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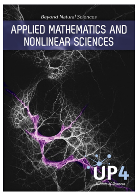

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Three-way weighted combination-entropies based on three-layer granular structures

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Abstract

Rough set theory is an important theory for the uncertain information processing. The information theoretic measures have been introduced into rough set theory and provided a new effective method in uncertainty measurement and attribute reduction. However, most of them did not consider the hierarchical structure of a decision table (D-Table). Thus, this paper concretely constructs three-way weighted combination-entropies based on the D-Table's three-layer granular structures and Bayes' theorem from a new perspective, and reveals the granulation monotonicity and systematic relationships of three-way weighted combination-entropies. The relevant conclusion provides a more complete and updated interpretation of granular computing for the uncertainty measurement, and it also establishes a more effective basis for the quantitative application in attribute reduction.

Keywords: Rough set; Granular computing; Three-layer granular structures; Three-way weighted combination-entropies; Three-way decisions; Bayes' theorem
AMS 2010 codes: 68T30, 68T37.

1 Introduction

Rough set theory, introduced by Pawlak [1], is a kind of important theory about uncertainty information processing. So far, it has been successfully applied in data analysis, pattern recognition, machine learning and knowledge discovery, artificial intelligence, and so on [2–7]. Using the tool of entropy to deal with the uncertainty problem in rough set theory has been already studied [8–11], and the combination entropy was proposed in the literature [12].

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As it is known that the rough set theory is mainly utilized to address the problem of information granules approximation problem based on a D-Table, and the granularity reflects the different levels of a given problem, then the granular computing (GrC) concerns the processing of complex information entities, information granules, which arise in the process of data abstraction and derivation of knowledge from D-Table. Hence, GrC is a kind of important structure technology, and can be able to resolve the hierarchies problems in rough set theory [13–17]. Note that three-way decisions serve as a fundamental methodology with extensive applications. Hu [18] discussed three-way decisions based on semi-three-way decision spaces, and Li et al. [17] adopted the multi-granularity to study three-way cognitive concept learning. In particular, Yao [19] pointed out that the three-level analysis falls into the category of three-way decisions, so the three-layer attribute reduction and relevant three-level measure construction become a typical case and a good example of three-way decisions.But the definition of combination entropy proposed in literature [12] did not consider the hierarchical structure of the a decision table (D-Table). Hence, on the basis of Ref. [12, 16], this paper concretely constructs three-way weighted combination-entropies based on the new perspective of a D-Table's three-layer granular structures and Bayes' theorem, and reveals the granulation monotonicity and systematic relationships of the weighted combination-entropies.

The relevant conclusions of the study has been deepened information theory of rough set theory, provides a more complete and updated interpretation of granular computing for the uncertainty measurement, and establish more effective basis of the quantitative application for attribute reduction.

2 Preliminaries

2.1 The three-layer granular structures and the three-way probabilities

This section reviews the three-layer granular structures of a given D-Table and the three-way probabilities in Ref. [16].

An information system is a pair $S = (U, AT)$, where,

1. U is a non-empty finite set of objects;
2. AT is a non-empty finite set of attributes;
3. for every $a \in AT$, there is a mapping f_a, $f_a : U \to V_{f_a}$, where V_{f_a} is called the value set of U.

The D-Table is a special type of information table with $AT = C \cup D$ and $C \cap D = \emptyset$, where C and D denote the sets of condition attribute and decision attribute, respectively.

Each subset of attributes $A \subseteq C$ determines a binary indistinguishable relation $IND(A)$ as follows $IND(A) = \{(u,v) \in U \times U | \forall a \in A, f_a(u) = f_a(v)\}$.

$IND(A)$ serves as an equivalence relation to cause C-Class $[x]_A$, which implies a type of basic granule. The classified structure $U/IND(A) = \{[x]_A : x \in U\}$ means knowledge or C-Classification. Suppose that $U/IND(A) = \{[x]_A^i : i = 1,2,\ldots,n\}$, thus $|U/IND(A)| = n$. Similarly, D can induce the equivalence relation $IND(D)$ and further D-Classification $U/IND(D) = \{X_j : j = 1,2,\ldots,m\}$, thus $|U/IND(D)| = m$.

Aiming at the D-Table $(U, C \cup D)$ according to the four basic notions of the D-Table and four granular notions presented in Table 1, the relevant classification and class lead to three-layer granular structures, as shown in Table 2.

The three-layer granular structures (Macro-Top, Meso-Middle, and Micro-Bottom) are mainly considered from a systematic viewpoint, the numeric result and hierarchical/granular relationships are described in Fig. 1.

