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## Analytical solutions of the relative orbital motion in unperturbed and in $J_2$ - perturbed elliptic orbits

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### Abstract

This paper introduces a different approach to obtain the exact solution of the relative equations of motion of a deputy (follower) satellite with respect to a chief (leader) satellite that both rotate about central body (Earth) in elliptic orbits by using Laplace transformation. Moreover, the paper will take the perturbation due to the oblateness of the Earth into consideration and simulate this problem with numerical example showing the effect of the perturbation on the Keplerian motion. The solution of such equations in this work is represented in terms of the eccentricity of the chief orbit and its true anomaly as the independent variable.

**Keywords:** Relative motion of two satellites; Tschauner Hempel equations;  $J_2$  Perturbation; Laplace transformation.

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## 1 Introduction

Solving and modeling the relative motion problem between satellites or space crafts is of great importance in the field of formation flying, rendezvous and disturbed satellite systems which in turn play a significant rule in space missions. Analytical expressions for propagating relative motion under the influence of  $J_2$  have been developed by several authors. Tschauner-Hempel [1] derived equations of relative motion valid for vehicles in nearby eccentric orbits. Melton [2] presents a time-explicit solution for relative motion for elliptic orbits using a perturbation method. Valaddi et al. [3] also presents a time-explicit perturbation solution to the relative motion in

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elliptic orbits by retaining quadratic nonlinearities of the gravitational field. Alternatively, Gim and Alfriend [4] provide a state transition matrix for the solution of the linearized relative motion problem using curvilinear coordinates. Garrison et al. [5] consider a geometric method for deriving the state transition matrix, utilizing small differences in orbital elements between two satellites. Also Srinivas R. Vadali [6] uses the geometric method but under the influence of  $J_2$ -perturbation. Schaub [7] presents analytical expressions for propagating the linearized orbital element differences using the true anomaly as the independent variable. Time-explicit solutions have been developed by Sabol et al. [8] using eccentricity expansions. This approach is valid for small eccentricities.

In this research article, the true anomaly of the Chief is used as the independent variable, rather than time in the modelling of the relative equations of motion of a deputy with respect to a chief satellite, such that both of them rotate under the gravitational field of the Earth in elliptic orbits, considering the effect of the Earth oblateness  $J_2$  gravity perturbation into our calculations. Then we use Laplace transformation to solve the equations of motion. Finally we will illustrate the effect of the perturbation part using numerical example.

## 2 Description of the problem

In order to describe the relative motion of the deputy satellite with respect to the chief satellite, two Cartesian coordinate systems are used as shown in Fig. 1. The first is the Earth centered inertial (ECI) coordinate frame (X, Y, Z) which is spanned by the unit vectors (I, J, K). In this system the X-axis points toward the vernal equinox, the Z-axis points toward the North-pole of the Earth, and the Y-axis is perpendicular to the other two and completes the right-handed coordinate system. The plane determined by I and J coincides with the equatorial plane. The chief orbit is defined in the ECI frame using orbital elements ( $r_C, \theta, i, \omega, \Omega$ ), which include geocentric distance  $r$ , argument of latitude  $\theta$ , inclination  $i$ , argument of the peri-apse  $\omega$ , and the right ascension of the ascending node  $\Omega$ .

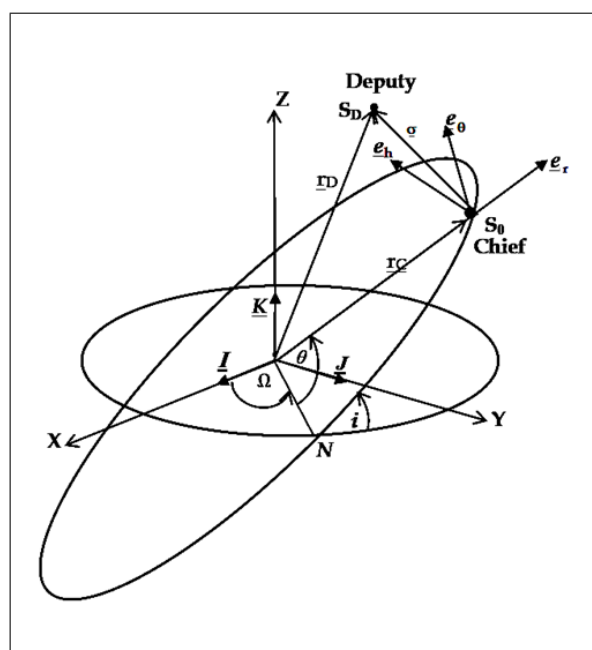


Fig. 1 ECI and LVLH frames.

