otherwise.

Finally, the Robin boundary condition is the mathematical formulation of Newton's law of cooling where the heat transfer coefficient α is utilized. The heat transfer coefficient is determined by details of the interface structure (sharpness, geometry) between two media. This law describes the boundary between metals and gas quite well and is good for the convective heat transfer.

5 Appendix

Here, we will give the definition of a Lipschitz domain and complete the proof of the equation (24) that appeared in the proof of Corollary 8.

Definition 2. Let Ω be a bounded domain in $\mathbb{R}^n$. Ω is a Lipschitz domain (sometimes called strongly Lipschitz or Lipschitz graph domains) with constants m and ρ when for each point p on the boundary of Ω there is a rectangular coordinate system $x = (x', x_n)$ and a Lipschitz function $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ verifying

$$\phi(0') = 0, \quad |\phi(x'_1) - \phi(x'_2)| \le m|x'_1 - x'_2|, \text{ for all } x'_1, x'_2 \in \mathbb{R}^{n-1}, \tag{35}$$

$p=(0',0)$ on this coordinate system and

$$\begin{aligned} Z_{m,\rho}\cap\Omega &= \{(x',x_n) : |x'|<\rho,\ \phi(x')<x_n<2m\rho\},\\ Z_{m,\rho}\cap\partial\Omega &= \{(x',\phi(x')) : |x'|<\rho\}, \end{aligned} \tag{36}$$

where $Z_{m,\rho}=B'_\rho\times(-2m\rho,2m\rho)$.

Proof. [Proof of (24)] For each $(p,\tau)\in\partial\Omega\times\mathbb{R}$ and fixed $\xi>0$, we define

$$\Gamma(p)=\{x\in\Omega : |x-p|\le(1+\xi)\,d(x,\partial\Omega)\},$$

$$\Gamma(p,\tau)=\{(x,t)\in\Omega\times(0,T) : |x-p|+\sqrt{|t-\tau|}\le(1+\xi)\,d(x,\partial\Omega)\}.$$

The later are called respectively elliptic and parabolic non-tangential approach regions from the interior of $\Omega\times(0,T)$ to (p,τ). In particular,

$$\Gamma(p)\times\{\tau\}\subset\Gamma(p,\tau),\ \text{for all}\ (p,\tau)\in\partial\Omega\times(0,T).$$

When $u:\Omega\longrightarrow\mathbb{R}$ or $u:\Omega\times(0,T)\longrightarrow\mathbb{R}$ (or $\mathbb{R}^n$), define the elliptic and parabolic non-tangential maximal function of u in $\partial\Omega\times(0,T)$ as

$$u^*(p)=\sup_{x\in\Gamma(p)}|u(x)|,\quad u^\sharp(p,\tau)=\sup_{(x,t)\in\Gamma(p,\tau)}|u(x,t)|,\ \text{when}\ p\in\partial\Omega\ \text{and}\ \tau\in(0,T).$$

Let $\eta>0$ be fixed such that $[T-l_1,T-l]\subset[2\eta,T-2\eta]$, with l and l_1 as defined in Corollary 8. Denote by u the solution to

$$\begin{cases} \partial_t u-\Delta u=0, & \text{in}\ \Omega\times(0,T),\\ u=g\chi_{\mathscr{M}}\equiv\gamma, & \text{on}\ \partial\Omega\times(0,T),\\ u(0)=u_0, & \text{in}\ \Omega. \end{cases}$$

(See the beginning of Section 3 for the definition of the solution to this equation.)

Let γ^ε in $C_0^1(\partial\Omega\times(0,T))$ be a regularization of γ in $\partial\Omega\times[0,T]$ such that

$$\|\gamma^\varepsilon\|_{L^\infty(\partial\Omega\times[0,T])}+\varepsilon\|\gamma^\varepsilon\|_{C^1(\partial\Omega\times[0,T])}\le\|\gamma\|_{L^\infty(\partial\Omega\times[0,T])},$$

$$\operatorname{supp}(\gamma^\varepsilon)\subset\partial\Omega\times[\eta,T-\eta]$$

and let v^ε be the solution to

$$\begin{cases} \partial_t v^\varepsilon-\Delta v^\varepsilon=0, & \text{in}\ \Omega\times(0,T),\\ v^\varepsilon=\gamma^\varepsilon, & \text{on}\ \partial\Omega\times(0,T),\\ v^\varepsilon(0)=0, & \text{in}\ \Omega. \end{cases}$$