At the Micro-Bottom, C-Class $[x]_A^i$ and D-Class X_j are of concern. They exist in approximate space (U, AT) and can produce some fundamental measures, including probabilities. By connecting the Meso-Middle and its reasoning mechanism, three-way probabilities become bottomed measures that underlie informational construction at higher levels.

Table 1 Conditional and decisional classifications and classes

Item	C-Classification	C-Class	D-Classification	D-Class
Mathematical symbol	$U/IND(A)$	$[x]_A^i, i = 1,2,\ldots,n$	$U/IND(D)$	X_j,j=1,2,...,m
Granular essence	Conditional granule set	Conditional granule	Decisional granule set	Decisional granule

Table 2 Basic descriptions of a D-Table's three-layer granular structures

Structure	Composition	Granular scale	Granular level	Simple name
(1)	$U/IND(A), U/IND(D)$	Macro	Top	Macro-Top
(2)	$U/IND(A), X_j$	Meso	Middle	Meso-Middle
(3)	$[x]_A^i, X_j$	Micro	Bottom	Micro-Bottom

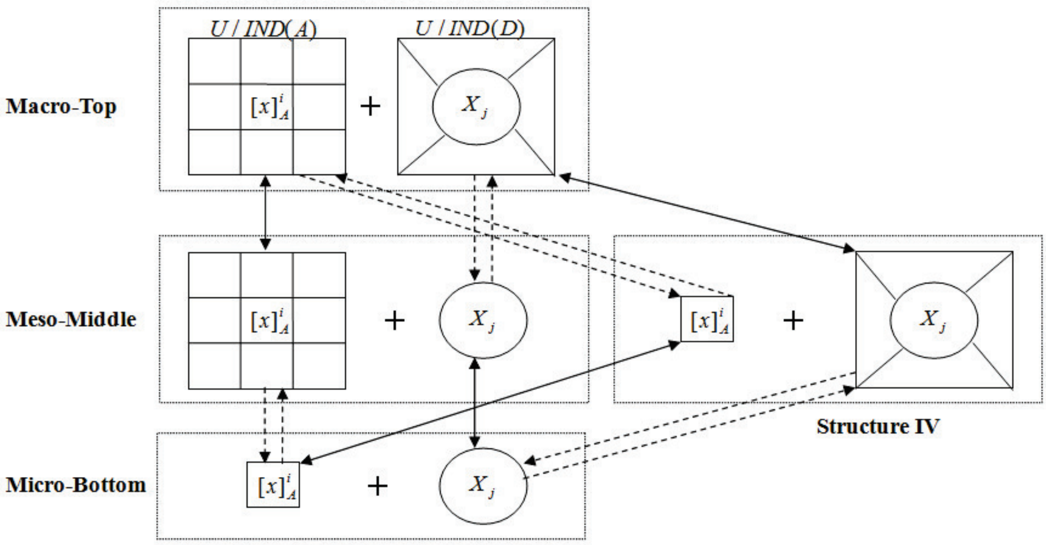

Fig. 1 Hierarchical/Granular relationships of the D-Table's three-layer granular structures.

Definition 1. At the Micro-Bottom, C-Class $[x]_A^i$ and D-Class X_j are of concern. The three-way probabilities are defined by:

$$P([x]_A^i) = \frac{|[x]_A^i|}{|U|}, P([x]_A^i/X_j) = \frac{|[x]_A^i \cap X_j|}{|X_j|}, P(X_j/([x]_A^i)) = \frac{|[x]_A^i \cap X_j|}{|[x]_A^i|}. \tag{1}$$

Theorem 1. *Three-way probabilities hold systematicness with regard to Bayes' theorem:*

$$P([x]_A^i/X_j) = \frac{P([x]_A^i) \times P(X_j/[x]_A^i)}{P(X_j)}. \tag{2}$$

2.2 Information theory of combination entropy

Regarding the D-Table, this subsection reviews the information theory of combination entropy on classifications by Ref. [12].

Definition 2. Let $K = (U,R)$ be an approximation space, $U/IND(R) = \{X_1, X_2, \ldots, X_m\}$ a partition of U. Combination entropy of R is defined as:

$$CE(R) = \sum_{i=1}^{m} \frac{\left|[x]_R^i\right|}{|U|} \frac{C_{|U|}^2 - C_{|[x]_R^i|}^2}{C_{|U|}^2} = \sum_{i=1}^{m} \frac{\left|[x]_R^i\right|}{|U|} \left(1 - \frac{C_{|[x]_R^i|}^2}{C_{|U|}^2}\right), \tag{3}$$

where $C_{|[x]_R^i|}^2 = \frac{|[x]_R^i| \times (|[x]_R^i| - 1)}{2}$, $\frac{|[x]_R^i|}{|U|}$ represents the probability of an equivalence X_i within the universe U, and $\frac{C_{|U|}^2 - C_{|[x]_R^i|}^2}{C_{|U|}^2}$ denotes the probability of pairs of the elements which are distinguishable each other within the whole number of pairs of the elements on the universe U.