The second frame is the Local-Vertical-Local-Horizontal (LVLH) frame ( $\xi, \eta, \zeta$ ) which is attached to the chief satellite  $S_0$  and spanned by the unit vectors ( $e_r, e_\theta, e_h$ ) where  $e_r$  lies in the chief's radial direction from the

Earth center,  $\underline{e}_h$  is parallel to the orbit angular momentum vector, and  $\underline{e}_\theta$  completes the right-handed orthogonal triad. Thus,

$$\underline{e}_r = \frac{\underline{r}_c}{|\underline{r}_c|}, \quad \underline{e}_h = \frac{\underline{h}_c}{|\underline{h}_c|}, \quad \underline{e}_\theta = \underline{e}_h \wedge \underline{e}_r \quad (1)$$

Such that that  $\underline{h}_c = \underline{r}_c \wedge \underline{v}_c$ , is the chief's angular momentum vector per unit mass in its orbit around the central body (Earth), and

$$r_c = \frac{p}{1 + e \cos f}, \quad p = a(1 - e^2) \quad (2)$$

where  $p$ ,  $a$  and  $e$  are the semi-latus rectum, semi-major axis and the eccentricity of the chief orbit, respectively.

The position vector of the deputy satellite relative to the chief satellite can be expressed by Cartesian coordinate vector  $\underline{\sigma}$ :

$$\underline{\sigma} = [\xi \quad \eta \quad \zeta]^T \quad (3)$$

Where  $\xi$ ,  $\eta$  and  $\zeta$  denote the components of the position vector along the radial, transverse, and the normal to the orbital plane of the chief directions respectively. The position vectors of the chief and deputy with respect to the Earth center are given by  $\vec{r}_C$  and  $\vec{r}_D$  respectively. Assuming a central gravity field, the chief frame rotates with an angular velocity  $\underline{\omega} = [0 \ 0 \ \dot{\theta}]^T$ , where  $\theta = \omega + f$ , is the argument of latitude,  $\omega$  is the argument of periapsis and  $f$  is the true anomaly. Under the assumption of a central gravity field,  $\omega$  is constant, and  $\dot{\theta} = \dot{f}$ .

From Figure 1, the deputy satellite position vector with respect to the Earth can be written as:

$$\vec{r}_D = \vec{r}_C + \vec{\sigma} = (r_C + \xi)\underline{e}_r + \eta\underline{e}_\theta + \zeta\underline{e}_h \quad (4)$$

### 3 Equations of motion

The equations of motion of the chief and deputy satellites with respect to the ECI-frame can be written respectively as:

$$\ddot{\vec{r}}_C = -\frac{\mu}{r_C^3} \vec{r}_C + \vec{a}_C(\vec{r}_C) \quad (5)$$

$$\ddot{\vec{r}}_D = -\frac{\mu}{r_D^3} \vec{r}_D + \vec{a}_D(\vec{r}_D) \quad (6)$$

where  $\vec{a}_C$  and  $\vec{a}_D$  are the perturbing accelerations acting on the chief and deputy satellites due to the Earth oblateness.

From equation 4, we can write

$$\ddot{\vec{r}}_D = \ddot{\vec{r}}_C + \ddot{\vec{\sigma}}_{ECI} \quad (7)$$

Now we need to transfer the acceleration  $\ddot{\vec{\sigma}}$  in the LVLH - frame to the acceleration in the ECI - frame (inertial frame). The transformation is well known, for example see Wiesel's book [9].

$$\ddot{\vec{\sigma}}_{ECI} = \ddot{\vec{\sigma}}_{LVLH} + 2\vec{\omega} \wedge \dot{\vec{\sigma}}_{LVLH} + \dot{\vec{\omega}} \wedge \vec{\sigma}_{LVLH} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{\sigma}_{LVLH}) \quad (8)$$

Where  $\vec{\omega}$  is the angular velocity of the leader satellite in the LVLH - frame with respect to the ECI - frame, where

$$\vec{\omega} = [0 \ 0 \ \dot{f}]^T, \quad \dot{\vec{\omega}} = [0 \ 0 \ \ddot{f}]^T, \quad \vec{r}_C = [r_C \ 0 \ 0]^T \quad (9)$$

Using equations 4 to 9, we can write

$$-\frac{\mu}{r_D^3} \vec{r}_D + \vec{a}_D(\vec{r}_D) = \vec{a}_C(\vec{r}_C) + \left( \ddot{\xi} - 2\dot{\eta}\dot{f} - \eta\ddot{f} - \xi\dot{f}^2 - \frac{\mu}{r_C^2} \right) \vec{e}_r + \left( \ddot{\eta} + 2\dot{\xi}\dot{f} + \xi\ddot{f} - \eta\dot{f}^2 \right) \vec{e}_\theta + \ddot{\zeta} \vec{e}_h \quad (10)$$