From [6, Theorem 3.2] and either [3, Theorem 6.1] or [4, Theorem 2.9]

$$\|v^\varepsilon\|_{L^\infty(\partial\Omega\times[0,T])}+\varepsilon\,\|(\nabla v^\varepsilon)^\sharp\|_{L^2(\partial\Omega\times[0,T])}\le\|\gamma\|_{L^\infty(\partial\Omega\times[0,T])}, \tag{37}$$

and the limits

$$\lim_{\substack{(x,t)\to(p,\tau)\\(x,t)\in\Gamma(p,\tau)}}\nabla v^\varepsilon(x,t)=\nabla v^\varepsilon(p,\tau)$$

exist and are finite for a.e. (p,τ) in $\partial\Omega\times(0,T)$. Also, $v_\varepsilon\in C(\overline{\Omega}\times[0,T])\cap C^\infty(\Omega\times[0,T])$, $v^\varepsilon=0$ for $t\le\eta$, and $v^\varepsilon=0$ on $\partial\Omega\times(T-\eta,T]$. Moreover, the Hölder regularity up to the boundary for bounded solutions to parabolic equations with zero local lateral Dirichlet data, shows that there are positive constants $N = N(m,\rho,\eta)$ and $\alpha = \alpha(m,\rho)$, with $\alpha \in (0,1)$, such that

$$|v^{\varepsilon}(x_1,t_1) - v^{\varepsilon}(x_2,t_2)| \le N\left[|x_1-x_2|^2 + |t_1-t_2|\right]^{\alpha/2} \|\gamma\|_{L^{\infty}(\partial\Omega\times[0,T])}, \tag{38}$$

when $x_1, x_2 \in \overline{\Omega}$, $T - \frac{\eta}{2} \le t_1, t_2 \le T$ [9, Theorems 6.28 and 6.32].

Let $\varphi(t) = e^{(T-t)\Delta}\varphi_T$, $t \in (0,T)$, where φ_T is in $L^2(\Omega)$. From the regularity of caloric functions [5, Theorem 1.7]

$$\varphi \in C([0,T];L^2(\Omega)) \cap C^{\infty}(\Omega\times[0,T)) \cap C(\overline{\Omega}\times[0,T)) \tag{39}$$

and from [6, Theorems 1.3 and 1.4] or the proof of (40) and (41) in this appendix, there are $N = N(m,\rho)$ and $\varepsilon = \varepsilon(m,\rho,n) > 0$ such that

$$\|(\nabla\varphi)^*\|_{L^{\infty}(0,T-\delta;L^{2+\varepsilon}(\partial\Omega))} \le Ne^{1/\delta}\|\varphi_T\|_{L^2(\Omega)}, \tag{40}$$

when $0 < \delta < T$ and the limit

$$\lim_{\substack{x\to p\\ x\in\Gamma(p)}} \nabla\varphi(x,\tau) = \nabla\varphi(p,\tau), \tag{41}$$

exists and is finite for a.e. $p \in \partial\Omega$ and for all $\tau \in (0,T)$. Now, let $\Omega_j \subset \overline{\Omega}_{j+1} \subset \Omega$, $j \ge 1$, be a sequence of C^{∞}-domains approximating Ω as in [3, Lemma 2.2]. Set, $u^{\varepsilon} = v^{\varepsilon} + e^{t\Delta}u_0$. By Green's formula,

$$\frac{d}{dt}\int_{\Omega_j} u^{\varepsilon}(t)\varphi(t)\,dx = \int_{\partial\Omega_j} \frac{\partial u^{\varepsilon}}{\partial\nu_j}\varphi - \frac{\partial\varphi}{\partial\nu_j}u^{\varepsilon}\,d\sigma_j.$$

Integrating the above identity over $[\delta, T-\delta]$ for a fixed $\delta \in (0,\frac{\eta}{2})$, we get

$$\begin{aligned}&\int_{\Omega_j} u^{\varepsilon}(T-\delta)\varphi(T-\delta)\,dx - \int_{\Omega_j} u^{\varepsilon}(\delta)\varphi(\delta)\,dx\\ &= \int_{\partial\Omega_j\times(\delta,T-\delta)} \frac{\partial u^{\varepsilon}}{\partial\nu_j}\varphi - \frac{\partial\varphi}{\partial\nu_j}u^{\varepsilon}\,d\sigma_j dt.\end{aligned} \tag{42}$$

Recall that $u^{\varepsilon}(\delta) = e^{\delta\Delta}u_0$ and let $j \to +\infty$ in (42) with ε and δ being fixed. Then, (37), (39), (41) and the dominated convergence theorem show that

$$\int_{\Omega} u^{\varepsilon}(T-\delta)\varphi(T-\delta)\,dx = \int_{\Omega}(e^{\delta\Delta}u_0)\varphi(\delta)\,dx - \int_{\partial\Omega\times(\delta,T-\delta)} \gamma^{\varepsilon}\frac{\partial\varphi}{\partial\nu}\,d\sigma dt.$$

Because γ^{ε} is supported in $[\eta, T-\eta]$, the later is the same as

$$\int_{\Omega} u^{\varepsilon}(T-\delta)\varphi(T-\delta)\,dx = \int_{\Omega}(e^{\delta\Delta}u_0)\varphi(\delta)\,dx - \int_{\partial\Omega\times(\eta,T-\eta)} \gamma^{\varepsilon}\frac{\partial\varphi}{\partial\nu}\,d\sigma dt, \tag{43}$$

when $0 < \delta < \eta/8$. Next, from (38),

$$u^{\varepsilon}(T-\delta) = v^{\varepsilon}(T-\delta) + e^{(T-\delta)\Delta}u_0 = v^{\varepsilon}(T) + e^{T\Delta}u_0 + O(\delta^{\alpha/2}),$$

uniformly for $x \in \overline{\Omega}$, when $0 < \delta < \eta/8$. Hence, after letting $\delta \to 0$ in (43), we get

$$\int_{\Omega}(v^{\varepsilon}(T) + e^{T\Delta}u_0)\varphi(T)\,dx = \int_{\Omega} u_0\varphi(0)\,dx - \int_{\partial\Omega\times(\eta,T-\eta)} \gamma^{\varepsilon}\frac{\partial\varphi}{\partial\nu}\,d\sigma dt.$$

