Applied Mathematics and Nonlinear Sciences 2(2) (2017) 519–528

Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

J-class abelian semigroups of matrices on $\mathbb{R}^n$

Habib Marzougui [†]

University of Carthage, Faculty of Science of Bizerte, Department of Mathematics, Jarzouna. 7021 Tunisia.

Submission Info

Communicated by F. Balibrea
Received 7th May 2017
Accepted 8th December 2017
Available online 8th December 2017

Abstract

We establish, for finitely generated abelian semigroups G of matrices on $\mathbb{R}^n$, and by using the extended limit sets (the J-sets), the following equivalence analogous to the complex case: (i) G is hypercyclic, (ii) $J_G(v_\eta) = \mathbb{R}^n$ for some vector v_η given by the structure of G, (iii) $\overline{G(v_\eta)} = \mathbb{R}^n$. This answer a question raised by the author. Moreover we construct for any $n \geq 2$ an abelian semigroup G of $\mathrm{GL}(n,\mathbb{R})$ generated by $n+1$ diagonal matrices which is locally hypercyclic (or J-class) but not hypercyclic and such that $\mathrm{J}_G(e_k) = \mathbb{R}^n$ for every $k = 1,\dots,n$, where $(e_1,\dots,e_n)$ is the canonical basis of $\mathbb{R}^n$. This gives a negative answer to a question raised by Costakis and Manoussos.

Keywords: Hypercyclic semigroup, locally hypercyclic, J-class operator, extended limit set, dense orbit, semigroup.
AMS 2010 codes: 47A16.

1 Introduction

Let $M_n(\mathbb{R})$ be the set of all square matrices over $\mathbb{R}$ of order $n \geq 1$ and by $\mathrm{GL}(n,\mathbb{R})$ the group of invertible matrices of $M_n(\mathbb{R})$. Let G be a finitely generated abelian sub-semigroup of $M_n(\mathbb{R})$. By a sub-semigroup of $M_n(\mathbb{R})$, we mean a subset which is stable under multiplication and contains the identity matrix. For a vector $v \in \mathbb{C}^n$, we consider the orbit of G through v: $G(v) = \{Av : A \in G\} \subset \mathbb{R}^n$. A subset $E \subset \mathbb{R}^n$ is called *G-invariant* if $A(E) \subset E$ for any $A \in G$. The orbit $G(v) \subset \mathbb{R}^n$ is *dense* in $\mathbb{R}^n$ if $\overline{G(v)} = \mathbb{R}^n$, where $\overline{E}$ denotes the closure of a subset $E \subset \mathbb{R}^n$. The semigroup G is called *hypercyclic* if there exists a vector $v \in \mathbb{R}^n$ such that $G(v)$ is dense in $\mathbb{R}^n$. We refer the reader to the recent books [4] and [9] for a thorough account on hypercyclicity. In [7], [8], Costakis and Manoussos localized the notion of hypercyclicity through the use of the J-sets, firstly for a single bounded linear operator T acting on a complex Banach space X, and secondly extended to that of semigroup. Recall

[†] Corresponding author.
Email address: habib.marzougui@fsb.rnu.tn

ISSN 2444-8656 doi:10.21042/AMNS.2017.2.00043

that T is called a J-class operator if for every non-zero vector x in X and for every open neighborhood $U \subset X$ of x and every non-empty open set $V \subset X$, there exists a positive integer n such that $T^n(U) \cap V = \emptyset$. Recently, there are much research that study or use J-class operators, We mention for instance, the series of papers by Azimi and Müller [3], Nasseri [10], Chan and Seceleanu [5]. Among properties, there are locally hypercyclic, non hypercyclic operators and that finite dimensional Banach spaces do not admit locally hypercyclic operators (see [7]).

In [8], Costakis and Manoussos extend the notion of J-class operator to that of a J-class of semigroup G as follows: Suppose that G is generated by p matrices $A_1, \dots, A_p$ $(p \geq 1)$ then we define the extended limit set $\mathrm{J}_G(x)$ of x under G to be the set of $y \in \mathbb{R}^n$ for which there exists a sequence $(x_m)_m \subset \mathbb{R}^n$ and sequences of non-negative integers $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ with $k_m^{(1)} + k_m^{(2)} + \dots + k_m^{(p)} \to +\infty$ such that $x_m \to x$ and $A_1^{k_m^{(1)}} A_2^{k_m^{(2)}} \dots A_p^{k_m^{(p)}} x_m \to y$.
This describes the asymptotic behavior of the orbits of nearby points to x. Note that the condition $k_m^{(1)} + k_m^{(2)} + \dots + k_m^{(p)} \to +\infty$ is equivalent to having at least one of the sequences $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ containing an increasing subsequence tending to $+\infty$. We say that G is *locally hypercyclic* (or J-class) if there exists a vector $v \in \mathbb{R}^n \backslash \{0\}$ such that $\mathrm{J}_G(v) = \mathbb{R}^n$. This notion is a “localization“ of the concept of hypercyclicity, this can be justified by the following: $\mathrm{J}_G(x) = \mathbb{R}^n$ if and only if for every open neighborhood $U_x \subset \mathbb{R}^n$ of x and every nonempty open set $V \subset \mathbb{R}^n$ there exists $A \in G$ such that $A(U_x) \cap V \neq \emptyset$.

