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New fixed point results in partial quasi-metric spaces

Zahra Mohammadi[‡] and Oscar Valero[⋆] [†]

[‡]Applied Mathematics Department, University of Western Ontario, London, ON N6A5B7, Canada
[⋆]Department of Mathematics and Computer Science, Universidad de las Islas Baleares,
Ctra. de Valldemossa km 7.5, Palma de Mallorca, 07122, Baleares, Spain

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Abstract

In 1970, D.S. Scott gave applications of Kleene's fixed point theorem to describe the meaning of recursive denotational specifications in programming languages. Later on, in 1994, S.G. Matthews and, in 1995, M.P. Schellekens gave quantitative counterparts of the Kleene fixed point theorem which allowed to apply partial metric and quasi-metric fixed point techniques to denotational semantics and asymptotic complexity analysis of algorithms in the spirit of Scott. Recently, in 2005, J.J. Nieto and R. Rodríguez-López made an in-depth study of how to reconcile order-theoretic and metric fixed point techniques in the classical metric case with the aim of providing the existence and uniqueness of solutions to first-order differential equations admitting only the existence of a lower solution. Motivated by the aforesaid fixed point results we prove a partial quasi-metric version, when the specialization order is under consideration, of the fixed point results of Nieto and Rodríguez-López in such a way that the results of Matthews and Schellekens can be retrieved as a particular case.

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1 Introduction

In 1970, D.S. Scott introduced a fixed point technique in order to describe the meaning of recursive denotational specifications in programming languages ([15]). In order to state the aforementioned fixed point result let us recollect some germane concepts about partially ordered sets.

As usual ([4]), a partial order on a nonempty set X is a reflexive, antisymmetric and transitive binary relation $\leq$ on X. A partially ordered set is a pair $(X,\leq)$ such that X is nonempty set and $\leq$ is a partial order

[†]Corresponding author.
Email address: o.valero@uib.es

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on X. Moreover, given a subset $Y \subseteq X$, an upper bound for Y in $(X,\leq)$ is an element $x \in X$ such that $y \leq x$ for all $y \in Y$. The supremum for Y in $(X,\leq)$, if exists, is an element $z \in X$ which is an upper bound for Y and, in addition, satisfies that $z \leq x$ provided that $x \in X$ is an upper bound for Y. We will denote by $\uparrow_{\leq} x$ ($\downarrow_{\leq} x$), with $x \in X$, the set $\{y \in X : x \leq y\}$ ($\{y \in X : y \leq x\}$). Furthermore, a sequence $(x_n)_{n\in\mathbb{N}}$ in $(X,\leq)$ is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integer numbers.

Following [3], a partially ordered set $(X,\leq)$ is said to be chain complete provided that every increasing sequence has a least upper bound, where a sequence $(x_n)_{n\in\mathbb{N}}$ is said to be increasing whenever $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

Of course, a mapping f from a partially ordered set $(X,\leq)$ into itself will be called monotone if $f(x) \leq f(y)$ whenever $x \leq y$. Besides, a mapping from a partially ordered set $(X,\leq)$ into itself is said to be $\leq$-continuous if the least upper bound of the sequence $(f(x_n))_{n\in\mathbb{N}}$ is $f(x)$ for every increasing sequence $(x_n)_{n\in\mathbb{N}}$ whose least upper bound exists and is x. Notice that every $\leq$-continuous function is always monotone.

In the light of the above notions the celebrated Kleene's fixed point theorem can be stated as follows (see, for instance, [3]):

Theorem 1. *Let $(X,\leq)$ be a chain complete partially ordered set and let f be a $\leq$-continuous mapping from $(X,\leq)$ into itself. If there exist $x_0 \in X$ such that $x_0 \leq f(x_0)$, then f has a fixed point x^* which satisfies that x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X,\leq_p)$ and, thus, $x^* \in \uparrow_{\leq_p} x_0$. Moreover, if there exists $y_0 \in X$ such that $x_0 \leq y_0$ and $f(y_0) \leq y_0$, then $x^* \in \downarrow y_0$.*

Since S. Banach proved his celebrated fixed point technique for self-mappings defined between complete metric spaces in 1922, a few extensions to different generalized metric frameworks have been provided in the literature in order to yield quantitative counterparts of the Kleene fixed point theorem which allow to apply metric fixed point techniques to denotational semantics in the spirit of Scott. Thus, in 1994, S.G. Matthews introduced the notion of partial metric space and developed a partial metric fixed point technique in order to develop suitable mathematical tools for describing the meaning of recursive specifications in denotational semantics for programming languages ([8]). With the aim of stating the fixed point result in which such a technique is based on, let us recall a few pertinent notions about partial metric spaces.

From now on, $\mathbb{R}^+$ will denote the set of nonnegative real numbers. Let us recall that, according to [8], a partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$:

(i) $p(x,x) = p(x,y) = p(y,y) \Leftrightarrow x = y$;
(ii) $p(x,x) \leq p(x,y)$;
(iii) $p(x,y) = p(y,x)$;
(iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Clearly a metric on a nonempty set X is a partial metric p on X satisfying, in addition, the following condition for all $x \in X$: (v) $p(x,x) = 0$.