In order to simplify the previous equation we will eliminate  $\ddot{f}$  using the equation of motion per unit mass of the chief in its Keplerian motion around the Earth in the form

$$\ddot{\vec{r}}_C = (\ddot{r}_C - r_C \dot{f}^2) \vec{e}_r + (r_C \ddot{f} + 2\dot{r}_C \dot{f}) \vec{e}_\theta = -\frac{\mu}{r_C^2} \vec{e}_r$$

Equating the components on both sides, we obtain the following relations:

$$\ddot{r}_C = r_C \dot{f}^2 - \frac{\mu}{r_C^2}, \quad \ddot{f} = -2 \frac{\dot{r}_C}{r_C} \dot{f} \quad (11)$$

Using equations 4 and 11, we can say that

$$\vec{a}_D(\vec{r}_D) - \vec{a}_C(\vec{r}_C) = \left( \ddot{\xi} - 2\dot{\eta}\dot{f} + 2\frac{\dot{r}_C}{r_C} \eta\dot{f} - \xi\dot{f}^2 - \frac{\mu}{r_C^2} + \frac{\mu}{r_D^3} (r_C + \xi) \right) \vec{e}_r + \left( \ddot{\eta} + 2\dot{\xi}\dot{f} - 2\frac{\dot{r}_C}{r_C} \xi\dot{f} - \eta\dot{f}^2 + \frac{\mu}{r_D^3} \eta \right) \vec{e}_\theta + \left( \ddot{\zeta} + \frac{\mu}{r_D^3} \zeta \right) \vec{e}_h \quad (12)$$

Or we can write 12 in its components form as the following

$$\begin{aligned} \ddot{\xi} - 2(\dot{\eta} - \frac{\dot{r}_C}{r_C} \eta)\dot{f} - \xi\dot{f}^2 - \frac{\mu}{r_C^2} + \frac{\mu}{r_D^3} (r_C + \xi) &= (a_D)_r - (a_C)_r \\ \ddot{\eta} + 2(\dot{\xi} - \frac{\dot{r}_C}{r_C} \xi)\dot{f} - \eta\dot{f}^2 + \frac{\mu}{r_D^3} \eta &= (a_D)_\theta - (a_C)_\theta \\ \ddot{\zeta} + \frac{\mu}{r_D^3} \zeta &= (a_D)_h - (a_C)_h \end{aligned} \quad (13)$$

### 3.1 The $J_2$ -Perturbation

The aspherical nature of the Earth, arising from the equatorial bulge, leads to a gravitational attraction on a body that is no longer directed towards the center of mass of the Earth. The potential of the Earth due to oblateness effects is presented in many text books and can be written up to  $J_2$ -order in the form

$$\Phi = \frac{\mu J_2 R_e^2}{2r^3} (1 - 3\sin^2 i \sin^2(\omega + f)) \quad (14)$$

where  $M_e$ , is the mass of the Earth,  $\mu = GM_e$ , is the gravitational parameter of the Earth  $R_e$ , is the mean radius of the Earth and  $J_2 = 1.082629 \times 10^{-3}$  is the second zonal harmonic of the Earth. We will consider the  $J_2$  effects on the chief satellite orbit only, which is reasonable because the distance between the deputy and the chief is too small relative to their distances from the Earth. The  $J_2$  perturbing force can be calculated as the gradient of the potential in equation 14 as Schweighart and Sedwick [10] :

$$\vec{a}_C = [J_{2r}, J_{2\theta}, J_{2h}]^T = \frac{3J_2 \mu R_e^2}{2r_C^4} \begin{bmatrix} 1 - 3\sin^2 i \sin^2(\omega + f) \\ \sin^2 i \sin 2(\omega + f) \\ \sin 2i \sin(\omega + f) \end{bmatrix} \quad (15)$$

### 3.2 Transforming the independent variable

Since the true anomaly of the chief,  $f$ , gives more details about its orbit. It is convenient to transform the independent variable from the time to  $f$  using the following equations:

$$(\dot{\cdot}) = (\cdot)' \dot{f} \quad \text{and} \quad (\ddot{\cdot}) = (\cdot)'' \dot{f}^2 + (\cdot)' \ddot{f} \dot{f} \quad (16)$$

where the prime means that the derivative is with respect to  $f$ .