Proposition 2. *Let $K_1 = (U,R)$ and $K_2 = (U,Q)$ be two approximation spaces, then $CE(P) > CE(Q)$ if $P \prec Q$.*

3 Three-way weighted combination-entropies at the Meso-Middle

Based on three-way probabilities at the Micro-Bottom, this subsection constructs three-way weighted combination-entropies at the Meso-Middle using the Bayes' theorem and discusses their granulation monotonicity and systematicness. Relevant results take a link function to underlie the latter informational construction at the Macro-Top.

A promotional measure at the Meso-Middle requires probability fusion when integrating C-Classes into C-Classification. And because Bayes' theorem provides systematicness of three-way probabilities. So it becomes the starting point. Herein, we first make a key transformation for Bayes' theorem. According to Theorem 1. with stable X_j,

Because, $P([x]_A^i/X_j) = \frac{P([x]_A^i) \times P(X_j/([x]_A^i))}{P(X_j)}, \forall i = 1, 2, \ldots, n$. So,

$$P(X_j)P(([x]_A^i)/X_j) = P([x]_A^i) \times P(X_j/([x]_A^i)). \tag{4}$$

Then, the Eq.(4) on both sides is multiplied by $\left(1 - \frac{C_{|[x]_A^i \cap X_j|}^2}{C_{|X_j|}^2}\right)$, we have:

$$P(X_j) \times P([x]_A^i)/X_j) \left(1 - \frac{C_{|[x]_A^i \cap X_j|}^2}{C_{|X_j|}^2}\right) = P([x]_A^i)P(X_j/[x]_A^i) \left(1 - \frac{C_{|[x]_A^i \cap X_j|}^2}{C_{|X_j|}^2}\right) \tag{5}$$

then, the right side of Eq.(5) can be calculated as follows:

$$
\begin{aligned}
&P([x]_A^i)P(X_j/[x]_A^i)\left(1-\frac{C^2_{|[x]_A^i\cap X_j|}}{C^2_{|X_j|}}\right)\\
&=P([x]_A^i)P\left(X_j/([x]_A^i)\right)-P([x]_A^i)P(X_j/([x]_A^i))\,\frac{C^2_{|[x]_A^i\cap X_j|}}{C^2_{|X_j|}}\\
&=P([x]_A^i)P(X_j/([x]_A^i))-P([x]_A^i)P(X_j/([x]_A^i))\frac{C^2_{\left|[x]_A^i\cap X_j\right|}}{C^2_{|X_j|}}\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{\left|[x]_A^i\right|}}\\
&=P([x]_A^i)P\left(X_j/([x]_A^i)\right)-P\left(X_j/([x]_A^i)\right)\left(\frac{C^2_{\left|[x]_A^i\cap X_j\right|}}{C^2_{\left|[x]_A^i\right|}}\right)P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)\\
&=P([x]_A^i)P(X_j/([x]_A^i))-P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)P(X_j/([x]_A^i))+P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)\left(P(X_j/([x]_A^i))-P(X_j/([x]_A^i))\left(\frac{C_{|[x]_A^i\cap X_j|^2}}{C^2_{\left|[x]_A^i\right|}}\right)\right)\\
&=P([x]_A^i)P(X_j/([x]_A^i))-P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)P(X_j/([x]_A^i))\left(\frac{C^2_{|U|}}{C^2_{|U|}}\right)+P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)\times\\
&\quad\left[P(X_j/([x]_A^i))-P(X_j/([x]_A^i))\left(\frac{C^2_{\left|[x]_A^i\cap X_j\right|}}{C^2_{|[x]_A^i|}}\right)\right]\\
&=P([x]_A^i)P(X_j/([x]_A^i))-P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|U|}}\right)P(X_j/([x]_A^i)\left(\frac{C^2_{|U|}}{C^2_{|X_j|}}\right)+P([x]_A^i)P(X_j/([x]_A^i))\left(\frac{C^2_{|U|}}{C^2_{|X_j|}}\right)-\\
&\quad P([x]_A^i)P(X_j/([x]_A^i))\frac{C^2_{|U|}}{C^2_{|X_j|}}+P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)\left[P(X_j/[x]_A^i)-P(X_j/[x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\cap X_j\right|}}{C^2_{\left|[x]_A^i\right|}}\right)\right]\\
&=P(X_j/[x]_A^i)\left(\frac{C^2_{|U|}}{C^2_{|X_j|}}\right)\left[P([x]_A^i)-P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|U|}}\right)\right]+P([x]_A^i)P(X_j/([x]_A^i))-P([x]_A^i)P(X_j/([x]_A^i))\left(\frac{C^2_{|U|}}{C^2_{|X_j|}}\right)+\\
&\quad P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)\left[P(X_j/[x]_A^i)-P(X_j/([x]_A^i))\left(\frac{C^2_{\left|[x]_A^i\cap X_j\right|}}{C^2_{|[x]_A^i|}}\right)\right],
\end{aligned}
\tag{6}
$$

