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A note on strongly nonlinear parabolic variational inequalities

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Abstract

We prove the existence of weak solutions of variational inequalities for general quasilinear parabolic operators of order $m = 2$ with strongly nonlinear perturbation term. The result is based on a priori bound for the time derivatives of the solutions.

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1 Introduction

Consider the parabolic initial-boundary value problem

$$\begin{cases} u_t + A(u) + G(u) = f & \text{in } Q_T; \\ u(0) = 0 & \text{in } \Omega; \\ D^\alpha u = 0 & \text{on } \partial\Omega \times]0,T[\text{ for } |\alpha| \leq m-1 \end{cases}$$

on a cylinder $Q_T = \Omega \times]0,T[$ over a bounded smooth domain $\Omega \subset \mathbb{R}^N$, where

$$A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,t;Du(x,t)), \quad G(u) = g(x,t;u) \tag{1}$$

and $Du = (D^\alpha u)_{|\alpha| \leq m}$. If the coefficients A_α satisfy at most polynomial growth conditions in u and its space derivatives while g obeys no growth in u, but merely a sign condition, Landes and Mustonen [6] proved that

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the usual truncation can be utilized to obtain weak solutions of (1) when $m = 1$. In [1], Brézis and Browder considered (1) but under stronger hypotheses on g. Roughly speaking, they required g to be controlled from above and below by the derivative of some convex function. In [5], Landes proved that this assumption is not necessary provided a certain a priori bound for the time derivative of solutions were needed. In [2], Browder and Breézis established an existence and uniqueness result for a general class of variational inequalities for (1) when g obeys no growth condition while A is a regular elliptic operator. Their proof is based on a type of compactness result. In this note, we extend the result of [5] to the corresponding class of variational inequalities under weaker assumptions.

2 Assumptions and the main result

We start by assuming the following hypotheses.

A_1 $A_\alpha(x,t,\xi) : \Omega\times]0,T[\times\mathbb{R}^s \to \mathbb{R}$ is continuous in t and ξ for almost all x and measurable in x for all t and ξ. Moreover, there exist a constant c_1 and a function $\lambda_1 \in L^{p'}(Q_T)$ with $p \in]1,\infty[, p' = \frac{p}{p-1}$ such that

$$|A_\alpha(x,t,\xi)| \leq c_1|\xi|^{p-1} + \lambda_1(x,t) \text{ for all } (x,t) \in Q_T \text{ and } \xi \in \mathbb{R}^s.$$

A_2 $\sum_{|\alpha|\leq m} [A_\alpha(x,t,\xi) - A_\alpha(x,t,\xi^*)]\,(\xi_\alpha - \xi^*_\alpha) \geq 0$ for all $(x,t) \in Q_T$ and $\xi \neq \xi^*$ in $\mathbb{R}^s$.

A_3 There exists a constant $c_2 > 0$ and a function $\lambda_2 \in L^2(Q_T)$ such that $\sum_{|\alpha|\leq m} A_\alpha(x,t,\xi)\xi_\alpha \geq c_1|\xi|^p - \lambda_2(x,t)$ for all $(x,t) \in Q_T$ and $\xi \in \mathbb{R}^s$.

A_4 There is a function $F(x,t,\xi)$ continuous in ξ, measurable in x and differentiable in t such that $\frac{\partial F}{\partial \xi_\alpha} = A_\alpha$ for all $(x,t) \in Q_T$ and all α with $|\alpha| \leq m$.

G (i) $g(x,t,r)\Omega\times]0,T[\times\mathbb{R} \to \mathbb{R}$ is continuous in t and r for almost all x and measurable in x for all t and ξ. Moreover,

$$|g(x,t,r)| \leq \lambda_4(x,t)\psi(r)$$

for some continuous function $\psi : \mathbb{R} \to \mathbb{R}$ and $\lambda_4 \in L^1(Q_T)$.

(ii) $g(x,t,r)r \geq -\lambda_5(x,t)$ for some function $\lambda_5 \in L^1(Q_T)$.

D There exists a function $\tilde{f} \in L^2(Q_T)$ such that $(f,v) = \int_{Q_T} \tilde{f}(x,t)v(x,t)dxdt$.

