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Conservation laws for a Boussinesq equation.

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Abstract

In this work, we study a generalized Boussinesq equation from the point of view of the Lie theory. We determine all the low-order conservation laws by using the multiplier method. Taking into account the relationship between symmetries and conservation laws and applying the multiplier method to a reduced ordinary differential equation, we obtain directly a second order ordinary differential equation and two third order ordinary differential equations.

Keywords: Generalized Boussinesq equation, Conservation Laws, Potential Symmetries, Double reduction
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1 Introduction

There have been several generalizations of the Boussinesq equation such as the modified Boussinesq equation, or the dispersive water wave. The following generalization of the Boussinesq equation

$$u_{tt} - au_{xx} - (u^{m+1})_{xx} - b\,(u(u^m)_{xx})_{xx} = 0 \tag{1}$$

was introduced in [1]by P. Rosenau to extend the Boussinesq equation

$$u_{tt} - u_{xx} - cu_{xxxx} - (u^2)_{xx} = 0, \tag{2}$$

in order to include nonlinear dispersion to the effect that the new equations support compact and semi-compact solitary structures in higher dimensions, where a and b are arbitrary constants. Eq.(1) describe for $a = 0$ the vibrations of a purely an harmonic lattice and support travelling structures with a compact support [1,2].

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In [3] Bruzón and Gandarias made a full analysis of equations (1), by using classical symmetries, nonclassical symmetries and nonclassical potential symmetries, and they obtained new solutions. The authors also obtained some Type-II hidden symmetries of equation (1) with $m = 1$ and $a = \lambda^2$.
The Boussinesq equation (2), where $u = u(x,t)$ is a sufficiently often differentiable function, belongs to the KdV family of equations and describes motions of long waves in shallow water under gravity propagating in both directions, which for $c = -1$ gives the good Boussinesq or well-posed equation, while for $c = 1$ the bad or ill-posed classical equation.
Clarkson *et al* studied the symmetry reductions and exact solutions of some Boussinesq equations [4–8]. In [6] the authors presented some new similarity reductions of equation (2). The symmetries were determined by using a direct method and cannot be obtained by using the Lie group method for finding group-invariant solutions.
Bruzón and Gandarias applied the theory of groups transformations and the nonclassical method to derive exact solutions of some Boussinesq equations [9–13].
Symmetry reductions and exact solutions have several different important applications in the context of differential equations. Since solutions of partial differential equations asymptotically tend to solutions of lower-dimensional equations obtained by symmetry reduction, some of these special solutions will illustrate important physical phenomena. In particular, exact solutions arising from symmetry methods can often be used effectively to study properties such as asymptotics and “blow-up”. Furthermore, explicit solutions (such as those found by symmetry methods) can play an important role in the design and testing of numerical integrators; these solutions provide an important practical check on the accuracy and reliability of such integrators, [8].
It is known that conservation laws play a significant role in the solution process of an equation or a system of differential equations. Although not all of the conservation laws of partial differential equations (PDEs) may have physical interpretation they are essential in studying the integrability of the PDEs. For variational problems, the Noether theorem can be used for the derivation of conservation laws. For non variational problems there are different methods for the construction of conservation laws. In [14, 15], Anco and Bluman gave a general algorithmic method to find all conservations laws for evolution equations like Eq. (1).
In this work, in order to derive conservation laws for equation (1) we will apply the multipliers method. We use these conservation laws to obtain associated potential systems. We use of the resulting potential systems to investigate nonlocal symmetries of equation (1). Furthermore, taking into account the relationship between symmetries and conservation laws and applying the multiplier method to a reduced ordinary differential equation, we obtain directly a second order ordinary differential equation and two third order ordinary differential equations.

2 Conservation laws

Anco and Bluman gave a general treatment of a direct conservation law method for partial differential equations expressed in normal form. An Nth-order PDE is in normal form if it can be expressed in a solved form for some leading derivative of u such that all the other terms in the equation contain neither the leading derivative nor its differential consequences. For (1) a conservation law can be expressed in an equivalent form by a divergence identity

$$D_tT + D_xX = (u_{tt} - au_{xx} - (u^{m+1})xx - b\,(u(u^m)_{xx})_{xx})Q$$

this identity is called the characteristic equation for the conserved density and flux. The nontrivial conservation laws are characterized by a multiplier

$$Q\,(t,x,u,u_t,u_x,u_{tx},u_{xx},u_{xxx})$$

with no dependence on u_{tt}. The corresponding conserved densities and fluxes depend at most on