Using 16, we can obtain the following relations:

$$\dot{f} = \frac{h}{r_C^2} = \frac{\sqrt{\mu}}{p^{3/2}}(1 + e \cos f)^2 \Rightarrow \dot{f}' = \frac{-2\sqrt{\mu}}{p^{3/2}} e \sin f (1 + e \cos f) \quad (17)$$

$$\frac{\dot{f}'}{\dot{f}} = \frac{-2e \sin f}{1 + e \cos f} \quad \text{and} \quad \frac{r'_C}{r_C} = \frac{e \sin f}{1 + e \cos f} \quad (18)$$

Substituting 17 and 18 into the system 13, we obtain the new system of the equations of motion

$$\begin{bmatrix} \xi'' & \eta'' & \zeta'' \end{bmatrix}^T = \begin{bmatrix} \frac{2e \sin f}{1+e \cos f} \xi' + 2\eta' + \xi - \frac{2e \sin f}{1+e \cos f} \eta + \frac{\mu}{r_C^2 \dot{f}^2} - \frac{\mu}{r_D^3 \dot{f}^2} (r_C + \xi) \\ -2\xi' + \frac{2e \sin f}{1+e \cos f} \eta' + \frac{2e \sin f}{1+e \cos f} \xi + \eta - \frac{\mu}{r_D^3 \dot{f}^2} \eta \\ \frac{2e \sin f}{1+e \cos f} \zeta' - \frac{\mu}{r_D^3 \dot{f}^2} \zeta \end{bmatrix} - \begin{bmatrix} \frac{J_{2r}}{\dot{f}^2} \\ \frac{J_{2\theta}}{\dot{f}^2} \\ \frac{J_{2h}}{\dot{f}^2} \end{bmatrix} \quad (19)$$

### 3.3 Linearization for the relative equations of motion

Since  $\sigma \ll r_D$ , we can get the linearized relative equations of motion using the following

$$r_D^2 = (\vec{r}_C + \vec{\sigma}) \bullet (\vec{r}_C + \vec{\sigma}) = r_C^2 + \sigma^2 + 2r_C \xi \approx r_C^2 \left( 1 + \frac{2\xi}{r_C} \right)$$

Hence

$$r_D^{-3} \approx r_C^{-3} \left( 1 - \frac{3\xi}{r_C} \right) \quad (20)$$

Equations 19 can be simplified to

$$\begin{bmatrix} \xi'' & \eta'' & \zeta'' \end{bmatrix}^T = \begin{bmatrix} \frac{2e \sin f}{1+e \cos f} \xi' + 2\eta' + \frac{3+e \cos f}{1+e \cos f} \xi - \frac{2e \sin f}{1+e \cos f} \eta \\ -2\xi' + \frac{2e \sin f}{1+e \cos f} \eta' + \frac{2e \sin f}{1+e \cos f} \xi + \frac{e \cos f}{1+e \cos f} \eta \\ \frac{2e \sin f}{1+e \cos f} \zeta' - \frac{1}{1+e \cos f} \zeta \end{bmatrix} - \begin{bmatrix} \frac{J_{2r}}{\dot{f}^2} \\ \frac{J_{2\theta}}{\dot{f}^2} \\ \frac{J_{2h}}{\dot{f}^2} \end{bmatrix} \quad (21)$$

### 3.4 Normalizing deputy relative position components

In order to write these equations with dimensionless coordinates, we introduce

$$\begin{bmatrix} \xi & \eta & \zeta \end{bmatrix}^T = r_C \begin{bmatrix} x & y & z \end{bmatrix}^T \quad (22)$$

And by these relations, we have

$\xi' = r'_C x + r_C x'$  and  $\xi'' = r''_C x + 2r'_C x' + r_C x''$ , with similar relations for  $\eta$  and  $\zeta$

After many simplifications including the perturbed part we can write 21 as

$$\begin{bmatrix} x'' & y'' & z'' \end{bmatrix}^T = \begin{bmatrix} 2y' + \frac{3}{1+e \cos f} x \\ -2x' \\ -z \end{bmatrix} + k(1 + e \cos f) \begin{bmatrix} b_r & b_\theta & b_h \end{bmatrix}^T \quad (23)$$

Where

$$k = \frac{-3J_2 R_e^2}{2a^2(1-e^2)^2} \begin{bmatrix} b_r & b_\theta & b_h \end{bmatrix}^T = \begin{bmatrix} 1 - 3 \sin^2 i \sin^2 f \\ \sin^2 i \sin 2f \\ \sin 2i \sin f \end{bmatrix} \quad (24)$$

Which is the final perturbed form of the system of linear relative equations of motion of a deputy with respect to a chief satellite, using the chief's true anomaly as the independent variable.