thus,

$$
\begin{aligned}
&P(X_j)\times P([x]_A^i)/X_j)\left(1-\frac{C^2_{|[x]_A^i\cap X_j|}}{C^2_{|X_j|}}\right)\\
&=P(X_j/[x]_A^i)\left(\frac{C^2_{|U|}}{C^2_{|X_j|}}\right)\left[P([x]_A^i)-P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|U|}}\right)\right]+P([x]_A^i)\left(\frac{C^2_{\left|[x]_A^i\right|}}{C^2_{|X_j|}}\right)[P(X_j/([x]_A^i))-\\
&\quad P(X_j/[x]_A^i)\frac{C^2_{\left|[x]_A^i\cap X_j\right|}}{C^2_{\left|[x]_A^i\right|})}]+P([x]_A^i)P(X_j/([x]_A^i))-P([x]_A^i)P(X_j/[x]_A^i)\frac{C^2_{|U|}}{C^2_{|X_j|}}.
\end{aligned}
\tag{7}
$$

According to the i-based summation,

$$
\begin{aligned}
&P(X_j)\times\sum_{i=1}^{m}P([x]_A^i)/X_j)\left(1-\frac{C^2_{|[x]_A^i\cap X_j|}}{C^2_{|X_j|}}\right)\\
&=\sum_{i=1}^{m}P(X_j/[x]_A^i)\frac{C^2_{|U|}}{C^2_{|X_j|}}\left[P([x]_A^i)\left(1-\frac{C^2_{|[x]_A^i|}}{C^2_{|U|}}\right)\right]+\sum_{i=1}^{m}P([x]_A^i)\frac{C^2_{|[x]_A^i|}}{C^2_{|X_j|}}\left[P(X_j/([x]_A^i)\left(1-\frac{C^2_{\left|[x]_A^i\cap X_j\right|}}{C^2_{|[x]_A^i|}}\right)\right]+\\
&\quad\sum_{i=1}^{m}P([x]_A^i)P(X_j/[x]_A^i))\left(1-\frac{C^2_{|U|}}{C^2_{|X_j|}}\right).
\end{aligned}
\tag{8}
$$

The final item in Eq. (8) becomes

$$\sum_{i=1}^{m} P([x]_A^i)P(X_j/([x]_A^i))\left(1-\frac{C_{|U|}^2}{C_{|X_j|}^2}\right) = P(X_j)\left(1-\frac{C_{|U|}^2}{C_{|X_j|}^2}\right). \tag{9}$$

The above step-by-step deduction implies the hierarchical evolution of Bayes' theorem. Bayes' theorem and its three-way probabilities at the Micro-Bottom evolve in the combination-entropy direction, and thus, weight-based combination-entropies and their relationships emerge at the Meco-Middle. Concretely, Eq. (9) provides a constant that is based on X_j, and thus, systematic Eq. (8) concerns three weighted and informational items. In Eq. (8) , except the final item, the others three terms derived from the combination entropy proposed in [12] are multiplied by the corresponding weight coefficients of specific probabilities. Next, we introduce the weighted combination-entropy. Suppose that (ξ, p_i) denotes a probability distribution and $\omega_i \geq 0$ means the weight, then, the weighted combination-entropy is defined as:

$$CE_\omega(\xi) = \sum_{i=1}^{n} \omega_i P_i(1-CP_i). \tag{10}$$

Definition 3. At the Meso-Middle, three-way weighted combination-entropies are defined by:

$$\begin{aligned} CE_\omega^{X_j}(A) &= \sum_{i=1}^{n} P(X_j/[x]_A^i)\frac{C_{|U|}^2}{C_{|X_j|}^2}\left[P([x]_A^i)\left(1-\frac{C_{\left|[x]_A^i\right|}^2}{C_{|U|}^2}\right)\right], \\ CE_\omega(A/X_j) &= P(X_j) \times \sum_{i=1}^{n} P([x]_A^i/X_j)\left(1-\frac{C_{\left|[x]_A^i \cap X_j\right|}^2}{C_{|X_j|}^2}\right), \\ C_\omega(X_j/A) &= \sum_{i=1}^{n} P([x]_A^i)\frac{C_{\left|[x]_A^i\right|}^2}{C_{|X_j|}^2}\left[P(X_j/[x]_A^i)\left(1-\frac{C_{\left|[x]_A^i \cap X_j\right|}^2}{C_{\left|[x]_A^i\right|}^2}\right)\right]. \end{aligned} \tag{11}$$