As we have mentioned above, in $\mathbb{C}^n$ or $\mathbb{R}^n$, no matrix can be locally hypercyclic. However, what is rather remarkable is that in $\mathbb{C}^n$ or $\mathbb{R}^n$, a pair of commuting matrices exists which forms a locally hypercyclic, non-hypercyclic semigroup (see [8]).

In the present work, we show that G is hypercyclic if and only if there exists a vector v in an open set V, defined according to the structure of G, such that $\mathrm{J}_G(v) = \mathbb{R}^n$ (Theorem 1). This answer the question 1 raised by the author in [2]. Furthermore, we construct for every $n \geq 2$, a locally hypercyclic abelian semigroup G generated by matrices $A_1, \dots, A_{n+1}$ which is non-hypercyclic whenever $\mathrm{J}_G(u_k) = \mathbb{R}^n$, $k = 1, \dots, n$, for a basis $(u_1, \dots, u_n)$ of $\mathbb{R}^n$ (Theorem 4), this answers negatively the question raised by Costakis and Manoussos in [8]. However, we prove that the question is true (see Proposition 5) for any abelian semigroup G consisting of lower triangular matrices on $\mathbb{R}^n$ with all diagonal elements equal.

Before stating our main results, let introduce the following notations and definitions.

Set $\mathbb{N}$ be the set of non negative integers and write $\mathbb{N}_0 = \mathbb{N} \backslash \{0\}$. Let $n \in \mathbb{N}_0$ be fixed. Denote by:
• $\mathscr{B}_0 = (e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$.
• I_n the identity matrix on $\mathbb{R}^n$.
For each $m = 1, 2, \dots, n$, denote by:
• $\mathbb{T}_m(\mathbb{R})$ the set of matrices over $\mathbb{R}$ of the form:

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix}$$

• $\mathbb{S}$ the set of matrices over $\mathbb{R}$ of the form

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

For each $1 \leq m \leq \frac{n}{2}$, denote by

- $\mathbb{B}_m(\mathbb{R})$ the set of matrices of $M_{2m}(\mathbb{R})$ of the form

$$\begin{bmatrix} C & & & 0 \\ C_{2,1} & C & & \\ \vdots & \ddots & \ddots & \\ C_{m,1} & \dots & C_{m,m-1} & C \end{bmatrix} : \; C,\, C_{i,j} \in \mathbb{S},\; 2 \le i \le m, 1 \le j \le m-1$$

Let $r,\ s \in \mathbb{N}$. By a partition of n we mean a finite sequence of positive integers

$$\eta = \begin{cases} (n_1, \dots, n_r;\ m_1, \dots, m_s) & \text{if } rs \neq 0, \\ (m_1, \dots, m_s) & \text{if } r = 0, \\ (n_1, \dots, n_r) & \text{if } s = 0 \end{cases}$$

such that $(n_1 + \dots + n_r) + 2(m_1 + \dots + m_s) = n$. In particular, we have $r + 2s \le n$. The number $r + 2s$ will be called the *length* of the partition.
Given a partition $\eta = (n_1, \dots, n_r;\ m_1, \dots, m_s)$, denote by:

- $\mathscr{K}_\eta(\mathbb{R}) := \mathbb{T}_{n_1}(\mathbb{R}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{R}) \oplus \mathbb{B}_{m_1}(\mathbb{R}) \oplus \dots \oplus \mathbb{B}_{m_s}(\mathbb{R})$.
- $\mathscr{K}^*_\eta(\mathbb{R}) := \mathscr{K}_\eta(\mathbb{R}) \cap \mathrm{GL}(n,\ \mathbb{R})$, it is a sub-semigroup of $\mathrm{GL}(n,\ \mathbb{R})$. In particular:

- If $r = 1,\ s = 0$ then $\mathscr{K}_\eta(\mathbb{R}) = \mathbb{T}_n(\mathbb{R})$ and $\eta = (n)$.
- If $r = 0,\ s = 1$ then $\mathscr{K}_\eta(\mathbb{R}) = \mathbb{B}_m(\mathbb{R})$ and $\eta = (m)$, $n = 2m$.
- If $r = 0,\ s > 1$ then $\mathscr{K}_\eta(\mathbb{R}) = \mathbb{B}_{m_1}(\mathbb{R}) \oplus \dots \oplus \mathbb{B}_{m_s}(\mathbb{R})$ and $\eta = (m_1, \dots, m_s)$.

For a row vector $v \in \mathbb{R}^n$, we will be denoting by v^T the transpose of v.

- $u_\eta = [e_{\eta,1}, \dots, e_{\eta,r}; f_{\eta,1}, \dots, f_{\eta,s}]^T \in \mathbb{R}^n$, where for every $k = 1, \dots, r;\ l = 1, \dots, s$, $e_{\eta,k} = [1, 0, \dots, 0]^T \in \mathbb{R}^{n_k}$, $f_{\eta,l} = [1, 0, \dots, 0]^T \in \mathbb{R}^{2m_l}$.
- We let

$$U := \prod_{k=1}^{r} (\mathbb{R}^* \times \mathbb{R}^{n_k - 1}) \times \prod_{l=1}^{s} \left((\mathbb{R}^2 \backslash \{(0,0)\}) \times \mathbb{R}^{2m_l - 2} \right).$$

It is plain that U is open and dense in $\mathbb{R}^n$.

Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$, we have the following "normal form of G":
There exists a $P \in \mathrm{GL}(n, \mathbb{R})$ such that $P^{-1}GP \subset \mathscr{K}_\eta(\mathbb{R})$ for some partition η of n (see Proposition 6).
Given two integers $r,\ s \in \mathbb{N}$, we shall say that the semigroup G has "a normal form of length $r + 2s$" if G has a normal form in $\mathscr{K}_\eta(\mathbb{R})$ for some partition η with length $r + 2s$. For such a choice of matrix P, we let:

- $v_\eta = Pu_\eta$.
- $V = P(U)$, it is a dense open set in $\mathbb{R}^n$.

Our principal results are the following:

Theorem 1. *Let G be a finitely generated abelian semigroup of matrices on $\mathbb{R}^n$. If $J_G(v) = \mathbb{R}^n$ for some $v \in V$ then $\overline{G(v_\eta)} = \mathbb{R}^n$.*

Corollary 2. *Under the hypothesis of Theorem 1, the following are equivalent:*

(i) G *is hypercyclic.*

(ii) $J_G(v_\eta) = \mathbb{R}^n$.

(iii) $\overline{G(v_\eta)} = \mathbb{R}^n$.

Corollary 3. *Under the hypothesis of Theorem 1, assume that G is not hypercyclic. Then*

$$E := \{x \in \mathbb{R}^n : J_G(x) = \mathbb{R}^n\} \subset \bigcup_{k=1}^{r} H_k \cup \bigcup_{l=1}^{s} F_l, \ (r+2s \leq n)$$

, where H_k *(resp.* F_l*) are G-invariant vector subspaces of* $\mathbb{R}^n$ *of dimension* $n-1$ *(resp.* $n-2$*).*

Remark. If $n = 1$, then $v_\eta = 1$ and by Corollary 2, a sub-semigroup G of $\mathbb{R}$ is hypercylic if and only if it is locally hypercyclic.

Theorem 4. *Let* $n \geq 2$ *be an integer. Then there exists an abelian semigroup G generated by diagonal matrices* $A_1, \dots, A_{n+1} \in GL(n, \mathbb{R})$ *which is not hypercyclic such that* $J_G(e_k) = \mathbb{R}^n$ *for every* $k = 1, \dots, n$.

Proposition 5. *Let G be a finitely generated abelian sub-semigroup of* $\mathbb{T}_n(\mathbb{R})$*. If there exists a basis* $(e'_1, \dots, e'_n)$ *of* $\mathbb{R}^n$ *such that* $J_G(e'_k) = \mathbb{R}^n$ *for every* $k = 1, \dots, n$*, then G is hypercyclic.*

2 Notation and some results

Recall first the following proposition.

Proposition 6. *([1], Proposition 2.2) Let G be an abelian sub-semigroup of* $M_n(\mathbb{R})$*. Then there exists a* $P \in \mathrm{GL}(n, \mathbb{R})$ *such that* $P^{-1}GP$ *is an abelian sub-semigroup of* $\mathscr{K}_\eta(\mathbb{R})$*, for some partition* η *of n.*

Notation.

- If G is a sub-semigroup of $\mathbb{T}_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), denote by

$$F_G = \mathrm{vect}\big(\{(B - \mu_B I_n)e_i \in \mathbb{K}^n : \ 1 \leq i \leq n-1, \ B \in G\}\big)$$

the vector subspace generated by the family of vectors $\big\{(B - \mu_B I_n)e_i \in \mathbb{K}^n : \ 1 \leq i \leq n-1, \ B \in G\big\}$.
- rank(F_G) the rank of F_G. We have rank$(F_G) \leq n-1$.

- If G is an abelian sub-semigroup of $\mathbb{B}_m(\mathbb{R})$, $(n = 2m)$, denote by:

$$F_G = \mathrm{vect}\Big(\{(\widetilde{B} - CI_n)e_i \in \mathbb{R}^n : \ 1 \leq i \leq n-1, \ \widetilde{B} \in G\}\Big)$$

, where

$$\widetilde{B} = \begin{bmatrix} C & & & 0 \\ C_{2,1} & C & & \\ \vdots & \ddots & \ddots & \\ C_{m,1} & \dots & C_{m,m-1} & C \end{bmatrix} : \ C, \ C_{i,j} \in \mathbb{S}, \ i = 2, \dots, m, j = 1, \dots, m-1.$$

• If G is an abelian sub-semigroup of $\mathscr{K}_n(\mathbb{R})$, then for every $B\in\mathscr{K}_\eta(\mathbb{R})$, we have $B=\mathrm{diag}(B_1,\dots,B_r;\,\widetilde{B_1},\dots,\widetilde{B_s})$ with $B_k\in\mathbb{T}_{n_k}(\mathbb{R})$, $\widetilde{B_l}\in\mathbb{B}_{m_l}(\mathbb{R})$, $k=1,\dots,r$, $l=1,\dots,s$. Denote by:

- $G_k=\{B_k\ :\ B\in G\}$, $k=1,\dots,r$, it is an abelian sub-semigroup of $\mathbb{T}_{n_k}(\mathbb{R})$.

- $\widetilde{G_l}=\{\widetilde{B_l}\ :\ B\in G\}$, $l=1,\dots,s$, it is an abelian sub-semigroup of $\mathbb{B}_{m_l}(\mathbb{R})$.