Of course, a partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

On account of [8], each partial metric p on X generates a T_0 topology $\mathscr{T}(p)$ on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Thus, a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X,p) converges to a point $x \in X$ with respect to $\mathscr{T}(p) \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n)$. Moreover, a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and it is finite. Besides, a partial metric space (X,p) is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to $\mathscr{T}(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Following [8], in every partial metric space (X,p) a partial order, the specialization order, $\leq_p$ can be induced as follows: $x \leq_p y \Leftrightarrow p(x,y) = p(x,x)$.

In the light of the exposed notions, the fixed point result which provided the basis for the aforesaid Matthews fixed point technique can be stated as follows (see [8]).

Theorem 2. *Let f be a mapping from a complete partial metric space (X,p) into itself such that there is $c \in [0,1[$ satisfying*

$$p(f(x), f(y)) \leq cp(x,y) \tag{1}$$

for all $x,y \in X$. Then f has a unique fixed point $x^* \in X$ which holds:

(1) x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X, \leq_p)$ for every $x_0 \in X$ and $x^* \in \uparrow_{\leq_p} x_0$ for every $x_0 \in X$.

(2) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^* with respect to $\mathscr{T}(d_p^s)$ for every $x_0 \in X$.

(3) $p(x^*, x^*) = 0$.

Later on, in 1995, M.P. Schellekens introduced a new extension of the classical Banach fixed point theorem. In particualr, he developed a quasi-metric fixed point technique which turned out to be useful in asymptotic complexity analysis of algorithms ([14]). In order to recall it, let us fix some notions about quasi-metric spaces that will be useful later on.

On account on [5, Chapter 11], a quasi-metric on a nonempty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$:

(i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$;
(ii) $d(x,y) \leq d(x,z) + d(z,y)$.

It is clear that a metric d on a nonempty set X is a quasi-metric which holds additionally the property (iv) $d(x,y) = d(y,x)$ for all $x,y \in X$.

A quasi-metric space is a pair (X,d) such that X is a nonempty set and d is a quasi-metric on X.

According to [5, Chapter 11], each quasi-metric d on X generates a T_0-topology $\mathscr{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X, then the function $d^s : X \times X \to \mathbb{R}^+$ is a metric, where $d^s(x,y) = \max(d(x,y), d(y,x))$ for all $x,y \in X$.

Following [12], a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X,d) is said to be left K-Cauchy if, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n \geq n_0$. Moreover, on account of [7], a quasi-metric space (X,d) is Smyth complete provided that every left K-Cauchy sequence is convergent with respect to $\tau(d^s)$.

Similar to the partial metric case, in each quasi-metric space (X,d) a partial order, the specialization order, $\leq_d$ can be defined as follows: $x \leq_d y \Leftrightarrow d(x,y) = 0$.

Taking into account the preceding notions, the aforementioned fixed point result of Schellekens can be stated as follows ([14]):

Theorem 3. *Let (X,d) be a Smyth complete quasi-metric space and let f be a mapping from X into itself such that there exists $c \in [0,1[$ with*

$$d(f(x), f(y)) \leq cd(x,y)$$

for all $x,y \in X$. Then f has a unique fixed point $x^* \in X$ which holds:

(1) the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^* with respect to $\mathscr{T}(d^s)$ for all $x_0 \in X$.

(2) if $x_0 \leq_d f(x_0)$ and then x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X, \leq_d)$ and, thus, $x^* \in \uparrow_{\leq_d} x_0$.

In [6], H.P.A. Künzi, H. Pajooshesh and M.P. Schellekens introduced the concept of partial quasi-metric space with the aim of providing a framework which generalizes both the partial metric space and the quasi-metric space.

Following [6], a partial quasi-metric on a nonempty set X is a function $q : X\times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$:

(i) $q(x,x) = q(x,y)$ and $q(y,y) = q(y,x) \Leftrightarrow x = y$;
(ii) $q(x,x) \leq q(x,y)$;
(iii) $q(x,y) \leq q(x,z)+q(z,y)-q(z,z)$.

It must be stressed that a partial quasi-metric satisfying the preceding assertions is called a lopsided partial quasi-metric in [6].

Notice that a partial metric on a set X is a partial quasi-metric q satisfying additionally the condition: (iv) $q(x,y) = q(y,x)$ for all $x,y \in X$. Besides a quasi-metric on a set X is a partial quasi-metric q satisfying in addition the condition: (iv) $q(x,x) = 0$ for all $x \in X$.

A partial quasi-metric space is a pair (X,q) such that X is a nonempty set and q is a partial quasi-metric on X.

A partial quasi-metric q induces a T_0-topology $\mathscr{T}(q)$ and a partial order $\leq_q$ on X in the same way how partial metrics make it. The completeness can be also literally adapted from the partial metric context to the partial quasi-metric one. A few fixed point theorems for self-mappings between complete partial quasi-metric spaces and applications to Computer Science were given in [2] and [9] recently.

In 2005, J.J. Nieto and R. Rodríguez-López made an in-depth study of how to reconcile order-theoretic and metric fixed point techniques in the classical metric case with the aim of providing the existence and uniqueness of solutions to first-order differential equations admitting only the existence of a lower solution [10] (see also [11]). Let us recall the fixed point results in which the aforementioned techniques are based on:

Theorem 4. *Let $(X,\leq)$ be a partially ordered set and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) There exists $x_0 \in X$ such that $x_0 \leq f(x_0)$.

