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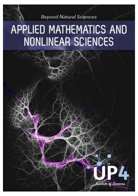

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An iterative method for non-autonomous nonlocal reaction-diffusion equations

Tomás Caraballo, Marta Herrera-Cobos, Pedro Marín-Rubio†.

Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, SPAIN

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Abstract

In this paper we provide a method to prove the existence of weak solutions for a type of non-autonomous nonlocal reaction-diffusion equations. Due to the presence of the nonlocal operator in the diffusion term, we cannot apply the Monotonicity Method directly. To use it, we build an auxiliary problem with linear diffusion and later, through iterations and compactness arguments, we show the existence of solutions for the nonlocal problem.

Keywords: Nonlocal diffusion; Non-autonomous reaction-diffusion equations; monotone, iterative and compactness arguments.
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1 Introduction and statement of the problem

The diffusion of a bacteria in a container or the study of the behaviour of a population which tends to move away from overpopulated areas are more accurate when the problem is modelled through a nonlocal diffusion equation (for more details see [9–12, 20]). In the last decades, many authors have been interested in studying a variant of the heat equation with a nonlocal diffusion term, namely

$$\frac{du}{dt} - a(l(u))\Delta u = f,$$

where the function a is continuous and greater than a positive constant and $l \in \mathscr{L}(L^2(\Omega), \mathbb{R})$. One of the difficulties of this kind of problems is that the existence of a Lyapunov structure is not always guaranteed, only in particular situations (for more details cf. [15, 16], for p-laplacian see [13]).

†Corresponding author.
Email address: pmr@us.es

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Despite the limitations, Chipot and his coworkers have dealt with this type of equations in the last decades. Namely, they have been interested in studying the asymptotic behaviour of the solutions. In [8,10,11], assuming that function a is locally Lipschitz, they analyzed results which establish order relationships between the solution of the evolution problem and the corresponding stationary solutions. In fact, under suitable assumptions, they are able to prove the strong convergence of the solution of the evolution problem towards a stationary solution. To obtain their goal, they apply a wide range of techniques. For instance, dynamical systems are used in [11], Maximum Principle is applied in [10], a Lyapunov structure and minimizers of energy are used in [15], and relationships between the Lipschitz constant of the function a and the lower bound m are established in [14].

Besides some authors deal with equations with more than one nonlocal operator, for instance

$$\frac{du}{dt} - A(l(u))u + a_0(l(u))u = f,$$

being $A = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ a uniformly elliptic operator (cf. for more details [7]). Issues as existence and uniqueness of weak solution and stationary solution, and the exponential decay of the evolution problem towards a stationary solution are analyzed. In addition, in a simpler framework

$$\frac{du}{dt} - a(l(u))\Delta u + u = f,$$

Chipot and Chang [7] also prove results which establish relationships between weak solutions and stationary solutions similarly to the cited above, and the strong convergence of the weak solution towards a stationary solution (cf. [7, Lemma 4.5] and [7, Theorem 4.2] respectively).

Moreover, in this setting where the forcing term is linear, there exist some partial results concerning the existence of global attractors for autonomous nonlocal problems developed by Lovat [20] and Andami Ovono [1].

Other authors have been interested in including nonlinear and time-dependent forcing terms and have analysed the non-autonomous nonlocal diffusion problem

$$\begin{cases} \dfrac{du}{dt} - a(l(u))\Delta u = f(u) + h(t) \quad \text{in } \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\ u(x,\tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where

$$a \in C(\mathbb{R}; [m, \infty)), \tag{2}$$

being $m > 0$, l a continuous linear form on $L^2(\Omega)$, i.e.

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx \quad \text{for some } g \in L^2(\Omega), \tag{3}$$

$u_\tau \in L^2(\Omega)$ and $h \in L^2_{loc}(\mathbb{R}; H^{-1}(\Omega))$. When the functions a and f are also globally Lipschitz, the existence and uniqueness of weak solution to (1) have been proved in [21] by Menezes making use of fixed point arguments. Later, this result is proved in [3] via the compactness method when a is locally Lipschitz and f is just continuous, sublinear and fulfils for some $\eta > 0$

$$(f(s) - f(r))(s - r) \leq \eta(s - r)^2 \quad \forall s, r \in \mathbb{R}. \tag{4}$$

There exist also results concerning the existence of pullback attractors in $L^2(\Omega)$ and the upper semicontinuity property of attractors in the multi-valued framework for a variation of (1) with perturbations (see [4] for more details). In this case, the function $f \in C(\mathbb{R})$ fulfils

$$-\kappa - \alpha_1|s|^p \leq f(s)s \leq \kappa - \alpha_2|s|^p \quad \forall s \in \mathbb{R}, \tag{5}$$

where α_1, α_2, and κ are positive constants and $p \geq 2$. Observe that from (5), we can deduce that there exists $\beta > 0$ such that

$$|f(s)| \leq \beta(|s|^{p-1}+1) \quad \forall s \in \mathbb{R}. \tag{6}$$

Interested in improving the assumptions made in [3], there exist recent results that show, via the compactness method, the existence and uniqueness of a weak solution to (1) when the function $f \in C(\mathbb{R})$ satisfies (4) and (5) (see [6] for more details).