The function spaces we shall deal with will be obtained by the completion of the space of smooth functions with respect to the appropriate norm. We denote by

$$X = L^p(0,T;W_0^{m,p}(\Omega)) = \mathscr{C}^1\overline{(0,T,\mathscr{C}_0^\infty(\Omega))}^{\|\ \|_{p;m,p}}, \qquad L^p(Q_T) = \overline{\mathscr{C}_0^\infty(\Omega)}^{\|\ \|_{p;p}},$$

where

$$\|u\|^p_{p;m,p} = \int_0^T \|u\|^p_{m,p}dt = \int_0^T \sum_{|\alpha|\leq m} \|D^\alpha u\|^p_p dt$$

$$\|u\|_{p;p} = \int_0^T \|u\|^P_P dt \text{ and } \|u\|^P_P = \int_\Omega |u|^p dx.$$

Put $W = X \cap C(0,T;L^2(\Omega))$. Finally, we choose a sequence $(\Phi_i)_{i=1}^\infty \subset \mathscr{C}_0^\infty(\Omega)$ such that $\cup_{n=1}^\infty V_n$ with $V_n =$ span$(\Phi_1,\Phi_2,\ldots,\Phi_n)$ is dense in $W^{j,p}(\Omega) : jp > mp+N$.

Denote by $Y_n = C(0,T;V_n)$. Since the closure of $\cup_{n=1}^\infty Y_n$ with respect to the C^m-topology contains $\mathscr{C}_0^\infty(Q_T)$, then for $f \in L^2(Q_T)$ there exists $f_k \in \cup_{n=1}^\infty Y_n$ such that $f_k \to f$ in $L^2(Q_T)$ [4]. For simplicity, we fix the constant c throughout this note. Now we are in a position to give our result.

Theorem. *Let K be a closed convex subset of $C(0,T;L^2(\Omega))$ with $0 \in K$. Let the hypotheses $A_1 - A_4, G$, and D be satisfied. Then for a given $f \in W^*$ there exists a weak solution $u \in W \cap K$ with $u(0) = 0$ such that*

$$\langle \dot{u}, v-u\rangle + \langle A(u), v-u\rangle + \int_{Q_T} g(x,t,u)(v-u)\,dxdt \geq \langle f, v-u\rangle \text{ for all } v \in C^1(0,T;\mathscr{C}_0^\infty(\Omega)) \cap K. \tag{2}$$

Proof. We shall give the proof in several steps. In many stages we may adopt the ideas of [5] and [6]. Let g_k be the truncation of g at level $k \in \mathbb{N}$:

$$g_k(x,t;u) = \begin{cases} k\frac{g(x,t;u)}{|g(x,t;u)|} & \text{if } |g(x,t;u)| > k \\ g(x,t;u) & \text{otherwise.} \end{cases}$$

Consider the truncated variational inequality

$$\langle \dot{u}, v-u\rangle + \langle A(u), v-u\rangle + \int_{Q_T} g_k(x,t,u)(v-u)\,dxdt \geq \langle \tilde{f}, v-u\rangle \text{ for all } v \in W \cap K. \tag{3}$$

Firstly, we show the existence of a Galerkin solution $u_\varepsilon \in Y_n \cap K$ of (3) with

$$\|\dot{u}_n\|_{2;2} + \|u_n\|_{2;2} + \|u_n\|_{p;m,p} \leq c, \tag{4}$$

where c is a constant not depending on ε, k, and n.

For this aim, let $A_{\alpha,\varepsilon}, g(\alpha,\varepsilon)$, and $\tilde{\tilde{f}}_\varepsilon$ be the Friedrich's mollification in the variables $(x,t) \in \mathbb{R}^{N+1}$ of A_α, g_k, and $\tilde{f}$, respectively. There exists a Galerkin solution $u_\varepsilon \in Y_n \cap K$ for the mollified variational inequality

$$\int_0^\tau (\dot{u}_\varepsilon, v-u_\varepsilon)dt + \int_0^\tau A_\varepsilon(u_\varepsilon, v-u_\varepsilon)dt + \int_{Q_T} g_{k,\varepsilon}(x,t,u_\varepsilon)(v-u_\varepsilon)\,dxdt \geq \int_{Q_T} \tilde{f}_\varepsilon(v-u_\varepsilon)\,dxdt \tag{5}$$

for all $v \in Y_n \cap K$ and all $\tau \in]0,T[$ with

$$\|u_\varepsilon(t)\|_2 \leq c$$

(see [3,4]). Put $v = 0$ in (5). Then we get from A_3 and G the estimate

$$\|u_\varepsilon\|_{p;p} \leq c.$$

On the other hand, given $h > 0, n \in \mathbb{N}$, and any $w_\varepsilon \in Y_n \cap K$, put $v = u_\varepsilon - hw_\varepsilon$ in (5). Then we get

$$\int_0^T (\dot{u}_\varepsilon, w_\varepsilon)dt + \int_0^T (A_\varepsilon(u_\varepsilon), w_\varepsilon)dt + \int_{Q_T} g_{k,\varepsilon}(x,t,u_{eps})w_\varepsilon\,dxdt \leq \int_{Q_T} f_\varepsilon w_\varepsilon\,dxdt.$$