For every $x=[x_1,\dots,x_r;\widetilde{x_1},\dots,\widetilde{x_s}]^T\in\mathbb{R}^n$, where $x_k=[a_{k,1},\dots,a_{k,n_k}]^T\in\mathbb{R}^*\times\mathbb{R}^{n_k-1}$, $\widetilde{x_l}=[b_{l,1},\dots,b_{l,m_l}]^T\in(\mathbb{R}^2\backslash\{(0,0)\})\times\mathbb{R}^{2m_l-2}$, we let:

- $H_{x_k}=\mathbb{R}x_k+F_{G_k}$, $k=1,\dots,r$.
- $\widetilde{H}_{\widetilde{x_l}}=\mathscr{C}\widetilde{x_l}+F_{\widetilde{G_l}}$, $l=1,\dots,s$, where $\mathscr{C}=\{\mathrm{diag}(C,\dots,C):\ C\in\mathbb{S}\}$.
- $H_x=\bigoplus\limits_{k=1}^{r}H_{x_k}\bigoplus\bigoplus\limits_{l=1}^{s}\widetilde{H}_{\widetilde{x_l}}$.

We start with the following lemmas:

Lemma 7. *Let G be an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$. Under the notations above, for every $x\in\mathbb{R}^n$, H_x is G-invariant.*

Proof. It suffices to prove that H_{x_k} is G_k-invariant (resp. $\widetilde{H}_{\widetilde{x_l}}$ is $\widetilde{G}_l$-invariant): write for short $G=G_k$ and $x=x_k$ (resp. $G=\widetilde{G_l}$ and $x=\widetilde{x_l}$). One has $H_x=\mathbb{R}x+F_G$ (resp. $\widetilde{H}_x=\mathscr{C}x+F_G$).

- If $w=[w_1,\dots,w_n]^T\in H_x$ and $B\in G$ with eigenvalue $\mu\in\mathbb{R}$, then $Bw=\mu w+(B-\mu I_n)w=\mu w+\sum\limits_{i=1}^{n-1}w_k(B-\mu I_n)e_i\in H_x$ (since w, $(B-\mu I_n)e_i\in H_x$ and H_x is a vector space).

- If $\widetilde{w}=[\widetilde{w_1},\dots,\widetilde{w_s}]^T\in\widetilde{H}_x$ and $B\in G$, then we have $B\widetilde{w}=C\widetilde{w}+(B-CI_n)\widetilde{w}=C\widetilde{w}+\sum\limits_{i=1}^{n-1}\widetilde{w_k}(B-CI_n)e_i\in\widetilde{H}_x$ (since $C\widetilde{w}$, $(B-CI_n)e_i\in\widetilde{H}_x$ and $\widetilde{H}_x$ is a vector space).

Proposition 8. *Let G be an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$. If $J_G(u)=\mathbb{R}^n$ for some $u\in U$, then* $\mathrm{rank}(F_{G_k})=n_k-1$ *and* $\mathrm{rank}(F_{\widetilde{G_l}})=2m_l-2$*, for every $k=1,\dots,r;\ l=1,\dots,s$.*

Proof. Let $u=[u_1,\dots,u_r;\ \widetilde{u_1},\dots,\widetilde{u_s}]^T\in\mathbb{R}^n$, where $u_k\in\mathbb{R}^{n_k}$, $\widetilde{u_l}\in\mathbb{R}^{2m_l}$, for every $k=1,\dots,r;\ l=1,\dots,s$.

i) First, we will show that $J_{G_k}(u_k)=\mathbb{R}^{n_k}$ and $J_{\widetilde{G_l}}(\widetilde{u_l})=\mathbb{R}^{2m_l}$. For this, let $x_k\in\mathbb{R}^{n_k}$, $\widetilde{x_l}\in\mathbb{R}^{2m_l}$ and set $y=[y_1,\dots,y_r;\widetilde{y_1},\dots,\widetilde{y_s}]^T\in\mathbb{R}^n$ such that

$$y_i=\begin{cases}0\in\mathbb{R}^{n_i}, & \text{if } i\neq k\\ x_k, & \text{if } i=k.\end{cases}$$

and

$$\widetilde{y_l}=\begin{cases}0\in\mathbb{R}^{2m_l}, & \text{if } i\neq l\\ \widetilde{x_l}, & \text{if } i=l.\end{cases}$$

As $J_G(u)=\mathbb{R}^n$, there exist two sequences $(z_m)_m\subset\mathbb{R}^n$ and $(B_m)_m\subset G$ such that

$$\lim_{m\to+\infty}z_m=u\quad\text{and}\quad\lim_{m\to+\infty}B_mz_m=y.\qquad(2)$$

Write $z_m = [z_{m,1}, \dots, z_{m,r}; \widetilde{z_{m,1}}, \dots, \widetilde{z_{m,s}}]^T$ with $z_{m,k} \in \mathbb{R}^{n_k}$, $\widetilde{z_{m,l}} \in \mathbb{R}^{2m_l}$, and $B_m = \mathrm{diag}(B_{m,1}, \dots, B_{m,r}; \widetilde{B_{m,1}}, \dots, \widetilde{B_{m,s}})$ with $B_{m,k} \in \mathbb{T}_{n_k}(\mathbb{R})$, $\widetilde{B_{m,l}} \in \mathbb{B}_{m_l}(\mathbb{R})$, $k = 1, \dots, r$, $l = 1, \dots, s$.