$$t,x,u,u_t,u_x,u_{xx},u_{tx},u_{xx},u_{xxx}$$

The determining equation for all multipliers $Q(t,x,u,u_t,u_x,u_{xx},u_{tx},u_{xx},u_{xxx})$ of low-order conservation laws admitted by the equation consists of

$$E_u\left(u_{tt}-au_{xx}-(u^{m+1})xx-b\left[u(u^m)_{xx}\right]_{xx}\right)Q=0$$

which arises from the property that the variational derivative (Euler operator)

$$E_u=\partial_u-D_x\partial_{u_x}-D_t\partial_{u_t}+D_x^2\partial_{u_{xx}}+D_t^2\partial_{u_{tt}}+D_xD_t\partial_{u_{xt}}-\ldots$$

annihilates an expression identically if and only if it is a space-time divergence.

- This condition can be split with respect to and all the t and x derivatives yielding an equivalent overdetermined system of equations on Q.
- For any solution Q of the multiplier determining equation, corresponding conserved density and fluxes can be recovered through integration of the characteristic equation.

There is a one-to-one correspondence between equivalence classes of non-trivial low-order conservation laws and non-zero low-order multipliers. We get the following classification result. The generalized Bousinesq equation admits the following low-order multipliers: For m arbitrary $m\neq 0$, $b\neq 0$

$$\begin{array}{c} Q_1=1,\ Q_2=t \\ Q_3=x,\ Q_4=tx \end{array} \tag{3}$$

Associated to the multipliers we obtain the corresponding conserved densities and fluxes are the following:

1. For the multiplier $Q_1=1$, we obtain the following conservation law:

$$\begin{aligned} &T_1=u_t,\\ &X_1=-mu^m bu_{xxx}-3\,\tfrac{u_{xx}bu_xm^2u^m}{u}+2\,\tfrac{u_{xx}bu_xmu^m}{u}-u^mu_x-au_x-\tfrac{bu_x{}^3m^3u^m}{u^2}+2\,\tfrac{bu_x{}^3m^2u^m}{u^2}-\tfrac{bu_x{}^3mu^m}{u^2}-mu^mu_x. \end{aligned}$$

2. For the multiplier $Q_2=t$, we obtain the following conservation law:

$$\begin{aligned} &T_2=tu_t-u,\\ &X_2=-\tfrac{tbu_x{}^3m^3u^m}{u^2}+2\,\tfrac{tbu_x{}^3m^2u^m}{u^2}-\tfrac{tbu_x{}^3mu^m}{u^2}-3\,\tfrac{tbu_xu_{xx}m^2u^m}{u}+2\,\tfrac{tmu^mbu_xu_{xx}}{u}-tmu^mbu_{xxx}-tmu^mu_x\\ &-tu^mu_x-tau_x. \end{aligned}$$

3. For the multiplier $Q_3=x$, we obtain the following conservation law:

$$\begin{aligned} &T_3=xu_t,\\ &X_3=-xbmu^mu_{xxx}-3\,\tfrac{u_{xx}xbu_xm^2u^m}{u}+2\,\tfrac{u_{xx}xbu_xmu^m}{u}+u^mbmu_{xx}-xmu^mu_x+\tfrac{bu_x{}^2m^2u^m}{u}-\tfrac{bu_x{}^2mu^m}{u}+2\,\tfrac{xbu_x{}^3m^2u^m}{u^2}\\ &-\tfrac{xbu_x{}^3mu^m}{u^2}-\tfrac{xbu_x{}^3m^3u^m}{u^2}-xu^mu_x-xau_x+uu^m+au. \end{aligned}$$

4. For the multiplier $Q_4=tx$, we obtain the following conservation law:

$$\begin{aligned} &T_4=x\left(tu_t-u\right),\\ &X_4=-\tfrac{txbu_x{}^3m^3u^m}{u^2}+2\,\tfrac{txbu_x{}^3m^2u^m}{u^2}-\tfrac{txbu_x{}^3mu^m}{u^2}-3\,\tfrac{tu_{xx}xbu_xm^2u^m}{u}+2\,\tfrac{tu_{xx}xbu_xmu^m}{u}-txbmu^mu_{xxx}+\tfrac{tbu_x{}^2m^2u^m}{u}\\ &-\tfrac{tbu_x{}^2mu^m}{u}+tu^mbmu_{xx}-txmu^mu_x-txu^mu_x-txau_x+tuu^m+aut. \end{aligned}$$