The weighted combination-entropy introduces weights into the combination entropy, where the weights refect the importance degrees for information receivers or attention degrees of information receivers. Concretely, $CE_\omega^{X_j}(A)$ improves absolute $\sum_{i=1}^{n} P([x]_A^i)\left(1-\frac{C_{|[x]_A^i|}^2}{C_{|U|}^2}\right)$ introducing relative $P(X_j/[x]_A^i)\frac{C_{|U|}^2}{C_{|X_j|}^2}$ to the importance weights, while $C_\omega(X_j/A)$ and $CE_\omega(A/X_j)$ respectively improve the relative $\sum_{i=1}^{n}\left[P(X_j/[x]_A^i)\left(1-\frac{C_{\left|[x]_A^i \cap X_j\right|}^2}{C_{\left|[x]_A^i\right|}^2}\right)\right]$ and $\sum_{i=1}^{n} P([x]_A^i/X_j)\left(1-\frac{C_{\left|[x]_A^i \cap X_j\right|}^2}{C_{|X_j|}^2}\right)$ by introducing absolute $P([x]_A^i)\frac{C_{\left|[x]_A^i\right|}^2}{C_{|X_j|}^2}$ and $P(X_j)$. In other words, three-way weighted combination-entropies inherit the essential uncertainty semantics of three-way properties by using different probability weights, and thus, can better describe the system regarding cause A and result X_j, hence, they become robust for uncertainty measurement. Next, we discuss their properties.

Theorem 3. *At the Meso-Middle, three-way weighted combination-entropies have granulation monotonicity. Concretely, if $P \preceq Q$, then, (1)$CE_\omega^{(X_j)}(P) \geq CE_\omega^{(X_j)}(Q)$, (2) $CE_\omega(P/X_j) \geq CE_\omega(Q/X_j)$, (3)$CE_\omega(X_j/P) \leq CE_\omega(X_j/Q)$.*

Proof. Since $P \preceq Q$, let $\cup_{t=1}^{k}[x]_P^t = [x]_Q$, then we have

(1)

$$
\begin{aligned}
CE_{\omega}^{X_j}(P) &= \sum_{t=1}^{k} P(X_j/([x]_P^t)\left(\frac{C_{|U|}^2}{C_{|X_j|}^2}\right)\left[P([x]_P^t)\left(1-\frac{C_{|[x]_P^t|}^2}{C_{|U|}^2}\right)\right]\\
&= \sum_{t=1}^{k} P(X_j/([x]_P^t)P([x]_P^t)\frac{C_{|U|}^2}{C_{|X_j|}^2} - \sum_{t=1}^{k} P(X_j/[x]_P^t)P([x]_P^t)\frac{C_{|U|}^2}{C_{|X_j|}^2}\frac{C_{|[x]_P^t|}^2}{C_{|U|}^2}\\
&= \sum_{t=1}^{k} P(X_j\cap[x]_P^t)\frac{C_{|U|}^2}{C_{|X_j|}^2} - \sum_{t=1}^{k} P(X_j\cap[x]_P^t)\frac{C_{|[x]_P^t|}^2}{C_{|X_j|}^2}\\
&= \sum_{t=1}^{k} P(X_j\cap[x]_P^t)\frac{C_{|U|}^2-C_{|[x]_P^t|}^2}{C_{|X_j|}^2}\\
&= P(X_j\cap[x]_P^1)\left(\frac{C_{|U|}^2-C_{|[x]_P^1|}^2}{C_{|X_j|}^2}\right)+P(X_j\cap[x]_P^2)\left(\frac{C_{|U|}^2-C_{|[x]_P^2|}^2}{C_{|X_j|}^2}\right)+\cdots+P(X_j\cap[x]_P^k)\left(\frac{C_{|U|}^2-C_{|[x]_P^k|}^2}{C_{|X_j|}^2}\right)\\
&\geq P(X_j\cap[x]_P^1)\left(\frac{C_{|U|}^2-C_{|[x]_Q|}^2}{C_{|X_j|}^2}\right)+P(X_j\cap[x]_P^2)\left(\frac{C_{|U|}^2-C_{|[x]_Q|}^2}{C_{|X_j|}^2}\right)+\cdots+P(X_j\cap[x]_P^k)\left(\frac{C_{|U|}^2-C_{|[x]_Q|}^2}{C_{|X_j|}^2}\right)\\
&= \left[P(X_j\cap[x]_P^1)+P(X_j\cap[x]_P^2)+\cdots+P(X_j\cap[x]_P^k)\right]\left(\frac{C_{|U|}^2-C_{|[x]_Q|}^2}{C_{|X_j|}^2}\right)\\
&= \sum_{t=1}^{k} P(X_j\cap[x]_P^t)\left(\frac{C_{|U|}^2-C_{|[x]_Q|}^2}{C_{|X_j|}^2}\right)\\
&= P(X_j\cap[x]_Q)\left(\frac{C_{|U|}^2-C_{|[x]_Q|}^2}{C_{|X_j|}^2}\right)\\
&= CE_{\omega}^{X_j}(Q).
\end{aligned}
\tag{12}
$$