In particular,

$$\begin{aligned} \int_0^T \left(\dot{u}_\varepsilon(t), \frac{u_\varepsilon(t+h)-u_\varepsilon(t)}{h}\right)dt &+ \int_0^T \left(A_\varepsilon(u_\varepsilon), \frac{u_\varepsilon(t+h)-u_\varepsilon(t)}{h}\right)dt \\ &+ \int_{Q_T} g_{k,\varepsilon}(x,t,u_\varepsilon(t))\left(\frac{u_\varepsilon(t+h)-u_\varepsilon(t)}{h}\right)dxdt \\ &\leq \int_0^T (\tilde{f}_\varepsilon(t), \frac{u_\varepsilon(t+h)-u_\varepsilon(t)}{h})dt. \end{aligned}$$

Allowing $h \to 0$, keeping ε fixed, we have

$$\begin{aligned} \int_0^T (\dot{u}_\varepsilon(t), \dot{u}_\varepsilon(t))dt + \int_0^T (A_\varepsilon(u_\varepsilon), \dot{u}_\varepsilon(t))dt + \int_{Q_T} g_{k,\varepsilon}(x,t,u_\varepsilon)\dot{u}_\varepsilon(t)dxdt \\ \leq \int_0^T (\tilde{f}_\varepsilon(t), \dot{u}_\varepsilon(t))dt. \end{aligned}$$

By A_4 and G, we obtain

$$\int_0^T \|\dot{u}_\varepsilon(t)\|_2^2\,dt + \int_0^T \frac{\partial}{\partial t}[F_\varepsilon(x,t;Du_\varepsilon(t)) + \Gamma_{k,\varepsilon}(x,t;u_\varepsilon(t))]\,dxdt \le c\int_0^T \|\dot{u}_\varepsilon(t)\|_2^2\,dt.$$

We may write this inequality in the form

$$\begin{aligned}\int_0^T \|\dot{u}_\varepsilon(t)\|_2^2\,dt &+ \frac{3}{2}\int_0^T \frac{\partial}{\partial t}\int_\Omega[F_\varepsilon(x,t;Du_\varepsilon(t)) + \Gamma_{k,\varepsilon}(x,t;u_\varepsilon(t))]\,dxdt\\ &\le \frac{1}{2}\int_0^T \frac{\partial}{\partial t}\int_\Omega[F_\varepsilon(x,t;Du_\varepsilon(t)) + \Gamma_{k,\varepsilon}(x,t;u_\varepsilon(t))]\,dxdt + c\int_0^T \|\dot{u}_\varepsilon(t)\|_2^2\,dt.\end{aligned}$$

From the mean value theorem for definite integrals, we get

$$\begin{aligned}\int_0^T \|\dot{u}_\varepsilon(t)\|_2^2\,dt &+ cT\int_\Omega\left[\frac{\partial}{\partial t}F_\varepsilon(x,t;Du_\varepsilon(t)) + \frac{\partial}{\partial t}\Gamma_{k,\varepsilon}(x,t;u_\varepsilon(t))\right]dx\\ &\le c + c\int_0^T\int_\Omega\left[\frac{\partial}{\partial t}F_\varepsilon(x,t;Du_\varepsilon(t)) + \frac{\partial}{\partial t}\Gamma_{k,\varepsilon}(x,t;u_\varepsilon(t))\right]dxdt, \quad 0<t<T,\end{aligned}$$

where

$$\Gamma_{k,\varepsilon}(x,t;\rho) = \int_0^\rho g_{k,\varepsilon}(x,t;r)dr.$$

We may invoke Gronwall's lemma to get the estimate

$$\int_\Omega\left[\frac{\partial}{\partial t}F_\varepsilon(x,t;Du_\varepsilon(t)) + \frac{\partial}{\partial t}\Gamma_{k,\varepsilon}(x,t;u_\varepsilon(t))\right]dx \le c.$$

Therefore,

$$\|\dot{u}_\varepsilon\|_{L^2(Q)} \le c.$$

and consequently,

$$\|\dot{u}_\varepsilon\|_{2;2} + \|u_\varepsilon\|_{2;2} + \|u_\varepsilon\|_{p;m,p} \le c, \tag{6}$$

where the constant c is independent of ε, k, and n.