#### 4 Solving the linearized relative equations of motion

Starting from the second equation of 23, which is

$$y'' = -2x' + kA(1 + e \cos f) \sin 2f, \text{ where } A = \sin^2 i$$

Integrating with respect to the true anomaly  $f$ , we obtain:

$$y' = -2x + kA \left( \sin^2 f - \frac{2e}{3} \cos^3 f \right) + d, \text{ where } d = y'_0 + 2x_0 + \frac{2}{3}ekA \quad (25)$$

Substitution into the first equation in 23 gives

$$\begin{aligned} (1 + e \cos f)x'' = & -(1 + 4e \cos f)x + k(1 + e \cos f)^2 [1 - 3 \sin^2 i \sin^2 f] \\ & + 2k \sin^2 i (1 + e \cos f) \left( \sin^2 f - \frac{2e}{3} \cos^3 f \right) + 2d(1 + e \cos f) \end{aligned} \quad (26)$$

Which can be written as

$$\begin{aligned} (1 + e \cos f)x'' = & -(1 + 4e \cos f)x + \frac{k}{12} (1 + e \cos f) (12 - 6A + 3e(4 - 7A) \cos f \\ & + 6A \cos 2f + 5eA \cos 3f) + 2d(1 + e \cos f) \end{aligned} \quad (27)$$

Applying Laplace transformation on 27, such that  $\mathcal{L}\{g(f)\} = G(s)$ , where  $g(f)$ , is any function of  $f$ . We can construct table 1., taking into account the well known relation  $\cos f = \frac{e^{if} + e^{-if}}{2}$

**Table 1** Laplace transformation on the relative equation of motion of the Deputy in x-direction

No.	$x(f)$	$\mathcal{L}\{g(f)\} = G(s)$
1	$2d(1 + e \cos f)$	$\frac{2d}{s} + \frac{2eds}{s^2+1}$
2	$x''$	$s^2X(s) - sx_0 - x'_0$
3	$(4e \cos f)x$	$2e[X(s-i) + X(s+i)]$
4	$(e \cos f)x''$	$\frac{e}{2}(s-i)^2X(s-i) + \frac{e}{2}(s+i)^2X(s+i) - ex_0s - ex'_0$

And for the perturbed terms that contain  $k$ , we have:

$$\begin{aligned} \mathcal{L} \left\{ \frac{k}{12} (1 + e \cos f) (12 - 6A + 3e(4 - 7A) \cos f + 6A \cos 2f + 5eA \cos 3f) \right\} \\ = \frac{k}{24} \left( \frac{12(2-A)+3e^2(4-7A)}{s} + \frac{48e(1-A)s}{1+s^2} + \frac{4(e^2(3-4A)+3A)s}{4+s^2} + \frac{16eAs}{9+s^2} + \frac{5e^2As}{16+s^2} \right) \end{aligned} \quad (28)$$

After some simplifications, we can get

$$\begin{aligned} X(s) = & - \left[ \frac{(s-i)^2+4}{2(s-i)(s+i)} \right] eX(s-i) - \left[ \frac{(s+i)^2+4}{2(s-i)(s+i)} \right] eX(s+i) + \frac{2d}{s(s^2+1)} \\ & + \frac{(1+e)x_0s}{s^2+1} + \frac{(1+e)x'_0}{s^2+1} + \frac{k}{24(1+s^2)} \left[ \frac{12(2-A)+3e^2(4-7A)}{s} \right. \\ & \left. + \frac{48e(1-A)s}{1+s^2} + \frac{4(e^2(3-4A)+3A)s}{4+s^2} + \frac{16eAs}{9+s^2} + \frac{5e^2As}{16+s^2} \right] \end{aligned} \quad (29)$$

And now applying the inverse Laplace transformation, recalling that

$$\mathcal{L}^{-1}\{G(s-a)\} = e^{af}g(f)$$

We can get after simplifying

$$\begin{aligned} 2e \int_0^f x(\tau) d\tau + (\csc f + e \cot f)x = & 2d \csc f + [(1+e)x_0 - 2d] \cot f + [(1+e)x'_0 + e d f] \\ & + \frac{k}{72 \sin f} [9(4(2-A) + (4-7A)e^2) + 6(A(8+e+8e^2) - 4(3+e^2)) \cos f \\ & + 4((4A-3)e^2 - 3A) \cos 2f - Ae(6 \cos 3f + e \cos 4f) + 72(1-A)ef \sin f] \end{aligned} \quad (30)$$

By differentiating this equation with respect to  $f$ , we can get

$$\begin{aligned} x' + \left[ \frac{2e - \csc f \cot f - e \csc^2 f}{\csc f + e \cot f} \right] x = & \frac{ed - 2d \csc f \cot f - [(1+e)x_0 - 2d] \csc^2 f}{\csc f + e \cot f} \\ & + \frac{k \sin f}{12(1+e \cos f)} [(4e^2 + A(4-5e^2)) \cos f \\ & + \frac{4(3+e^2-2A(1+e^2))}{1+\cos f} + e(12-12A+4A \cos 2f + Ae \cos 3f)] \end{aligned} \quad (31)$$