Also, from (37) and (38), v^ε converges uniformly over $\overline{\Omega}\times[T-\eta/2,T]$ to some continuous function $\widetilde{v}$ as $\varepsilon\to 0$. We claim that $\widetilde{v}=v$. If it is the case, we get after letting $\varepsilon\to 0$ in the last equality, that

$$\int_\Omega u(T)\varphi(T)\,dx=\int_\Omega u_0\varphi(0)\,dx-\int_{\partial\Omega\times(\eta,T-\eta)}\gamma\frac{\partial\varphi}{\partial\nu}\,d\sigma dt,$$

because $\gamma^\varepsilon(p,\tau)\to\gamma(p,\tau)$ for a.e. $(p,\tau)\in\partial\Omega\times(0,T)$, (39) and

$$\operatorname{supp}(\gamma^\varepsilon)\cup\operatorname{supp}(\gamma)\subset\partial\Omega\times[\eta,T-\eta].$$

Recalling that $\gamma=g\chi_{\mathscr{M}}$, we get

$$\int_\Omega u(T)\varphi(T)\,dx=\int_\Omega u_0\varphi(0)\,dx-\int_{\partial\Omega\times(0,T)}g\chi_{\mathscr{M}}\frac{\partial\varphi}{\partial\nu}\,d\sigma dt.$$

Hence, (24) is proved.

To verify that $\widetilde{v}=v$ over $\overline{\Omega}\times[0,T]$, observe that because $v^\varepsilon-v$ is the unique solution to

$$\begin{cases}\partial_t u-\Delta u=0, & \text{in }\Omega\times(0,T),\\ u=\gamma^\varepsilon-\gamma, & \text{on }\partial\Omega\times(0,T),\\ u(0)=0, & \text{in }\Omega,\end{cases}$$

whose parabolic non-tangential maximal function is in $L^2(\partial\Omega\times(0,T))$ (See [6, Theorem 3.2]), it holds that

$$\|(v^\varepsilon-v)^\sharp\|_{L^2(\partial\Omega\times(0,T))}\le N\|\gamma^\varepsilon-\gamma\|_{L^2(\partial\Omega\times(0,T))}. \tag{44}$$

For fixed p in $\partial\Omega$, we may assume that $p=(0',0)$ and that near p,

$$\Omega\cap Z_{m,\rho}=\{(x',x_n):\phi(x')<x_n<2m\rho,\ |x'|\le\rho\},$$

with ϕ as in (35) and (36). Then,

$$\begin{aligned}&\int_0^T\int_{B'_\rho}\int_{\phi(y')}^{\phi(y')+m\rho}|F(y',y_n,t)|^2\,dy'dy_ndt\\ &\le m\rho\int_0^T\int_{B'_\rho}F^\sharp(y',y_n,t)^2\,dy'dt\le m\rho\int_{\partial\Omega\times(0,T)}F^\sharp(p,t)^2\,d\sigma dt,\end{aligned}$$

for all functions F. The above estimate, a covering argument and (44) show that

$$\|v^\varepsilon-v\|_{L^2(\Omega_{m\rho}\times(0,T))}\le N\|\gamma^\varepsilon-\gamma\|_{L^2(\partial\Omega\times(0,T))}, \tag{45}$$

with $\Omega_\eta=\{x\in\Omega:d(x,\partial\Omega)\le\eta\}$. Recalling that $v^\varepsilon=v=0$ for $t\le\eta$, the local boundedness properties of solutions to parabolic equations [9, Theorem 6.17] show that,

$$|(v^\varepsilon-v)(x,\tau)|\le\left(\fint_{B_{\frac{R}{20}}(x)\times[\tau-\frac{R^2}{20^2},\tau]}|v^\varepsilon-v|^2\,dyds\right)^{1/2},$$

when $x\in\partial\Omega_R$, $0\le\tau\le T$, and taking $R<\frac{m\rho}{20}$ above, we find from (45) that

$$\|v^\varepsilon-v\|_{L^\infty(\Omega^R\times\{0\}\cup\partial\Omega^R\times[0,T])}\le N_R\|\gamma^\varepsilon-\gamma\|_{L^2(\partial\Omega\times(0,T))}.$$

By the maximum principle and the above estimate

$$\|v^\varepsilon-v\|_{L^\infty(\Omega^R\times[0,T])}\le N_R\|\gamma^\varepsilon-\gamma\|_{L^2(\partial\Omega\times(0,T))}\longrightarrow 0,\ \text{ as }\ \varepsilon\to 0,$$

which shows that $\widetilde{v}=v$ in $\overline{\Omega}\times[0,T]$.

□

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