(2) There exist a metric d on X and $c \in [0,1[$ such that (X,d) is a complete metric space and

$$d(f(x),f(y)) \leq cd(x,y) \tag{2}$$

for all $x,y \in X$ with $y \leq x$.

(3) f is monotone and continuous.

Then there exists $x^* \in X$ such that $x^* \in Fix(f)$.

In the preceding result the continuity of the mapping can be interchanged by a kind of chain-completeness of the partially ordered set obtaining as a consequence the next result.

Theorem 5. *Let $(X,\leq)$ be a partially ordered set and let f be a mapping from X into itself. Assume that the following assertions hold:*

(1) There exists $x_0 \in X$ such that $x_0 \leq f(x_0)$.

(2) There exist a metric d on X and $c \in [0,1[$ such that (X,d) is a complete metric space and

$$d(f(x),f(y)) \leq cd(x,y) \tag{3}$$

for all $x,y \in X$ with $y \leq x$.

(3) f is monotone.

(4) If $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq)$ which converges to $x\in X$ with respect to $\tau(d)$, then x is an upper bound of $(x_n)_{n\in\mathbb{N}}$.

Then there exists $x^*\in X$ such that $x^*\in Fix(f)$.

Motivated by Theorems 4 and 5, our propose is prove counterparts of such results in the context of partial quasi-metric spaces, when the specialization order is considered, in such a way that Theorems 2 and 3 can be retrieved as a particular case. Thus the remainder of the paper is organized in the following way. In Section 2 we introduce a new notion of completeness for partial quasi-metric spaces that generalizes at the same time the completeness used in Theorems 2 and 3 and, in addition, we prove existence of fixed point results for self-mappings satisfying a new contractive condition which is inspired in the contractive condition given in Theorems 4 and 5. Moreover, examples that show that the assumptions in the news results cannot be weakened are given. Section 3 is devoted to discuss the uniqueness of the fixed point guaranteed by the results yielded in Section 2. Hence sufficient conditions that ensure the uniqueness, local and global, of fixed point are provided.

2 Existence of fixed point results

In this section our aim is to prove a partial quasi-metric version of Theorems 4 and 5 in such a way Theorems 2 and 3 can be retrieved as a corollary. To this end, we introduce a new natural notion, different from those used in [2] and [9], of completeness for partial quasi-metric spaces.

According to [6], every partial quasi-metric q on a nonempty set X induces a quasi-metric d_q defined by

$$d_q(x,y)=q(x,y)-q(x,x)$$

for all $x,y\in X$.

From now on, a partial quasi-metric space (X,q) will be said to be Smyth complete provided that the induced quasi-metric space (X,d_q) is Smyth complete. Note that the new notion of completeness coincides with the Smyth completeness and completeness when the partial quasi-metric is exactly a quasi-metric and a partial metric, respectively.

In order to prove a partial quasi-metric version of Theorems 4 and 5 we need to manage a contractive notion that involves the order. Taking this into account we introduce the concept of order-contraction.

In the following, given a partial quasi-metric space (Xq) and $x,y\in X$, we will write $x\bowtie_q y$ whenever either $x\leq_q y$ or $y\leq_q x$.

In the sequel, we will say that a mapping f from a partial quasi-metric space (X,q) into itself is order-contractive provided that there exists $c\in[0,1[$ such that

$$q(f(x),f(y))\leq cq(x,y), \tag{4}$$

for all $x\bowtie_q y$.

Theorem 6. *Let f be an order-contractive mapping from a Smyth complete partial quasi-metric space (X,q) into itself. If there exists $x_0\in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq_q)$, then there exists $x^*\in X$ which satisfies:*

1) $x^*\in\uparrow_{\leq_q}x_0$.

2) x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X,\leq_q)$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_q^s)$.

4) $x^* \in Fix(f)$ if and only if $q(x^*,x^*) = 0$.

Proof. Since $(f^n(x_0))_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq_q)$ we have that

$$q(f^n(x_0), f^m(x_0)) = q(f^n(x_0), f^n(x_0))$$

for all $n,m \in \mathbb{N}$ with $m \geq n$. Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is left K-Cauchy sequence in (X,d_q), since

$$d_q(f^n(x_0), f^m(x_0)) = q(f^n(x_0), f^m(x_0)) - q(f^n(x_0), f^n(x_0)) = 0$$

for all $n,m \in \mathbb{N}$ with $m \geq n$. The Smyth completeness of (X,q) provides the existence of $x^* \in X$ such that $(f^n(x_0))_n \in \mathbb{N}$ converges to x^* in (X,d_q^s). Whence

$$\lim_{n\to\infty} q(x^*, f^n(x_0)) = q(x^*,x^*)$$

and

$$\lim_{n\to\infty} q(f^n(x_0),x^*) = \lim_{n\to\infty} q(f^n(x_0), f^n(x_0))$$

By (4) we deduce that

$$q(f^n(x_0), f^n(x_0)) \leq c^n q(x_0,x_0)$$

for all $n \in \mathbb{N}$. Hence we obtain that $\lim_{n\to\infty} q(f^n(x_0), f^n(x_0)) = 0$. Thus we have that $\lim_{n\to\infty} q(f^n(x_0),x^*) = 0$.