Which is in the common form of the linear first order differential equation. To solve such type, we need to get the integrating factor as following

$$\exp \left\{ \int \left[ \frac{2e - \csc f \cot f - e \csc^2 f}{\csc f + e \cot f} \right] df \right\} = \frac{1}{\sin f(1+e \cos f)} \quad (32)$$

Thus, the solution of 31 can be written as

$$\begin{aligned} \frac{x(f)}{\sin f(1+e \cos f)} = & \int \frac{ed \sin^2 f - 2d \cos f - [(1+e)x_0 - 2d]}{\sin^2 f(1+e \cos f)^2} df + \int \frac{k}{12(1+e \cos f)^2} [(4e^2 + A(4-5e^2)) \cos f \\ & + \frac{4(3+e^2-2A(1+e^2))}{1+\cos f} + e(12-12A+4A \cos 2f + Ae \cos 3f)] df + c_1 \end{aligned} \quad (33)$$

In order to accomplish the integral, we use the eccentric anomaly  $E$ , rather than the true anomaly  $f$ , defined by  $E = 2 \tan^{-1} \left[ \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} \right]$ , then we can obtain  $x(f)$  as:

$$\begin{aligned} x(f) = & \frac{x_0}{2(1+e)} (1 + \cos f) (1 + e \cos f) + \frac{e^2}{(1-e)^2} \left[ \frac{(d+de-ex_0)}{(1+e)} + \frac{k(1+e-2Ae)}{3} \right] \sin^2 f \\ & + \sin f (1 + e \cos f) \left\{ \frac{3e(ex_0-d-de)}{2(1-e)^{5/2}(1+e)^{3/2}} E + \frac{(4d-(1+e)x_0)}{2(1-e)^2} \tan \frac{f}{2} \right. \\ & \left. + \frac{1}{3} k \left[ A \sin f + \frac{3e(e(2A-1)-1)}{(1-e)^2 \sqrt{1-e^2}} E + \frac{(3+e^2-2A(1+e^2))}{(1-e)^2} \tan \frac{f}{2} \right] + c_1 \right\} \end{aligned} \quad (34)$$

To get the value of  $c_1$ , we can find  $\left. \frac{dx}{df} \right|_{f=0} \Rightarrow c_1 = \frac{x'_0}{1+e}$ , then

$$\begin{aligned} x(f) = & (1 + e \cos f) \left[ \frac{x_0(1+\cos f)}{2(1+e)} + \frac{kA \sin^2 f}{3} \right] + \frac{e^2 \sin^2 f}{(1-e)^2} \left[ d - \frac{ex_0}{1+e} + \frac{k(1+e-2Ae)}{3} \right] \\ & + \sin f (1 + e \cos f) \left\{ \frac{eE}{(1-e)^{5/2}(1+e)^{1/2}} \left[ \frac{3ex_0}{2(1+e)} - \frac{3}{2}d + k(e(2A-1)-1) \right] \right. \\ & \left. + \frac{1}{2(1-e)^2} \tan \frac{f}{2} [(4d - (1+e)x_0) + \frac{2k}{3}(3+e^2-2A(1+e^2))] + \frac{x'_0}{1+e} \right\} \end{aligned} \quad (35)$$

Now to obtain the solution for  $y$ , we substitute the obtained solution of  $x(f)$  into equation 25, and integrating with respect to  $f$ , we obtain

$$\Rightarrow y = \int \left[ -2x + kA (\sin^2 f - \frac{2e}{3} \cos^3 f) + d \right] df + c_2 \quad (36)$$

To evaluate  $\int E \sin f (1 + e \cos f) df$ , we again use the eccentric anomaly

$$I = \int E \sin f (1 + e \cos f) df = (1 - e^2)^2 \int \frac{E \sin E}{(1 - e \cos E)^3} dE \quad (37)$$

Integrating 37 by parts, we get

$$I = \frac{1}{2e} \left[ \sqrt{1 - e^2} (f + e \sin f) - E (1 + e \cos f)^2 \right] \quad (38)$$