By (2), we have

$$\lim_{m\to+\infty} z_{m,k} = u_k, \quad \lim_{m\to+\infty} \widetilde{z_{m,l}} = \widetilde{u_l}$$

and

$$\lim_{m\to+\infty} B_{m,k} z_{m,k} = y_k = x_k, \quad \lim_{m\to+\infty} \widetilde{B_{m,l}}\widetilde{z_{m,l}} = \widetilde{y_l} = \widetilde{x_l}.$$

Therefore $x_k \in J_{G_k}(u_k)$ and $\widetilde{x_l} \in J_{\widetilde{G_l}}(\widetilde{u_l})$. It follows that $J_{G_k}(u_k) = \mathbb{R}^{n_k}$ and $J_{\widetilde{G_l}}(\widetilde{u_l}) = \mathbb{R}^{2m_l}$.

ii) Second, one can then suppose that $G \subset \mathbb{T}_n(\mathbb{R})$ and $u \in \mathbb{R}^* \times \mathbb{R}^{n-1}$ (resp. $G \subset \mathbb{B}_m(\mathbb{R})$) and $u \in (\mathbb{R}^2\backslash\{(0,0)\}) \times \mathbb{R}^{2m-2}$. It is clear that $u \notin F_G$.

• Assume that $G \subset \mathbb{T}_n(\mathbb{R})$. By Lemma 7, $H_{e_1} = \mathbb{R}e_1 + F_G$ is G-invariant.

Suppose that $\mathbb{R}^n\backslash H_{e_1} \neq \emptyset$, so there exist $y \in \mathbb{R}^n\backslash H_{e_1}$ and two sequences $(x_m)_m \subset \mathbb{R}^n$ and $(B_m)_m \subset G$ such that $\lim_{m\to+\infty} x_m = e_1$ and $\lim_{m\to+\infty} B_m x_m = y$. Let $H_{x_m} = \mathbb{R}x_m + F_G$ for every $m \in \mathbb{N}$. By Lemma 7, H_{x_m} is G-invariant, so $B_m x_m \in H_{x_m}$, for every $m \in \mathbb{N}$. Write $B_m x_m = \alpha_m x_m + z_m$, $\alpha_m \in \mathbb{R}$ and $z_m \in F_G$. We distinguish two cases:

- If $(\alpha_m)_m$ is bounded, one can suppose by passing to a subsequence, that $(\alpha_m)_{m\geq 1}$ is convergent, say $\lim_{m\to+\infty} \alpha_m = a \in \mathbb{R}$. It follows that $\lim_{m\to+\infty} z_m = y - au \in F_G$ and so $y \in H_{e_1}$, a contradiction.

- If $(\alpha_m)_m$ is not bounded, one can suppose by passing to a subsequence, that $\lim_{m\to+\infty} |\alpha_m| = +\infty$, then $\lim_{m\to+\infty} \frac{1}{\alpha_m} z_m = -u \in F_G$, a contradiction. We conclude that $H_{e_1} = \mathbb{R}^n$ and so $\dim(F_G) = n-1$.

• Assume that $G \subset \mathbb{B}_m(\mathbb{R})$, $(n = 2m)$. By Lemma 7, $H_{e_1} = \mathscr{C}e_1 + F_G$ is G-invariant. Suppose that $\mathbb{R}^n\backslash H_{e_1} \neq \emptyset$ and let $y \in \mathbb{R}^n\backslash H_{e_1}$. Then there exist two sequences $(x_k)_k \subset \mathbb{R}^n$ and $(B_k)_k \subset G$ such that $\lim_{k\to+\infty} x_k = e_1$ and $\lim_{k\to+\infty} B_k x_k = y$. Let $H_{x_k} = \mathscr{C}x_k + F_G$ for every $k \in \mathbb{N}$. By Lemma 7, H_{x_k} is G-invariant, so $B_k x_k \in H_{x_k}$, for every $k \in \mathbb{N}$. Write $B_k x_k = C_k x_k + z_k$, with $C_k = \mathrm{diag}(R_k, \dots, R_k)$, $R_k = \begin{bmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{bmatrix}$, and $z_k \in F_G$. Take $\|C_k\| = \|R_k\| = \sup\,(|\alpha_k|, |\beta_k|)$. We distinguish two cases:

- If $(\|C_k\|)_k$ is bounded, one can suppose by passing to a subsequence, that $(\alpha_k)_k$ and $(\beta_k)_k$ converge, say $\lim_{k\to+\infty} \alpha_k = \alpha \in \mathbb{R}$ and $\lim_{k\to+\infty} \beta_k = \beta \in \mathbb{R}$. As $\lim_{k\to+\infty} C_k e_1 = \alpha e_1 - \beta e_2 = Ce_1$, where $C = \mathrm{diag}(R, \dots, R)$ with $R = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, and as $B_k x_k = C_k x_k + z_k = C_k(x_k - e_1) + C_k e_1 + z_k$, then $\lim_{k\to+\infty} z_k = y - Ce_1 \in F_G$ and therefore $y \in Ce_1 + F_G \subset H_{e_1}$, a contradiction.

- If $(\|C_k\|)_k$ is not bounded, one can suppose by passing to a subsequence, that $\lim_{k\to+\infty} \|C_k\| = +\infty$, then C_k is invertible for k large, so from $B_k x_k = C_k x_k + z_k = C_k(x_k - e_1) + C_k e_1 + z_k$, we have $\frac{1}{\|C_k\|} C_k e_1 + \lim_{k\to+\infty} \frac{1}{\|C_k\|} z_k = 0$. In particular, we get $\lim_{k\to+\infty} \frac{\alpha_k}{\|C_k\|} = \lim_{k\to+\infty} \frac{\beta_k}{\|C_k\|} = 0$, this is a contradiction with $\|C_k\| = \sup\,(|\alpha_k|, |\beta_k|)$.
We conclude that $H_{e_1} = \mathbb{R}^n$ and so $\dim(F_G) = n-1$.