(2)

$$
\begin{aligned}
&CE_{\omega}(P/X_j) = P(X_j)\times\sum_{t=1}^{k} P([x]_P^t/X_j)\left(1-\frac{C_{\left|[x]_P^t\cap X_j\right|}^2}{C_{|X_j|}^2}\right)\\
&= P(X_j)\times\left[P([x]_Q/X_j)-\sum_{t=1}^{k} P([x]_P^t/X_j)\frac{C_{\left|[x]_P^t\cap X_j\right|}^2}{C_{|X_j|}^2}\right]\\
&= P(X_j)\times P([x]_Q/X_j)-P(X_j)\times\left[P([x]_P^1/X_j)\frac{C_{\left|[x]_P^1\cap X_j\right|}^2}{C_{|X_j|}^2}+P([x]_P^2/X_j)\frac{C_{\left|[x]_P^2\cap X_j\right|}^2}{C_{|X_j|}^2}+\cdots+P([x]_P^k/X_j)\frac{C_{\left|[x]_P^k\cap X_j\right|}^2}{C_{|X_j|}^2}\right]\\
&\geq P(X_j)\times P([x]_Q/X_j)-P(X_j)\times\left[P([x]_P^1/X_j)\frac{C_{\left|[x]_Q\cap X_j\right|}^2}{C_{|X_j|}^2}+P([x]_P^2/X_j)\frac{C_{\left|[x]_Q\cap X_j\right|}^2}{C_{|X_j|}^2}+\cdots+P([x]_P^k/X_j)\frac{C_{\left|[x]_Q\cap X_j\right|}^2}{C_{|X_j|}^2}\right]\\
&= P(X_j)\times P([x]_Q/X_j)-P(X_j)\times\left[P([x]_P^1/X_j)+P([x]_P^2/X_j)+\cdots+P([x]_P^k/X_j)\right]\left(\frac{C_{\left|[x]_Q\cap X_j\right|}^2}{C_{|X_j|}^2}\right)\\
&= P(X_j)\times P([x]_Q/X_j)-P(X_j)\times\sum_{i=1}^{k} P([x]_P^i/X_j)\left(\frac{C_{\left|[x]_Q\cap X_j\right|}^2}{C_{|X_j|}^2}\right)\\
&= P(X_j)\times P([x]_Q/X_j)-P(X_j)\times P([x]_Q/X_j)\left(\frac{C_{\left|[x]_Q\cap X_j\right|}^2}{C_{|X_j|}^2}\right)\\
&= P(X_j)\times P([x]_Q/X_j)\left(1-\frac{C_{\left|[x]_Q\cap X_j\right|}^2}{C_{|X_j|}^2}\right) = CE_{\omega}(Q/X_j).
\end{aligned}
\tag{13}
$$

(3)