From (6) and in view of Arzelà-Ascoli's theorem, we get

$$u_\varepsilon \to u_n \text{ strongly in } Y_n \text{ and } \dot{u}_\varepsilon \to \dot{u}_n \text{ (weakly) in } L^2(Q_T).$$

Therefore, (5) yields

$$\int_0^T (\dot{u}_n, v-u_n)dt + \int_0^T (A(u_n), v-u_n)dt + \int_{Q_T} g_k(x,t,u_n)(v-u_n)dxdt \ge \int_{Q_T}\tilde{f}(v-u_n)\,dxdt, \tag{7}$$

where $v \in Y_n \cap K$ and $u_n(0) = 0$. By the lower-semicontinuity property of the norms in (6), we get (4).

From (4) and the fixes level of truncation, we get

$$\begin{cases}\dot{u}_n \rightharpoonup \dot{u}_k \text{ (weakly) in } L^2(Q_T)\\ u_n \to u_k \text{ strongly in } L^p(0,T;W_0^{m-1,p}(\Omega)) \text{ and weakly in } C(0,T,L^2(\Omega))\\ A_\alpha(x,t;Du_n) \rightharpoonup h_\alpha(x,t) \text{ (weakly) in } L^{p'}(Q_T), |\alpha| \le m\\ g_k(x,t;u_n) \to g_k(x,t;u_k) \text{ strongly in } L^{p'}(Q_T).\end{cases} \tag{8}$$

Secondly, we show that u_k is a weak solution of (3). For this purpose and in view of (6), it suffices to prove that

$$\limsup_n \int_0^T (A(u_n), u_n - u_k)dt \leq 0. \tag{9}$$

This inequality holds true at least for one subsequence $(v_k) \subset \cup_{n=1}^{\infty} Y_n$. From (7), for any fixed k, we have

$$\int_0^T (\dot{u}_n, v_k - u_n)dt + \int_0^T (A(u_n), v_k - u_n)dt + \int_{Q_T} g_k(x,t,u_n)(v_k - u_n)\,dxdt \geq \int_{Q_T} \tilde{f}(v_k - u_n)\,dxdt. \tag{10}$$

Let v_k be the truncation at level k and the mollification with respect to the time and space variables, respectively, of the Galerkin's solution u_n, i.e., $v_k = ((u_n^k)_\mu)_\sigma$. Letting $n \to \infty$ in (10), taking (8) into account, and the strong convergence of $((u_n^k)_\mu)_\sigma$ into u_k in X with respect to σ, μ, [6], we obtain (9) and hence, $u_k \in W \cap K$ is a weak solution of (3), i.e.,

$$\int_0^T (\dot{u}_k, v - u_k)dt + \int_0^T (A(u_k), v - u_k)dt + \int_{Q_T} g_k(x,t,u_k)(v - u_k)\,dxdt \geq \int_{Q_T} \tilde{f}(v - u_k)\,dxdt \tag{11}$$

for all $v \in W \cap K$. Finally, to show (2), it remains to prove the following assertions:

$$\dot{u}_k \rightharpoonup \dot{u} \text{ (weakly) in } L^2(Q_T), \tag{12}$$

$$u_k \to u \text{ (strongly) in } L^p(0,T;W_0^{m-1,p}(\Omega)), \tag{13}$$

$$\text{and (weakly) in } C(0,T;L^2(\Omega)), \tag{14}$$

$$g_k(x,t;u_k) \to g(x,t;u) \text{ (strongly) in } L^1(Q_T), \tag{15}$$

and

$$D^\alpha u_k(x,t) \to D^\alpha u(x,t) \text{a.e. in } Q_T \text{ for all } |\alpha| \leq m. \tag{16}$$

Assertions (12-15) follow similarly as above and as in [5]. To show 16, it suffices to show

$$\limsup_k \int_0^T (A(u_n), u_k - u)dt \leq 0. \tag{17}$$

Since for any $v \in X$ we may find a sequence (v_ℓ) converging weakly to u, we get from (11)

$$\begin{aligned} \int_0^T (\dot{u}_k, u_k)dt + \int_0^T (A(u_k), u_k - v_\ell)dt + \int_{Q_T} g_k(x,t;u_k)u_k\,dxdt \\ \leq \int_{Q_T} g_k(x,t;u_k)v_\ell\,dxdt + \int_0^T (\dot{u}_k, v_\ell)dt - \int_0^T (f, v_\ell - u_k)dt. \end{aligned}$$

Letting $k \to \infty$, keeping ℓ fixed, taking into account Fatou's lemma, and then allowing $\ell \to \infty$, we obtain (17) and consequently, (2) follows.

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