3 Proof of Theorem 1, Corollaries 2 and 3

Lemma 9. *Let G be an abelian sub-semigroup of $\mathbb{T}_n(\mathbb{K})$ $(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$ such that $\mathrm{rank}(F_G) = n-1$. Let $u,\ v \in \mathbb{K}^* \times \mathbb{K}^{n-1}$. If two sequences $(u_m)_{m\in\mathbb{N}}$ in $\mathbb{K}^* \times \mathbb{K}^{n-1}$ and $(B_m)_{m\in\mathbb{N}}$ in G such that $\lim_{m\to+\infty} u_m = u$ and*

$\lim_{m\to+\infty} B_m u_m = v$ *then* $(B_m)_{m\in\mathbb{N}}$ *is bounded.*

Proof. The proof is down in ([2], Lemma 3.4) if $\mathbb{K}=\mathbb{C}$ and the same proof works if $\mathbb{K}=\mathbb{R}$.

Now consider the following basis change:
Assume that $n=2m$, $m\in\mathbb{N}_0$. For every $k=1,\dots,m$, we let:
$e'_k=\frac{e_{2k-1}-ie_{2k}}{2}$ and $\mathscr{C}_0=(e'_1,\dots,e'_m,\ \overline{e'_1},\dots,\overline{e'_m})$, where $\overline{u}=[\overline{z_1},\dots,\overline{z_m}]^T$ is the conjugate of $u=[z_1,\dots,z_m]^T$. Then $\mathscr{C}_0$ is a basis of $\mathbb{C}^{2m}$. Denote by $Q\in \mathrm{GL}(2m,\mathbb{C})$ the matrix of basis change from $\mathscr{B}_0$ to $\mathscr{C}_0$. Then a simple calculation yields that:

Lemma 10. *Under the notation above, for every* $B\in\mathbb{B}_m(\mathbb{R})$, $Q^{-1}BQ=\mathrm{diag}(B'_1,\ \overline{B'_1})$, *where* $B'_1\in\mathbb{T}_m(\mathbb{C})$.

Lemma 11. *Let G be an abelian sub-semigroup of $\mathbb{B}_m(\mathbb{R})$ such that* $\mathrm{rank}(F_G)=2m-2$. *Let* $u,\ v\in(\mathbb{R}^2\backslash\{(0,0)\})\times\mathbb{R}^{2m-2}$. *If two sequences* $(u_m)_{m\in\mathbb{N}}\subset(\mathbb{R}^2\backslash\{(0,0)\})\times\mathbb{R}^{2m-2}$ *and* $(B_m)_{m\in\mathbb{N}}\subset G$ *such that* $\lim_{m\to+\infty}u_m=u$ *and* $\lim_{m\to+\infty}B_mu_m=v$ *then* $(B_m)_{m\in\mathbb{N}}$ *is bounded.*

Proof. For $B\in G$, we have $Q^{-1}BQ=\mathrm{diag}(B'_1,\ \overline{B'_1})$, where $B'_1\in\mathbb{T}_m(\mathbb{C})$. Write $G'_1=\{B'_1 : B\in G\}$ and $\overline{G'_1}=\{\overline{B'_1} : B\in G\}$. Then G'_1 (resp. $\overline{G'_1}$) is an abelian sub-semigroup of $\mathbb{T}_m(\mathbb{C})$.

First we prove that $\mathrm{rank}(F_{G'_1})=m-1$. Write $C=\mathrm{diag}(R,\dots,R)$ with $R=\begin{bmatrix}\alpha&\beta\\-\beta&\alpha\end{bmatrix}$ and $\mu=\alpha+i\beta$. One has

$$\begin{aligned}Q^{-1}(F_G)&=\mathrm{vect}\left(\{(Q^{-1}BQ-Q^{-1}CQI_{2m})Q^{-1}e_i : 1\le i\le n-1,\ B\in G\}\right)\\&=\mathrm{vect}\left(\{(\mathrm{diag}(B'_1,\ \overline{B'_1})-\mathrm{diag}(\mu I_m,\overline{\mu}I_m))e'_i : 1\le i\le n-1,\ B'_1\in G'_1\}\right)\\&=F_{G'_1}\oplus F_{\overline{G'_1}}\end{aligned}$$

As $\mathrm{rank}(F_G)=2m-2$ then $\mathrm{rank}(Q^{-1}(F_G))=2m-2=2\mathrm{rank}(F_{G'_1})$. So $\mathrm{rank}(F_{G'_1})=m-1$.