Next we show that in $x^* \in\uparrow x_0$. Since $x_0 \leq_q f^n(x_0)$ for all $n \in \mathbb{N}$ we have that $q(x_0, f^n(x_0)) = q(x_0,x_0)$ for all $n \in \mathbb{N}$. Now, let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $q(f^n(x_0),x^*) - q(f^n(x_0), f^n(x_0)) < \varepsilon$ for all $n \geq n_0$, since $\lim_{n\to\infty} q(f^n(x_0),x^*) = \lim_{n\to\infty} q(f^n(x_0), f^n(x_0)) = 0$. Thus

$$q(x_0,x^*) - q(x_0,x_0) \leq$$

$$q(x_0, f^n(x_0)) + q(f^n(x_0),x^*) - q(f^n(x_0), f^n(x_0)) - q(x_0,x_0) < \varepsilon$$

for all $n \geq n_0$. Whence we obtain that $q(x_0,x^*) - q(x_0,x_0) = 0$ and, hence, that $x_0 \leq_q x^*$, i.e., $x^* \in\uparrow x_0$.

Next we show that x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X,\leq_q)$.
First of all we prove that x^* is an upper bound of $(f^n(x_0))_{n\in\mathbb{N}}$. Since

$$q(f^n(x_0),x^*) - q(f^n(x_0), f^n(x_0)) \leq q(f^m(x_0),x^*)$$

for all $m,n \in \mathbb{N}$ with $n \leq m$ and $\lim_{m\to\infty} q(f^m(x_0),x^*) = 0$, we deduce that $q(f^n(x_0),x^*) - q(f^n(x_0), f^n(x_0)) = 0$ for all $n \in \mathbb{N}$. Consequently, $f^n(x_0) \leq_q x^*$ for all $n \in \mathbb{N}$.

Assume that there exists $y \in X$ such that $f^n(x_0) \leq_q y$ for all $n \in \mathbb{N}$. Then $q(f^n(x_0),y) = q(f^n(x_0), f^n(x_0))$ for all $n \in \mathbb{N}$. Hence

$$q(x^*,y) - q(x^*,x^*) \leq$$

$$q(x^*, f^n(x_0)) + q(f^n(x_0),y) - q(f^n(x_0), f^n(x_0)) - q(x^*,x^*) =$$

$$q(x^*, f^n(x_0)) - q(x^*,x^*).$$

It follows, from $\lim_{n\to\infty} q(x^*, f^n(x_0)) = q(x^*,x^*)$, that $q(x^*,y) = q(x^*,x^*)$ and, hence, that $x^* \leq_q y$. Whence x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$.

Next we assume that $q(x^*,x^*)=0$. In the following our aim is to show that $x^* \in Fix(f)$. To this end we note that

$$q(x^*,f(x^*)) = q(x^*,x^*) = 0,$$

since

$$q(x^*,f(x^*)) \leq q(x^*,f^n(x_0)) + q(f^n(x_0),f(x^*)) - q(f^n(x_0),f^n(x_0))$$

and, in addition, we have, on one hand, by (4) that

$$q(f^n(x_0),f(x^*)) \leq cq(f^{n-1}(x_0),x^*)$$

for all $n \in \mathbb{N}$ and, on the other hand, that

$$\lim_{n\to\infty} q(f^n(x_0),x^*) = \lim_{n\to\infty} q(f^n(x_0),f^n(x_0)) = 0.$$

Thus we deduce that $x^* \leq_q f(x^*)$.

Next we prove that $f(x^*) \leq_q x^*$. On one hand, we have that $q(f^n(x_0),x^*) = q(f^n(x_0),f^n(x_0))$ for all $n \in \mathbb{N}$. On the other hand, we have that

$$\lim_{n\to\infty} q(x^*,f^n(x_0)) = q(x^*,x^*) = 0.$$

Thus we deduce that $q(f(x^*),x^*) = q(f(x^*),f(x^*))$ because

$$q(f(x^*),x^*) - q(f(x^*),f(x^*)) \leq$$

$$q(f(x^*),f^n(x_0)) + q(f^n(x_0),x^*) - q(f^n(x_0),f^n(x_0)) - q(f(x^*),f(x^*)) \leq$$

$$cq(x^*,f^n(x_0)).$$

Consequently, $x^* = f(x^*)$ and, thus, $x^* \in Fix(f)$.

Suppose that $x^* \in Fix(f)$. Next we show that $q(x^*,x^*)=0$. Assume for the purpose of contradiction that $0 < q(x^*,x^*)$. Then, by (4), we have that

$$q(f(x^*),f(x^*)) \leq cq(x^*,x^*).$$

Whence we deduce that $1 \leq c$ which is a contradiction. So $q(x^*,x^*)=0$.

In the next example we show that the Smyth completeness of the partial quasi-metric space cannot be deleted in Theorem 6 in order to guarantee the existence of fixed point.