The constant of integration  $c_2$  in 36, can be calculated from  $y|_{f=0} \Rightarrow c_2 = y_0 - \frac{2x'_0}{1+e}$ . The solution for  $y(f)$  can now be expressed as:

$$\begin{aligned} y(f) = & \frac{x'_0}{1+e} (2 \cos f - e \sin^2 f) - \frac{x_0}{4(1+e)} (2f(2+e) + 4(1+e) \sin f + e \sin 2f) \\ & - \frac{kA}{18} (6f + 3e \sin f - 3 \sin 2f - e \sin 3f) - \frac{e^2}{2(1-e)^2} \left( d - \frac{ex_0}{1+e} + \frac{k(1+e-2Ae)}{3} \right) (2f - \sin 2f) \\ & - \frac{1}{4(1-e)^2} (4d - (1+e)x_0 + \frac{2k}{3} (3 + e^2 - 2A(1+e^2))) (2f(2-e) - 4(1-e) \sin f - e \sin 2f) \\ & - \frac{1}{\sqrt{(1-e)^5} \sqrt{1+e}} \left( \frac{3ex_0}{2(1+e)} - \frac{3d}{2} + k(e(2A-1) - 1) \right) \left( \sqrt{1-e^2} (f + e \sin f) - E (1 + e \cos f)^2 \right) \\ & - \frac{1}{36} Ak (9(-2f + \sin 2f) + 2e(9 \sin f + \sin 3f)) + df + y_0 - \frac{2x'_0}{1+e} \end{aligned} \quad (39)$$

Finally for the third equation in 23,

$$z'' + z = k(1 + e \cos f) \sin 2i \sin f$$

Applying Laplace transformation such that  $\mathcal{L}\{z(f)\} = Z(s)$ , we can get

$$Z(s) = k \sin(2i) \left[ \frac{1}{(s^2 + 1)^2} + \frac{e}{(s^2 + 1)(s^2 + 4)} \right] + \frac{z_0 s}{s^2 + 1} + \frac{z'_0}{s^2 + 1} \quad (40)$$

The inverse Laplace transform of equation 40, gives the solution of  $z(f)$  as:

$$z = z'_0 \sin f + z_0 \cos f + \frac{k}{6} (\sin 2i) (3 \sin f - 3f \cos f + 2e \sin f - e \sin 2f) \quad (41)$$

Equations 35, 39 and 41 are the solution of the equations of motion of the deputy relative to the chief including perturbation due to oblateness of the Earth.

## 5 Numerical Example:

In this section, we will present a simulation of the relative equations of motion of the deputy with respect to the chief satellite in both the unperturbed and the perturbed cases and we will show the difference between them. Where the unperturbed solution can be written as:

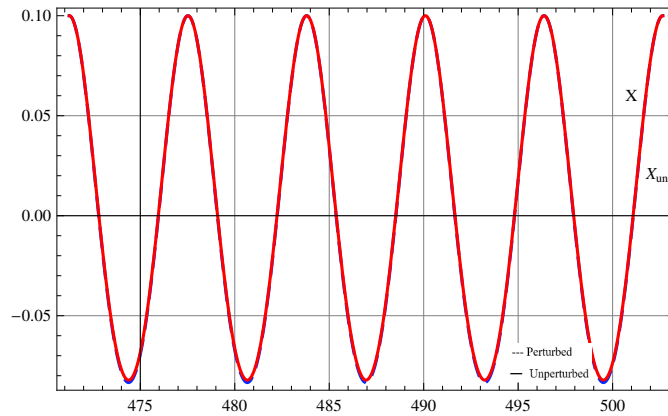
$$\begin{aligned} x_{unp}(f) = & \frac{x_0}{2(1+e)} (1 + \cos f) (1 + e \cos f) + \frac{e^2}{(1-e)^2} \left( d - \frac{ex_0}{1+e} \right) \sin^2 f \\ & + \sin f (1 + e \cos f) \left\{ \frac{3e(ex_0 - d - de)}{2(1-e)^{5/2}(1+e)^{3/2}} E + \frac{(4d - (1+e)x_0)}{2(1-e)^2} \tan \frac{f}{2} + \frac{x'_0}{1+e} \right\} \end{aligned} \quad (42)$$

$$\begin{aligned} y_{unp}(f) = & \frac{x'_0}{1+e} (2 \cos f - e \sin^2 f) - \frac{x_0}{4(1+e)} (2f(2+e) + 4(1+e) \sin f + e \sin 2f) \\ & - \frac{(d + de - ex_0)e^2}{2(1-e)^2} (2f - \sin 2f) - \frac{(4d - (1+e)x_0)}{4(1-e)^2} (2f(2-e) - 4(1-e) \sin f - e \sin 2f) \\ & - \frac{3(ex_0 - d - de)}{2(1-e)^{5/2} \sqrt{1+e}} \left( \sqrt{1-e^2} (f + e \sin f) - E (1 + e \cos f)^2 \right) + df + y_0 - \frac{2x'_0}{1+e} \end{aligned} \quad (43)$$