Second, we let $u'_m=Q^{-1}u_m=[u'_{m,1},u'_{m,2}]^T$, $u'=Q^{-1}u=[u'_1,u'_2]^T$ and $v'=Q^{-1}v=[v'_1,v'_2]^T$, where $u'_{m,i},u'_i,v'_i\in\mathbb{C}^m$, $i=1,2$. As $u,v,u_m\in(\mathbb{R}^2\backslash\{(0,0)\})\times\mathbb{R}^{2m-2}$ then $u'_1,v'_1,u'_{m,1}\in\mathbb{C}^*\times\mathbb{C}^{m-1}$. We have $\lim_{m\to+\infty}u'_m=u'$ and so $\lim_{m\to+\infty}u'_{m,1}=u'_1$. Take $B'_m=Q^{-1}B_mQ$. By Lemma 10, $B'_m=\mathrm{diag}(B'_{m,1},\ \overline{B'_{m,1}})$, where $B'_{m,1}\in\mathbb{T}_m(\mathbb{C})$. Then $B'_{m,1}\in G'_1$ and $\lim_{m\to+\infty}B'_mu'_m=v'$. So $\lim_{m\to+\infty}B'_{m,1}u'_{m,1}=v'_1$. By Lemma 9, $(B'_{m,1})_{m\in\mathbb{N}}$ is bounded, so is $(B'_m)_{m\in\mathbb{N}}$. We conclude that $(B_m)_{m\in\mathbb{N}}$ is bounded.

It follows from Lemmas 9 and 11, the following:

Corollary 12. *Let G be a finitely generated abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$. Suppose that* $\mathrm{rank}(F_{G_k})=n_k-1$, $\mathrm{rank}(F_{\widetilde{G}_l})=2m_l-2$, $k=1,\dots,r$; $l=1,\dots,s$. *If* $x,\ y\in U$ *and two sequences* $(B_m)_{m\in\mathbb{N}}\subset G$ *and* $(x_m)_{m\in\mathbb{N}}\subset\mathbb{R}^n$ *such that* $\lim_{m\to+\infty}x_m=x$ *and* $\lim_{m\to+\infty}B_mx_m=y$ *then* $(B_m)_{m\in\mathbb{N}}$ *is bounded.*

Proposition 13. *[8] Let G be a finitely generated abelian sub-semigroup of $M_n(\mathbb{R})$. Then G is hypercyclic if and only if $J_G(x)=\mathbb{R}^n$ for every $x\in\mathbb{R}^n$.*

Proof. [Proof of Theorem 1] One can assume, by Proposition 6, that G is a sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$. Suppose that $J_G(u)=\mathbb{R}^n$. Then by Proposition 8, $\mathrm{rank}(F_{G_k})=n_k-1$ and $\mathrm{rank}(F_{\widetilde{G}_l})=2m_l-2$, for every $k=1,\dots,r$; $l=1,\dots,s$. Let $y\in U$, then there exist two sequences $(B_m)_{m\in\mathbb{N}}\subset G$ and $(x_m)_{m\in\mathbb{N}}\subset\mathbb{R}^n$ satisfying:

$$\lim_{m\to+\infty}x_m=u\quad\text{and}\quad\lim_{m\to+\infty}B_mx_m=y.$$

So by Corollary 12, $(B_m)_{m\in\mathbb{N}}$ is bounded: $\|B_m\| \leq M$ for some $M > 0$, where $\|.\|$ is the Euclidean norm on $\mathbb{R}^n$. Then

$$\begin{aligned}\|B_m u - y\| &= \|B_m u - B_m x_m + B_m x_m - y\| \\ &\leq \|B_m u - B_m x_m\| + \|B_m x_m - y\| \\ &\leq \|B_m\|\|u - x_m\| + \|B_m x_m - y\| \\ &\leq M\|u - x_m\| + \|B_m x_m - y\|\end{aligned}$$

Thus $\lim\limits_{m\to+\infty} B_m u = y$ and so $y \in \overline{G(u)}$. It follows that $U \subset \overline{G(u)}$. Since $\overline{U} = \mathbb{R}^n$, we get $\overline{G(u)} = \mathbb{R}^n$.

Proof of Corollary 2. $(i) \Longrightarrow (ii)$ follows from Proposition 13.
$(ii) \Longrightarrow (iii)$ results from Theorem 1. $(iii) \Longrightarrow (i)$ is clear. □

Proof of Corollary 3. If G is not hypercyclic then by Theorem 1,
$J_G(v) \neq \mathbb{R}^n$ for any $v \in V$. Thus $V \cap E = \emptyset$ and therefore $E \subset \mathbb{R}^n \backslash V = P(\bigcup\limits_{k=1}^{r} H_k \cup \bigcup\limits_{l=1}^{s} F_l)$, where

$$H_k := \left\{u = [u_1,\dots,u_r;\ \widetilde{u}_1,\dots,\widetilde{u}_s]^T \in \mathbb{R}^n;\ u_j \in \mathbb{R}^{n_j}, u_k \in \{0\} \times \mathbb{R}^{n_k-1},\ 1 \leq j \neq k \leq r\right\}$$

and $F_l := \left\{u = [u_1,\dots,u_r;\ \widetilde{u}_1,\dots,\widetilde{u}_s]^T \in \mathbb{R}^n :\ \widetilde{u}_l \in \{(0,0)\} \times \mathbb{R}^{2m_l-2},\ 1 \leq l \leq s\right\}$. □

4 Proof of Theorem 4 and Proposition 5

Lemma 14 ([6], Lemma 2.6)**.** *Let $a, b \in \mathbb{R}$ with $-1 < a < 0$, $b > 1$ and $\dfrac{\log|a|}{\log b}$ is irrational. Then the set $\{a^k b^l : k, l \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Proof. [Proof of Theorem 4] Consider the abelian sub-semigroup G of $GL(n,\mathbb{R})$ generated by $B, A_1, \dots, A_n$, where $B = bI_n$ and $A_k = \mathrm{diag}(\underbrace{1,\dots\dots,1}_{(k-1)-terms}, a, 1\dots,1)$, $k = 1,\dots,n$. Then G is a sub-semigroup of $\mathscr{K}^*_\eta(\mathbb{R})$ with $r = n$ and $\eta = (1,\dots,1)$. One has $u_\eta = [1,\dots,1]^T$.