Example 7. *Let Σ be a nonempty alphabet. Denote by Σ^F the set of finite sequences over Σ and by $l(x)$ the length of $x \in \Sigma^F$. Given $x,y \in \Sigma^F$, we will write $x \sqsubseteq y$ provided that there exists $n_0 \in \mathbb{N}$ such that $l(x) \leq l(y)$, $n_0 \leq l(x)$ and $x(k) \leq x(k)$ for all $k \leq n_0$. Consider the partial quasi-metric $q_{l_\leq} : \Sigma^F \times \Sigma^F \to \mathbb{R}^+$ defined by $q_{l_\leq}(x,y) = 2^{-l_\leq(x,y)}$, where $l_\leq(x,y) = \sup\{n \in \mathbb{N} : x(k) \leq y(k) \text{ for all } k \leq n\}$ whenever $x \sqsubseteq y$, and $l_\leq(x,y) = 0$ otherwise. It is easy to check that $x \leq_{q_{l_\leq}} y \Leftrightarrow x \sqsubseteq y$ and $l_\leq(x,y) = l(x)$.*

It is clear that the partial quasi-metric space $(\Sigma^F, q_{l_\leq})$ is not Smyth complete. Indeed, fix $a \in \Sigma$ and take the sequence $(x_n)_{n\in\mathbb{N}}$ in Σ^F given by $x_n = \overbrace{aa\ldots aa}^{n}$ for all $n \in \mathbb{N}$. Clearly the sequence is a left K-Cauchy sequence and, in addition, it is not convergent with respect to $\tau(d^s_q)$.

Define the monotone mapping $f_a : \Sigma^F \to \Sigma^F$ by $f_a(x) = ax$, where ax denotes the concatenation of a and x. It is clear that $a \leq_{q_{l_\leq}} f_a(a)$. Moreover, we have that the mapping f is order-contractive, since

$$q_{l_\leq}(f_a(x),f_a(y)) \leq \frac{1}{2} q_{l_\leq}(x,y)$$

for all $x,y \in \Sigma^F$ with $x \bowtie_q y$. Nevertheless, f is fixed point free.

The next example shows that the hypothesis of existence of an element $x_0 \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq_q)$ cannot be removed from the statement of Theorem 6.

Example 8. *Let $\Sigma = \{0,1\}$. Denote by Σ^∞ the set of infinite sequences over Σ and let $\Sigma_\infty = \Sigma^F \cup \Sigma^\infty$, where Σ^∞, following Example 7, denotes the set of finite sequences over Σ. Consider the partial quasi-metric space (Σ_∞, q_l), where $q_l(x,y) = 2^{-l(x,y)}$ for all $x,y \in \Sigma_\infty$, where $l(x,y)$ denotes the longest common prefix of x and y. Then the partial quasi-metric space (Σ_∞, q_l) is Smyth-complete. Clearly $x \leq_{q_l} y \Leftrightarrow x$ is a prefix of y.*

Consider the mapping $f : \Sigma_\infty \times \Sigma_\infty$ defined by

$$f(x) = \begin{cases} 0x & if\, first(x) = 1 \\ 1x & if\, first(x) = 0 \end{cases},$$

where $first(x)$ denotes the first element of the sequence x. It is clear that f is order-contractive. Indeed,

$$q_l(f(x),f(y)) \leq \frac{1}{2} q_l(x,y)$$

for all $x,y \in \Sigma_\infty$ such that $x \bowtie_{q_l} y$. It is obvious that does not exist $x_0 \in \Sigma_\infty$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq_{q_l})$. Of course, f is fixed point free.

The next example evidences that the order-contractive condition of the self-mapping cannot be deleted in the statement of Theorem 6 in order to assure the existence of fixed point.

Example 9. *Define the function $q : [0,1]\times[0,1] \to \mathbb{R}^+$ by*

$$q(x,y) = \max\{y-x, 0\} + x$$

for all $x,y \in [0,1]$. It is not hard to check that q is a partial quasi-metric on X and that $x \leq_q y \Leftrightarrow y \leq x$. Moreover, an straightforward computation shows that $([0,1],q)$ is Smyth complete. Define the mapping $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \frac{x}{2} & if\, x \neq 0 \\ \\ \frac{1}{2} & if\, x = 0 \end{cases}.$$

Clearly the sequence $(f^n(\frac{1}{2}))_{n\in\mathbb{N}}$ is increasing in $([0,1],\leq_q)$. However f does not hold the order-contractive condition, since there is no $c \in [0,1[$ such that

$$\frac{1}{2} = q(\frac{1}{2},\frac{1}{2}) = q(f(0),f(0)) \leq cq(0,0) = 0.$$

Moreover, f is fixed point free.

When the order-contractive mapping is, in addition, monotone we obtain the next result.

Corollary 10. *Let f be a monotone order-contractive mapping from a Smyth complete partial quasi-metric space (X,q) into itself and let $x_0 \in X$ such that $x_0 \leq f(x_0)$. Then there exists $x^* \in X$ which satisfies:*

1) x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X,\leq_q)$ and, thus, $x^* \in \uparrow_{\leq_q} x_0$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_q^s)$.

4) $x^* \in Fix(f)$ if and only if $q(x^*,x^*) = 0$.

Proof. It is sufficient to show that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is increasing in $(X,\leq_q)$ and then apply Theorem 6. Since f is monotone and $x_0 \leq f(x_0)$ we immediately obtain that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is increasing in $(X,\leq_q)$.

The next result can be stated when one considers a quasi-metric space, a particular instance of partial quasi-metric spaces, in the statement of Theorem 6.

Corollary 11. *Let f be a monotone mapping from a Smyth complete quasi-metric space (X,q) into itself such that*

$$q(f(x),f(y)) \leq cq(x,y)$$

for all $x,y \in X$ with $y \leq_q x$. If there exists $x_0 \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq_q)$, then there exists $x^* \in X$ which satisfies:

1) $x^* \in \uparrow_{\leq_q} x_0$.
2) *x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X,\leq_q)$.*
3) *The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_q^s)$.*
4) $x^* \in Fix(f)$.