$$\begin{aligned}
C_\omega(X_j/P) &= \sum_{t=1}^{k} P([x]_P^t)\frac{C^2_{|[x]_P^t|}}{C^2_{|X_j|}}\left[P(X_j/[x]_P^t)\left(1-\frac{C^2_{|[x]_P^t\cap X_j|}}{C^2_{|[x]_P^t|}}\right)\right]\\
&= \sum_{t=1}^{k} p([x]_p^t)p(X_j/[x]_p^t)\left[\frac{C^2_{|[x]_p^t|}}{C^2_{|X_j|}}\left(1-\frac{C^2_{|X_j\cap[x]_p^t|}}{C^2_{|[x]_p^t|}}\right)\right]\\
&= \sum_{t=1}^{k} p(X_j\cap[x]_p^t)\left(\frac{C^2_{|[x]_p^t|}-C^2_{|X_j\cap[x]_p^t|}}{C^2_{|X_j|}}\right)\\
&= p(X_j\cap[x]_p^1)\left(\frac{C^2_{|[x]_p^1|}-C^2_{|X_j\cap[x]_p^1|}}{C^2_{|X_j|}}\right)+p(X_j\cap[x]_p^2)\left(\frac{C^2_{|[x]_p^2|}-C^2_{|X_j\cap[x]_p^2|}}{C^2_{|X_j|}}\right)+\cdots+p(X_j\cap[x]_p^k)\left(\frac{C^2_{|[x]_p^k|}-C^2_{|X_j\cap[x]_p^k|}}{C^2_{|X_j|}}\right)\\
&\le p(X_j\cap[x]_p^1)\left(\frac{C^2_{|[x]_Q|}-C^2_{|X_j\cap[x]_Q|}}{C^2_{|X_j|}}\right)+p(X_j\cap[x]_p^2)\left(\frac{C^2_{|[x]_Q|}-C^2_{|X_j\cap[x]_Q|}}{C^2_{|X_j|}}\right)+\cdots+p(X_j\cap[x]_p^k)\left(\frac{C^2_{|[x]_Q|}-C^2_{|X_j\cap[x]_Q|}}{C^2_{|X_j|}}\right)\\
&= \sum_{t=1}^{k} p(X_j\cap[x]_p^t)\left(\frac{C^2_{|[x]_Q|}-C^2_{|X_j\cap[x]_Q|}}{C^2_{|X_j|}}\right)\\
&= \left[\sum_{t=1}^{k} p(X_j\cap[x]_p^t)\right]\left(\frac{C^2_{|[x]_Q|}-C^2_{|X_j\cap[x]_Q|}}{C^2_{|X_j|}}\right)\\
&= p(X_j\cap[x]_Q)\left(\frac{C^2_{|[x]_Q|}-C^2_{|X_j\cap[x]_Q|}}{C^2_{|X_j|}}\right) = CW_\omega(X_j/Q).\square
\end{aligned} \tag{14}$$

Theorem 4. *Three-way weighted combination-entropies have systematicness:*

$$CE_\omega(A/X_j) = CE_\omega^{X_j}(A) + CE_\omega(X_j/A) + P(X_j)\left(1-\frac{C^2_{|U|}}{C^2_{|X_j|}}\right). \tag{15}$$

Theorem 4 provides an important relationship for the three-way weighted combination-entropies. In other words, $CE_\omega(A/X_j)$ is a linear translation of the sum of $CE_\omega^{X_j}(A)$ and $CE_\omega(X_j/A)$, where $P(X_j)[1-(C^2_{|U|})/(C^2_{|X_j|})]$ is a constant at the Meso-Middle. And it develops Bayes' theorem at the Micro-Bottom to establish a systematic equation of three-way weighted combination-entropies. Furthermore, eliminating the conversion distance can produce a new measure to simplify the systematic equation.

Definition 4. At the Meso-Middle, the linear weighted combination-entropy with regard to the weighted combination-entropy $CE_\omega(X_j/A)$ is defined as:

$$CE_\omega^{lin}(X_j/A) = CE_\omega(X_j/A) + P(X_j)\left(1-\frac{C^2_{|U|}}{C^2_{|X_j|}}\right). \tag{16}$$

Corollary 5. *At the Meso-Middle, the linear weighted combination-entropy has granulation monotonicity. Concretely, if $P \preceq Q$, then, $CE_\omega^{lin}(X_j/P) \le CE_\omega^{lin}(X_j/Q)$.*

Corollary 6. *Three-way weighted combination-entropies have the equivalent systematicness:*

$$CE_\omega(A/X_j) = CE_\omega^{X_j}(A) + CE_W^{lin}(X_j/A). \tag{17}$$

The linear weighted combination-entropy $CE_\omega^{lin}(X_j/A)$ corresponds to $CE_\omega(X_j/A)$ by virtue of a specific linear transformation. The former uses the superscript *lin* (which means linear) to different from the latter, but both are viewed as only one item for three-way weighted combination-entropies. In contrast to $CE_\omega(X_j/A)$,

$CE_{\omega}^{lin}(X_j/A)$ exhibits same granulation monotonicity, and it simplfies the systematicness of three-way weighted combination-entropies.

In summary, this section at the Meso-Middle becomes important to link the Micro-Bottom and Macro-Top. Bayes' theorem provides three-way probabilities systematicness, and it further plays a fundamental role in the informational evolution of weighted combination-entropies. It induces essential measures and systematic equations of three-way weighted combination-entropies. Next, three-way weighted combination-entropies are promoted from the Meso-Middle to the Macro-Top. combination

4 Three-way weighted combination-entropies at the macro-top

For three-way weighted combination-entropies at the Meso-Middle, their monotonicity and systematicness are established. They can hierarchically evolve to Macro-Top by using the natural sum integration with regard to multiple D-Classes. This subsection constructs three-way weighted combination-entropies at the Macro-Top and offers their monotonicity and systematicness.