• First, we will show that G is not hypercyclic: for this, it is equivalent to prove, by Corollary 2, that $\overline{G(u_\eta)} \neq \mathbb{R}^n$: We have

$$G(u_\eta) = \left\{[b^m a^{k_1};\ b^m a^{k_2}; \dots\dots;\ b^m a^{k_n}]^T :\ m, k_1, \dots, k_n \in \mathbb{N}\right\}$$

Observe that for every $x = [x_1,\dots,x_n]^T \in G(u_\eta)$, we have $\dfrac{x_1}{x_2} = a^{k_1-k_2}$. Since the set $\{a^p :\ \ p \in \mathbb{Z}\}$ is not dense in $\mathbb{R}$, the orbit $G(u_\eta)$ cannot be dense in $\mathbb{R}^n$.

• Second, we will show that $J_G(e_1) = \mathbb{R}^n$ (the other e_k work in the same way). Fix a vector $y = [y_1,\dots,y_n]^T \in \mathbb{R}^n$ such that $y_1 \neq 0$. By Lemma 14, choose two sequences of positive integers $(i_m)_{m\in\mathbb{N}}$ and $(j_m)_{m\in\mathbb{N}}$ with $i_m,\ j_m \to +\infty$ such that $\lim\limits_{m\to+\infty} a^{i_m} b^{j_m} = y_1$. Let $x^{(m)} = (1, x_2^{(m)}, \dots, x_n^{(m)})$ with $x_k^{(m)} = \frac{y_k}{a^{i_m} b^{j_m} a^m}$, $k = 2,\dots,n$. Since $-1 < a < 0$ and $y_1 \neq 0$, we see that $\lim\limits_{m\to+\infty} x^{(m)} = e_1$. On the other hand, consider
$B_m := B^{j_m} A_1^{i_m} A_2^{i_m+m} A_3^{i_m+m} \dots A_n^{i_m+m}$. Then $B_m x^{(m)} = \left[a^{i_m} b^{j_m}, y_2, \dots, y_n\right]^T$ and so, $\lim\limits_{m\to+\infty} B_m x^{(m)} = y$. We conclude that $y \in J_G(e_1)$ and therefore $J_G(e_1) = \mathbb{R}^n$.

Proof of Proposition 5. Since $(e'_1,\dots,e'_n)$ is a basis of $\mathbb{R}^n$, there exists $i_0 \in \{1,\dots,n\}$ such that $e'_{i_0} \in \mathbb{R}^* \times \mathbb{R}^{n-1}$. As $V = U = \mathbb{R}^* \times \mathbb{R}^{n-1}$ and

$J_G(e'_{i_0}) = \mathbb{R}^n$ then by Theorem 1, $\overline{G(e'_{i_0})} = \mathbb{R}^n$ and hence G is hypercyclic. $\square$

Acknowledgements. The author would like to thank the referee for valuable comments and suggestions. This work was supported by the research unit: “Dynamical systems and their applications”, UR17ES21, Ministry of Higher Education and Scientific Research, Tunisia.

References

[1] A. Ayadi and H. Marzougui, *Hypercyclic abelian semigroups of matrices on* $\mathbb{R}^n$, Topology Appl., **210** (2016), 29–45. doi 10.1016/j.topol.2016.07.007
[2] A. Ayadi and H. Marzougui, *J-class abelian semigroups of matrices on* $\mathbb{C}^n$ *and Hypercyclicity*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **108** (2014), 557–566. doi 10.1007/s13398-013-0126-6
[3] M.R. Azimi and V. Muller, *A note on J-sets of linear operators*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **105** (2011), 449-453. doi 10.1007/s13398-011-0042-6
[4] F. Bayart and E. Matheron, Dynamics of Linear Operators. Cambridge Tracts in Math., 179, Cambridge University Press, (ISBN 978-0-521-51496-5/hbk). xiv, 337 p. (2009).
[5] K. Chan and I. Seceleanu, *Hypercyclicity of shifts as a zero-one law of orbital limit points*, J. Operator Theory, **67** (2012), 257–277.
[6] G. Costakis, D. Hadjiloucas, and A. Manoussos, *Dynamics of tuples of matrices*, Proc. Amer. Math. Soc. **137**, (2009) 1025–1034. doi 10.1090/S0002-9939-08-09717-7
[7] G. Costakis and A. Manoussos, *J -class operators and hypercyclicity*, J. Operator Theory, **67** (2012), 101–119.
[8] G. Costakis, D. Hadjiloucas and A. Manoussos, *On the minimal number of matrices which form a locally hypercyclic, non-hypercyclic tuple*. J. Math. Anal. Appl. **365** (2010) 229–237. doi 10.1016/j.jmaa.2009.10.020
[9] K.G. Grosse-Herdmann and A. Peris Manguillot, Linear Chaos. Universitext, Springer, 2011. doi 10.1007/978-1-4471-2170-1
[10] A.B. Nasseri, *On the existence of J-class operators on Banach spaces*, Proc. Amer. Math. Soc. **140** (2012), 3549–3555. doi 10.1090/S0002-9939-2012-11200-6