3 Classical Potential Symmetries

Associated to conservation laws (T_1,X_1), (T_2,X_2), (T_3,X_3) and (T_4,X_4) we get the corresponding associated systems:

$$\begin{aligned} &v_x = -u_t, \\ &v_t = au_x + \left(u^{m+1}\right)_x + b\left[u\left(u^m\right)_{xx}\right]_x. \end{aligned} \tag{4}$$

$$\begin{aligned} &v_x = -tu_t + u, \\ &v_t = \frac{tbu_x{}^3m^3u^m}{u^2} - 2\,\frac{tbu_x{}^3m^2u^m}{u^2} + \frac{tbu_x{}^3mu^m}{u^2} + 3\,\frac{tbu_xu_{xx}m^2u^m}{u} - 2\,\frac{tmu^mbu_xu_{xx}}{u} + tmu^mbu_{xxx} + tmu^mu_x \\ &+ tu^mu_x + tau_x. \end{aligned} \tag{5}$$

$$\begin{aligned} &v_x = -xu_t, \\ &v_t = xbmu^mu_{xxx} + 3\,\frac{u_{xx}xbu_xm^2u^m}{u} - 2\,\frac{u_{xx}xbu_xmu^m}{u} - u^mbmu_{xx} + xmu^mu_x - \frac{bu_x{}^2m^2u^m}{u} + \frac{bu_x{}^2mu^m}{u} \\ &- 2\,\frac{xbu_x{}^3m^2u^m}{u^2} + \frac{xbu_x{}^3mu^m}{u^2} + \frac{xbu_x{}^3m^3u^m}{u^2} + xu^mu_x + xau_x + uu^m + au. \end{aligned} \tag{6}$$

$$\begin{aligned} &v_x = -x\left(tu_t - u\right), \\ &v_t = \frac{txbu_x{}^3m^3u^m}{u^2} - 2\,\frac{txbu_x{}^3m^2u^m}{u^2} + \frac{txbu_x{}^3mu^m}{u^2} + 3\,\frac{tu_{xx}xbu_xm^2u^m}{u} - 2\,\frac{tu_{xx}xbu_xmu^m}{u} + txbmu^mu_{xxx} \\ &- \frac{tbu_x{}^2m^2u^m}{u} + \frac{tbu_x{}^2mu^m}{u} - tu^mbmu_{xx} + txmu^mu_x + txu^mu_x + txau_x + tuu^m - aut. \end{aligned} \tag{7}$$

The basic idea for obtaining *classical potential* symmetries is to require that the infinitesimal generator

$$X = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \phi_1(x,t,u,v)\frac{\partial}{\partial u} + \phi_2(x,t,u,v)\frac{\partial}{\partial v} \tag{8}$$

leaves invariant the set of solutions of (4). This yields to an overdetermined, non linear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi_1(x,t,u,v)$ and $\phi_2(x,t,u,v)$. We obtain classical potential symmetries if

$$(\xi_v)^2 + (\tau_v)^2 + (\phi_{1,v})^2 \neq 0. \tag{9}$$

The classical method applied to (4), (5), (6) and (7) leads to the classical symmetries.

4 Nonclassical Potential Symmetries

The basic idea for obtaining *nonclassical potential* symmetries is that the potential system (4) is augmented with the invariance surface conditions

$$\xi u_x + \tau u_t - \phi_1 = 0, \qquad \xi v_x + \tau v_t - \phi_2 = 0, \tag{10}$$

which is associated with the vector field

$$X_1 = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \phi_1(x,t,u,v)\frac{\partial}{\partial u} + \phi_2(x,t,u,v)\frac{\partial}{\partial v}. \tag{11}$$

By requiring that both (4) and (10) are invariant under the transformations with infinitesimal generator (11) one obtains an overdetermined, nonlinear system of equations for the infinitesimals $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi_1(x,t,u,v)$ and $\phi_2(x,t,u,v)$.
In the case $\tau \neq 0$, without loss of generality, we may set $\tau(x,t,v) = 1$. The nonclassical method applied to (4) yields to the classical symmetries.
In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and we obtain overdetermined non linear system of equations for the infinitesimals ϕ_1 and ϕ_2 which is solve by making ansatz on the form of $\phi_1(x,t,u,v)$ and $\phi_2(x,t,u,v)$. In this way we have found one solution.
For $a = 0$ and $m = -1$ we obtain the infinitesimal generators

$$\xi = 1, \;\; \tau = 0, \;\; \phi_1 = k\,u\,\psi(v), \;\; \phi_2 = \omega(x,v),$$

where k is constant and ω and ψ satisfies $-k\,\psi\,\omega + \frac{\partial\omega}{\partial x} + \omega\frac{\partial\omega}{\partial v} = 0$.
In the case that $\omega = \omega(v)$ the infinitesimal generators are:

$$\xi = 1, \;\; \tau = 0, \;\; \phi_1 = u\frac{d\omega}{d\,v}, \;\; \phi_2 = \omega(v).$$