Definition 5. At Macro-Top, three-way weighted combination-entropies are defined by:

$$\begin{aligned} CE_{\omega}^{D}(A) &= \sum_{j=1}^{m} CE_{\omega}^{X_j}(A), \\ CE_{\omega}(A/D) &= \sum_{j=1}^{m} CE_{\omega}(A/X_j), \\ CE_{\omega}(D/A) &= \sum_{j=1}^{m} CE_{\omega}(X_j/A). \end{aligned} \tag{18}$$

Corollary 7. *$CE_{\omega}^{lin}(D/A)$ is a linear transformation of $CE_{\omega}(D/A)$. Thus,*

$$CE_{\omega}^{lin}(D/A) = \sum_{j=1}^{m} CE_{\omega}(X_j/A) + \sum_{j=1}^{m} P(X_j)\left(1 - \frac{C_{|U|}^{2}}{C_{|X_j|}^{2}}\right) = CE_{\omega}(D/A) + CE(D), \tag{19}$$

where combination entropy $CE(D) = \sum_{j=1}^{m} P(X_j)[1-(C_{|U|}^{2})/(C_{|X_j|}^{2})]$ is a constant.

$CE_{\omega}^{lin}(D/A)$ and $CE_{\omega}(D/A)$ exhibit a linear transformation to be viewed as only one item. Three-way weighted combination-entropies at Macro-Top depend on the sum integration to naturally inherit monotonicity and systematicness at the Meso-Middle, and the relevant features are presented as follows.

Theorem 8. *At Macro-Top, three-way weighted combination-entropies have granulation monotonicity. Concretely, if $P \preceq Q$, then, $CE_{\omega}^{D}(P) \geq CE_{\omega}^{D}(Q)$; $CE_{\omega}(P/D) \geq CE_{\omega}(Q/D)$, $CE_{\omega}(D/P) \leq CE_{\omega}(D/Q)$, $CE_{\omega}^{lin}(D/P) \leq CE_{\omega}^{lin}(D/Q)$.*

Theorem 9. *Three-way weighted combination-entropies have systematicness:*

$$CE_{\omega}(A/D) = CE_{\omega}^{D}(A) + CE_{\omega}(D/A) + CE(D) = CE_{\omega}^{D}(A) + CE_{\omega}^{lin}(D/A) \tag{20}$$

At Macro-Top, Theorem 9 describes an important relationship of the three-way weighted combination-entropies by introducing $CE(D)$. Thus, $CE_{\omega}(A/D)$ is a linear translation of the summation of $CE_{\omega}^{D}(A)$ and $CE_{\omega}^{lin}(D/A)$ or the difference between $CE_{\omega}(D/A)$ and $CE_{\omega}^{lin}(D/A)$.

With regard to the Meso-Middle, the Macro-Top exhibits the hierarchical promotion and systematic integration from D-Classes to D-Classification. Accordingly, three-way weighted combination-entropies at Macro-Top are interestedly fused by three-way weighted combination-entropies at the Meso-Middle, and they exhibit a type of informational summation. The relevant results are well clarified in a relationship as shown Fig. 2.

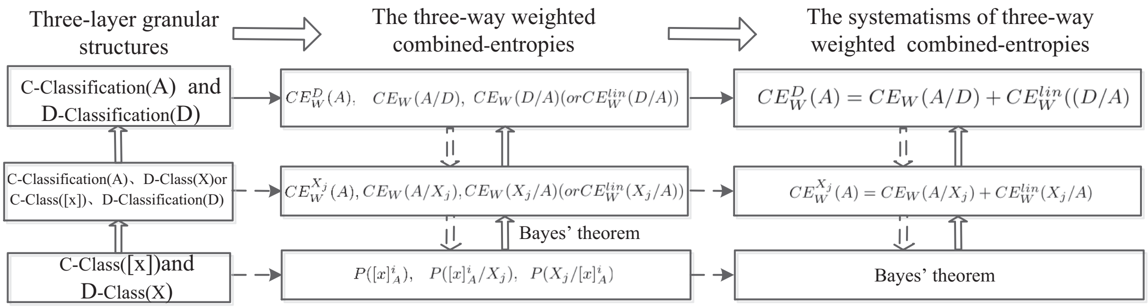

Fig. 2 Three-way weighted combination-entropies based on three-layer granular structures.

5 Conclusion

In summary, based on the new perspective of three-layer granular structures and Bayes' theorem, this paper concretely constructed three-way weighted combination-entropies, and revealed the granulation monotonicity and systematic relationships of the weighted combination-entropies. The relevant conclusion provided a more complete and updated the interpretation of granular computing for the uncertainty measurement, and established more effective basis of the quantitative application with attribute reduction.

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