Proof. It is obvious that f is monotone if and only if

$$q(f(x),f(y)) \leq q(x,y)$$

for all $x \leq_q y$. Taking into account the preceding fact and the assumption that

$$q(f(x),f(y)) \leq cq(x,y)$$

for all $x,y \in X$ with $y \leq_q x$, we deduce that f is an order-contractive mapping. Moreover, $q(x,x) = 0$ for all $x \in X$ whenever q is a quasi-metric on X. Applying Theorem 6 we obtain the desaired conclusions.

The below result is obtianed immediately from Corollary 11.

Corollary 12. *Let f be a monotone mapping from a Smyth complete quasi-metric space (X,q) into itself such that*

$$q(f(x),f(y)) \leq cq(x,y)$$

for all $x,y \in X$ with $y \leq_q x$. If there exists $x_0 \in X$ such that $x_0 \leq_q f(x_0)$, then there exists $x^* \in X$ which satisfies:

1) *x^* is the supremum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X,\leq_q)$ and, thus, $x^* \in \uparrow_{\leq_q} x_0$.*
3) *The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_q^s)$.*
4) $x^* \in Fix(f)$.

When the order-contractivity is replaced by the quasi-metric version of the classical Banach contractivity, Theorem 6 provides the following result.

Corollary 13. *Let f be a mapping from a Smyth complete quasi-metric space (X,q) into itself such that*

$$q(f(x),f(y)) \leq cq(x,y)$$

for all $x,y \in X$, where $c \in [0,1[$. If there exists $x_0 \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}}$ is an increasing sequence in $(X,\leq_q)$, then there exists $x^* \in X$ which satisfies:

1) $x^* \in \uparrow_{\leq_q} x_0$.

2) x^* *is the supremum of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \leq_q)$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to* x^* *with respect to* $\tau(d_q^s)$.

4) $x^* \in Fix(f)$.

Proof. We first note that f is order-contractive, since

$$q(f(x), f(y)) \leq cq(x,y)$$

for all $x, y \in X$. Moreover, $q(x,x) = 0$ for all $x \in X$. Consequently, Theorem 6 gives the desired concluions.

According to [8], a quasi-metric space (X,d) is said to be weightable provided the existence of a function (the weight) $w_d : X \to \mathbb{R}^+$ such that

$$d(x,y) + w_d(x) = d(y,x) + w_d(y)$$

for all $x, y \in X$. On account of [7], every bicomplete weightable quasi-metric space is always Smyth complete. Following [13], every partial metric space (X,q) induces a weightable quasi-metric space (X, d_q), where the quasi-metric d_q is defined by

$$d_q(x,y) = q(x,y) - q(x,x)$$

for all $x, y \in X$ and the weight w_{d_q} is given by $w_{d_q}(x) = q(x,x)$ for all $x \in X$.

The next result can be stated when one considers a partial metric space, a particular instance of partial quasi-metric spaces, in the statement of Theorem 6.

Corollary 14. *Let* f *be an order-contractive mapping from a complete partial metric space* (X,q) *into itself. If there exists* $x_0 \in X$ *such that* $(f^n(x_0))_{n\in\mathbb{N}}$ *is an increasing sequence in* $(X, \leq_q)$*, then there exists* $x^* \in X$ *which satisfies:*

1) $x^* \in \uparrow_{\leq_q} x_0$.

2) x^* *is the supremum of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \leq_q)$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$*converges to* x^* *with respect to* $\tau(d_q^s)$.

4) $q(x^*, x^*) = 0$ *and* $x^* \in Fix(f)$.

Proof. By Lemma 3 in [13], a partial metric space is complete if and only if the induced quasi-metric space is bicomplete. It follows that the partial quasi-metric (X, d_q) is bicomplete. Since (X, d_q) is weigthable we deduce that it is Smyth complete. Following the arguments applied to the proof of Theorem 6 we obtain that $\lim_{n\to\infty} q(f^n(x_0), f^n(x_0)) = 0$, $\lim_{n\to\infty} q(f^n(x_0), x^*) = 0$ and that $\lim_{n\to\infty} q(x^*, f^n(x_0)) = q(x^*, x^*)$. The symmetry of q yields that $q(x^*, f^n(x_0)) = q(f^n(x_0), x^*)$ for all $n \in \mathbb{N}$ and, thus, that $q(x^*, x^*) = 0$. So, by Theorem 6, we immediatelly deduce the assertions in the statement.

As a particular case of Corollary 14 we get the following result.

Corollary 15. *Let* f *be a monotone order-contractive mapping from a complete partial metric space* (X,q) *into itself. If there exists* $x_0 \in X$ *such that* $x_0 \leq_q f(x_0)$*, then there exists* $x^* \in X$ *which satisfies:*

1) x^* *is the supremum of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X, \leq_q)$ *and, thus,* $x^* \in \uparrow_{\leq_q} x_0$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$*converges to* x^* *with respect to* $\tau(d_q^s)$.

4) $q(x^*,x^*) = 0$ *and* $x^* \in Fix(f)$.

When the order-contractivity is replaced by the partial metric contractive condition (1), Theorem 6 provides the following two results whose easy proof we omit.