We obtain the nonclassical potential symmetry reduction

$$z = t, \;\; u = \exp\left(kx\frac{d\omega}{d\,v}\right)h_1(t)$$

and v is given by $\displaystyle\int \frac{dv}{\omega(v)} = kx + h_2(t).$

5 Double reduction

Conservation laws that are symmetry invariant have some important applications. It is well known that when a differential equation admits a Noether symmetry, a conservation law is associated with this symmetry, and furthermore that a double reduction can be achieved as a result of this association. Moreover, any symmetry-invariant conservation law will reduce to a first integral for the ODE obtained by symmetry reduction of the given PDE when symmetry-invariant solutions $u(x,t)$ are sought. In [16–18] the relationship between symmetries and conservation laws, has been used to find a double reduction of partial differential equations with two independent variables. This provides a direct reduction of order of the ODE.
A powerful application of conservation laws taking into account the relationship between Lie symmetries and conservation laws it is the so called double reduction method [18]. This method allow us to reduce directly Eq. (1) to a third order ordinary differential equation. In [18] Sjöberg introduced a method in order to get solutions of a qth partial differential equation from the solutions of an ordinary differential equation of order $q-1$ called *double reduction method*. This method can be applied when a symmetry $\mathbf{v}$ is associated to a conserved vector T [16, 18]. In [19] a further connection between symmetries and conservation laws by focusing on conservation laws that are invariant (or, more generally, homogeneous) under the action of a given set of symmetries has been explored. Some applications of symmetry-invariant conservation laws will also be discussed.
The notion of symmetry invariance of a conservation law has be defined and studied. This main result yields a direct condition for invariance (and homogeneity) formulated in terms of multipliers. Some applications to finding symmetry-invariant conservation laws and finding symmetry-invariant solutions of PDEs has been outlined. In [19], simple conditions are given for characterizing when a conservation law and its associated conserved quantity are invariant (and, more generally, homogeneous) under the action of a symmetry.
In the case $m = 1$ the reduced ODE by traveling waves $u = h(x-ct)$ is

$$-bhh'''' - 2bh'h''' - bh''^2 + c^2h'' - ah'' - 2hh'' - 2h'^2 = 0$$

Searching for multipliers $Q(z,h,h')$ we find two multipliers $Q_1 = z$ $Q_2 = 1$ and the corresponding conservation laws

$$D_z(h^2 + \left(\left(-bh''' - 2h'\right)z + bh'' - c^2 + a\right)h - zh'\left(bh'' - c^2 + a\right) = 0,$$

$$D_z(\left(-bh'' + c^2 - a - 2h\right)h' - bhh''') = 0.$$

Consequently

$$(h^2 + \left(\left(-bh''' - 2h'\right)z + bh'' - c^2 + a\right)h - zh'\left(bh'' - c^2 + a\right) - k_1 = 0$$

and

$$(\left(-bh'' + c^2 - a - 2h\right)h' - bhh''') - k_2 = 0.$$

Solving both equations in h''' we get directly the reduced second order ODE

$$h'' = \frac{c^2}{b} - \frac{a}{b} - \frac{k_2 z}{bh} - \frac{h}{b} + \frac{k_1}{bh}.$$

Setting $k_1 = k_2 = 0$ we get the exact solution

$$h(z) = \mathrm{e}^{\frac{z}{\sqrt{b}}} c_2 + \mathrm{e}^{-\frac{z}{\sqrt{b}}} c_1 - c^2 - a.$$

6 Conclusions

For the generalized Boussinesq equation (1) we have derived all the low-order conservation laws by using the multiplier method. Moreover we have considered potential and nonclassical potential symmetries for some of the associated systems. Taking into account the relationship between symmetries and conservation laws and applying the multiplier method to a reduced ordinary differential equation, we have obtained a second order ordinary differential equation and two third order ordinary differential equations.

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