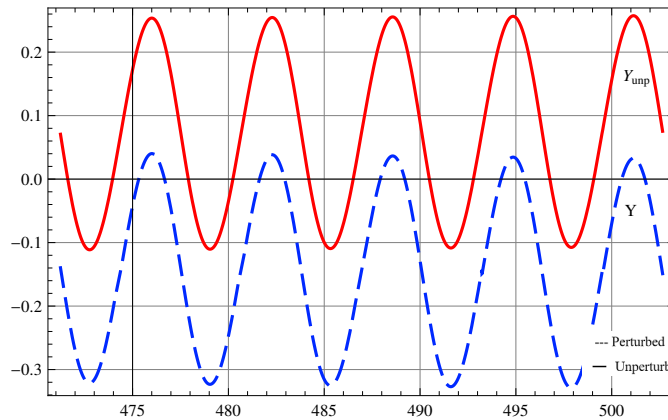


$$z_{unp} = z'_0 \sin f + z_0 \cos f \quad (44)$$

According to the following initial conditions [11]  $e = 0.1$ ,  $x_0 = 0.1$ ,  $x'_0 = 0$ ,  $y_0 = 0$ ,  $y'_0 = \frac{-21}{110}$ ,  $z_0 = 0.08$  and  $z'_0 = 0$ ,  $a = 7000\text{km}$ , we can get the relations between the chief true anomaly with the x, y and z components of the deputy relative position in both perturbed and unperturbed cases as shown in figures 2, 3 and 4 respectively. Also in figure 5 these three components are parametrized with the chief true anomaly.



**Fig. 2** Perturbed and unperturbed x-component of the deputy relative position with the Chief true anomaly.

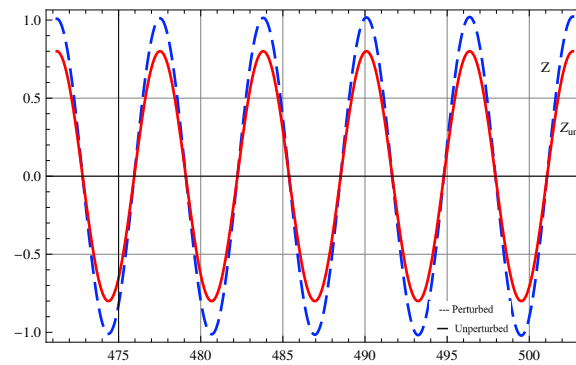


**Fig. 3** Perturbed and unperturbed y-component of the deputy relative position vs. Chief true anomaly.

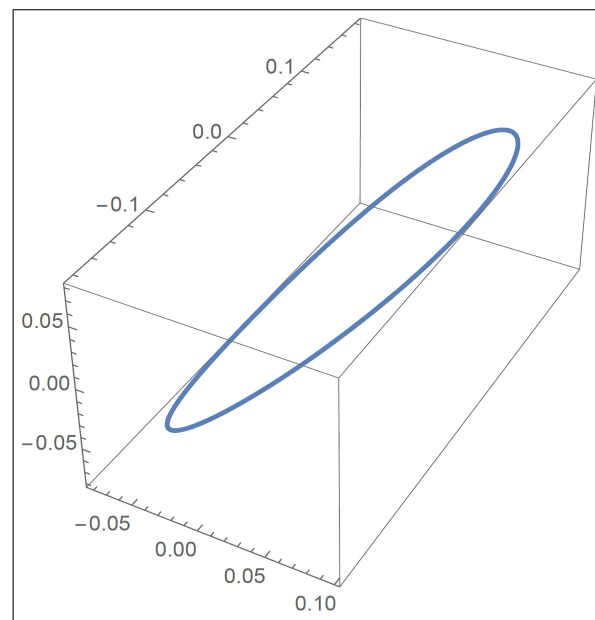
## 6 Conclusion

An explicit solution including the perturbation due to the oblateness of the Earth of the relative equations of motion of a deputy or follower satellite relative to a chief or leader satellite is expressed in terms of the eccentricity of the chief orbit and its true anomaly as the independent variable. We have the following notes:

1. Since the in-plane solution  $[x(f) \text{ and } y(f)]$  contains the eccentric anomaly  $E = 2 \tan^{-1} \left[ \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} \right]$ , therefore we have singularity when  $f$  is a multiple of  $\pi$ . But it is very clear that we can eliminate it by choosing the initial conditions such that  $e x_0 - d - d e \approx 0 \Rightarrow \frac{y'_0}{x_0} + \frac{2e k A}{3x_0} = -\frac{2+e}{1+e}$ , or  $\frac{y'_0}{x_0} = -\frac{2+e}{1+e}$  in the absence of the Earth oblateness effect, to obtain a periodic motion for the deputy around the chief.



**Fig. 4** Perturbed and unperturbed z-component of the deputy relative position vs. Chief true anomaly.



**Fig. 5** Deputy 3D parametric representation.

2. By setting  $k = 0$ , we can obtain the solution of our equations of motion in the unperturbed case in equations 42, 43 and 44.

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