Corollary 16. *Let* f *be a mapping from a complete partial metric space* (X,q) *into itself such that*

$$q(f(x), f(y)) \leq cq(x,y)$$

for all $x,y \in X$*, where* $c \in [0,1[$*. If there exists* $x_0 \in X$ *such that* $(f^n(x_0))_{n\in\mathbb{N}}$ *is an increasing sequence in* $(X,\leq_q)$*, then there exists* $x^* \in X$ *which satisfies:*

1) $x^* \in \uparrow_{\leq_q} x_0$.

2) x^* *is the supremum of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X,\leq_q)$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$*converges to* x^* *with respect to* $\tau(d_q^s)$.

4) $q(x^*,x^*) = 0$ *and* $x^* \in Fix(f)$.

Corollary 17. *Let* f *be a monotone mapping from a complete partial metric space* (X,q) *into itself such that*

$$q(f(x), f(y)) \leq cq(x,y)$$

for all $x,y \in X$*, where* $c \in [0,1[$*. If there exists* $x_0 \in X$ *such that* $x_0 \leq_q f(x_0)$*, then there exists* $x^* \in X$ *which satisfies:*

1) x^* *is the supremum of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X,\leq_q)$ *and, thus,* $x^* \in \uparrow_{\leq_q} x_0$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$*converges to* x^* *with respect to* $\tau(d_q^s)$.

4) $x^* \in Fix(f)$ *if and only if* $q(x^*,x^*) = 0$.

We end this subsection noting that dual versions of our main results, Theorem 6 and Corollary 10, can be obtained following similar arguments to those given in the proof of the aforesaid resutls.

Theorem 18. *Let* f *be an order-contractive mapping from a Smyth complete partial quasi-metric space* (X,q) *into itself. If there exists* $x_0 \in X$ *such that* $(f^n(x_0))_{n\in\mathbb{N}}$ *is a decreasing sequence in* $(X,\leq_q)$*, then there exists* $x^* \in X$ *which satisfies:*

1) $x^* \in \downarrow_{\leq_q} x_0$.

2) x^* *is the infimum of* $(f^n(x_0))_{n\in\mathbb{N}}$ *in* $(X,\leq_q)$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to* x^* *with respect to* $\tau(d_q^s)$.

4) $x^* \in Fix(f)$ *if and only if* $q(x^*,x^*) = 0$.

Proof. We only show that the sequence $\{f^n(x_0)\}_{n\in\mathbb{N}}$ is left K-Cauchy and thus, converges to x^* with respect to $\mathscr{T}(d_q^s)$. With this aim, let $x_0 \in X$ such that $\{f^n(x_0)\}_{n\in\mathbb{N}}$ is decreasing. By (4) we have that

$$q(f^n(x_0), f^{n+1}(x_0)) \leq c^n q(f(x_0), x_0)$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} q(f^n(x_0), f^{n+k}(x_0)) &\leq q(f^n(x_0), f^{n+1}(x_0)) + \ldots + q(f^{m+k-1}(x_0), f^{n+k}(x_0)) \\ &\leq \left(c^n + \ldots + c^{n+k-1}\right) q(f(x_0), x_0) \\ &\leq \tfrac{c^n}{1-c} q(f(x_0), x_0) \end{aligned}$$

for all $k \in \mathbb{N}$. Whence we obtain that $\{f^n(x_0)\}_{n\in\mathbb{N}}$ is a left K-Cauchy sequence in (X,q), since

$$q(f^n(x_0), f^{n+k}(x_0)) - q(f^n(x_0), f^n(x_0)) \leq q(f^n(x_0), f^{n+k}(x_0))$$

for all $k,n \in \mathbb{N}$. The Smyth completeness of (X,q) guarantees the existence of $x^* \in X$ such that $\lim_{n\to\infty} f^n(x_0) = x^*$ with respect to $\mathscr{T}(d_q^s)$.

The proof of the remainder assertions in statement of the result runs as in the proof of Theorem 6.

Corollary 19. *Let f be a monotone order-contractive mapping from a Smyth complete partial quasi-metric space (X,q) into itself. If there exists $x_0 \in X$ such that $f(x_0) \leq_q x_0$, then there exists $x^* \in X$ which satisfies:*

1) x^* is the infimum of $(f^n(x_0))_{n\in\mathbb{N}}$ in $(X,\leq_q)$ and, thus, $x^* \in\downarrow_{\leq_q} x_0$.

3) The sequence $(f^n(x_0))_{n\in\mathbb{N}}$converges to x^* with respect to $\tau(d_q^s)$.

4) $x^* \in Fix(f)$ if and only if $q(x^*,x^*) = 0$.

We end this section discussing whether the main conclusions of our main results, Theorems 6 and 18, can be retrieved as a particular case of Kleene's theorem (Theorem 1):

Fortunately, the answer to the posed question is negative. Although every partially ordered set $(X,\leq_q)$ induced by a Smyth complete partial quasi-metric space (X,q) is always countably chain complete, there exist order-contractive mappings that are not $\leq_q$-continuous. The following example illustrates the preceding fact.

Example 20. *Consider the Smyth complete partial quasi-metric space $([0,1],q)$ where q is the partial quasi-metric on $[0,1]$ given by $q(x,y) = \max\{y-x,0\}+x$ for all $x,y \in [0,1]$. Thus $x \leq_q y \Leftrightarrow y \leq x$. Define the mapping $f : [0,1] \to [0,1]$ by*

$$f(x) = \begin{cases} \frac{x}{2} \text{ if } x \in [0,\frac{1}{2}] \\ \\ \frac{x}{8} \text{ if } x \in]\frac{1}{2},1] \end{cases}.$$

It is nor hard to check that

$$q(f(x),f(y)) \leq \frac{1}{2} q(x,y)$$

for all $x,y \in [0,1]$ such that $x \bowtie_q y$ and, thus, that f is order-contractive. Nevertheless, f is not monotone because $1 \leq_q \frac{1}{2}$ but $\frac{1}{4} = f(\frac{1}{2}) \leq f(1) = \frac{1}{8}$. Since f is not monotone we conclude that it is not $\leq_q$-continuous.

3 Uniqueness of fixed point results

A natural question that one can wonders is whether Theorem 6 guarantees the uniqueness of fixed point in general. However, the next example gives a negative answer to the posed question .

Example 21. *Consider the Smyth complete partial quasi-metric space (X,q) where $X = \{(0,1),(1,0)\}$ and q is the partial quasi-metric on X given by $q((x,y),(w,r)) = \max\{x-w,0\} + \max\{r-y,0\}$ for all $(x,y),(w,r) \in X$. Define the mapping $f : X \to X$ by $f(x,y) = (x,y)$. It is obvious that $(x,y) \leq_q (w,r) \Leftrightarrow (x,y) = (w,r)$. Thus the sequence $(f^n(x,y))_{n\in\mathbb{N}}$ is increasing in $(X,\leq_q)$ for all $(x,y) \in X$. Besides,*

$$q(f(x,y),f(w,r)) \leq \frac{1}{2}q((x,y),(w,r))$$

for all $(x,y),(w,r) \in X$ with $(x,y) \bowtie_q (w,r)$. However, $Fix(f) = \{(0,1),(1,0)\}$.

Inspired by the above example we discuss conditions that guarantee the uniqueness of fixed point in the following results. With this aim we observe that Example 7 shows that there are order-contractive mappings f such that $Fix(f) = \emptyset$.

In the next result we give sufficient conditions in order to guarantee a local uniqueness.

Proposition 22. *Let f be an order-contractive mapping from a partial quasi-metric space (X,q) into itself such that $Fix(f) \neq \emptyset$. If $x^* \in Fix(f)$, then $Fix(f) \cap \uparrow_{\leq_q} x^* = x^*$ and $Fix(f) \cap \downarrow_{\leq_q} x^* = x^*$.*

Proof. Suppose that there exists $x^*, y^* \in Fix(f)$ such that $y^* \in Fix(f) \cap \uparrow_{\leq_q} x^*$. Then $x^* \leq y^*$. Consider for the purpose of contradiction that $q(x^*,y^*) \neq 0$. The fact that f is order-contractive yields that

$$q(x^*,y^*) = q(f(x^*),f(y^*)) \leq cq(x^*,y^*).$$

It follows that $1 \leq c$. So $q(x^*,x^*) = q(x^*,y^*) = 0$. Similarly we can show that $q(y^*,x^*) = q(y^*,y^*) = 0$. Therefore $x^* = y^*$. So we conclude that $Fix(f) \cap \uparrow_{\leq_q} x^* = x^*$. Similar arguments allow to prove that $Fix(f) \cap \downarrow_{\leq_q} x^* = x^*$.

The next result provides sufficient conditions in order to guarantee a global uniqueness. In order to state the aforesaid result, let us recall that a partially ordered set $(X,\leq)$ is called directed provided that for each $x,y \in X$ there exists $z \in X$ such that $x \leq_q z$ and $y \leq_q z$ (see, for instance, [1]).

Proposition 23. *Let f be a monotone order-contractive mapping from a partial quasi-metric space (X,q) into itself. If $(X,\leq_q)$ is a directed set with respect to $\leq_q$ and $Fix(f) \neq \emptyset$, then $card(Fix(f)) = 1$.*

Proof. Let $x^* \in Fix(f)$. Assume that $y^* \in Fix(f)$ with $x^* \neq y^*$. Since $(X,\leq_q)$ is directed there exists $z \in X$ such that $x^* \leq_q z$ and $y^* \leq_q z$. Since f is monotone we have that $f^n(x^*) \leq_q f^n(z)$ and $f^n(y^*) \leq_q f^n(z)$. It follows, by (4), that

$$\begin{aligned} q(x^*,y^*) = q(f^n(x^*),f^n(y^*)) &\leq q(f^n(x^*),f^n(z)) + q(f^n(z),f^n(y^*)) \\ &\leq c^n\left(q(y^*,z) + q(z,x^*)\right). \end{aligned}$$

Applying a similar reasoning we deduce that

$$q(y^*,x^*) \leq c^n\left(q(x^*,z) + q(z,y^*)\right).$$

It follows that $q(x^*,y^*) = q(y^*,x^*) = 0$. Thus $q(x^*,x^*) = q(y^*,y^*) = 0$, since $q(x^*,x^*) \leq q(x^*,y^*)$ and $q(y^*,y^*) \leq q(y^*,x^*)$. So we conclude that $x^* = y^*$.

Again Example 7 gives an instance of a monotone order-contractive mapping without fixed points defined from a partial quasi-metric. Moreover, observe that in the same example the induced partially ordered set is not directed.

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