The goal of this paper is to show the existence of weak solutions to (1) on a bounded open set $\Omega \subset \mathbb{R}^N$ under assumptions (2)–(5), via the monotonicity method (cf. [19, Chapitre 2]), together with an iterative procedure and compactness arguments. Although the idea of using iterations to deal with an equation with nonlocal diffusion was applied in the proof of [17, Theorem 1.1], as far as we know, there are no references that combine monotonicity, iterations and compactness, to prove existence results in the nonlocal setting. In addition, we would like to point out that although making use of compactness by translation the existence of weak solutions to (1) can be proved, due to the several difficulties existing with this type of problems we consider that it is interesting to be able to attack the equation in so many different ways as possible (in order to try to apply to new extensions). In particular here we combine iterations, compactness arguments and monotonicity as mentioned before. Observe that this new method allows to use the well-known monotonicity method for solving PDEs even in the nonlocal framework and for general linear operators (see Conclusions and final comments at the end of the manuscript for more details).

The structure of the paper is as follows. In Section 2 (and just for completeness to the reader), we recall the monotonicity method for solving nonlinear PDEs. After that, in Section 3, we prove the existence of weak solutions for a non-autonomous nonlocal reaction-diffusion problem combining the previous results from Section 2 with compactness arguments and iterations. To conclude, we provide some comments about how to extend this procedure to some other linear operators.

Let us introduce the notation used throughout the paper. As usual, the inner product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$ and by $|\cdot|$ the norm associated to it. The inner product in $H_0^1(\Omega)$, which is given by the product in $(L^2(\Omega))^N$ of the gradients, is represented by $((\cdot,\cdot))$ and by $\|\cdot\|$ its associated norm. By $\langle\cdot,\cdot\rangle$, we represent the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, and by $\|\cdot\|_*$ the norm in $H^{-1}(\Omega)$. We identify $L^2(\Omega)$ with its dual, and therefore we have the chain of compact and dense embeddings $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. Observe that as a result of the previous identification, we can abuse of the notation considering $l \in L^2(\Omega)$ but continue denoting (l,u) as $l(u)$. The duality product between $L^p(\Omega)$ and $L^q(\Omega)$ (where q is the conjugate exponent of p) will be denoted by $(\cdot,\cdot)$, and the norm in $L^s(\Omega)$ will be represented by $\|\cdot\|_{L^s(\Omega)}$ with $s \geq 1$. We also denote by $\langle\cdot,\cdot\rangle$ the duality product between $H^{-1}(\Omega)+L^q(\Omega)$ and $H_0^1(\Omega)\cap L^p(\Omega)$.

Definition 1. A weak solution to (1) is a function $u \in L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$ for all $T > \tau$, with $u(\tau) = u_\tau$, such that for all $v \in H_0^1(\Omega)\cap L^p(\Omega)$

$$\frac{d}{dt}(u(t),v) + a(l(u(t)))((u(t),v)) = (f(u(t)),v) + \langle h(t),v\rangle, \tag{7}$$

where the previous equation must be understood in the sense of the distributions $\mathscr{D}'(\tau,\infty)$.

Remark 1. If u is a weak solution to (1), then taking into account that the function a is continuous, $l \in L^2(\Omega)$, (6) and (7), it fulfils for any $T > \tau$ that $u' \in L^2(\tau,T;H^{-1}(\Omega)) + L^q(\tau,T;L^q(\Omega))$, and therefore $u \in C([\tau,\infty);L^2(\Omega))$ (cf. [19, Théorème 2, p. 575]). Observe that now the initial datum in (1) makes sense. In addition, it holds

$$|u(t)|^2 + 2\int_s^t a(l(u(r)))\|u(r)\|^2 dr = |u(s)|^2 + 2\int_s^t (f(u(r)),u(r))dr + 2\int_s^t \langle h(r),u(r)\rangle dr$$

for all $\tau \leq s \leq t$ (cf. [18, Théorème 2, p. 575] or [22, Lemma 3.2, p. 71]).

2 Brief recalling of the monotonicity method for solving nonlinear PDEs

In this section we provide a short summary about the requirements to apply the monotonicity method for solving nonlinear PDEs (see [19, Chapitre 2] for more details).

Consider a separable Hilbert space H, whose norm is denoted by $|\cdot|$. Moreover, suppose given V_i, $i = 1,\dots,m$, with $m \geq 1$, separable and reflexive Banach spaces, such that $\bigcup_{i=1}^m V_i \subset H$, $\bigcap_{i=1}^m V_i$ is dense in H, and $V_i \subset H$ with continuous injection for all $i = 1,\dots,m$.

We denote by $\|\cdot\|_i$ and $\|\cdot\|_{*_i}$ the norms in V_i and V_i' respectively ($i = 1,\dots,m$). By V we represent the space $\bigcap_{i=1}^m V_i$. In addition, $\langle\cdot,\cdot\rangle$ denotes the duality product between V_i' and V_i for all $i = 1,\dots,m$. Finally, H is identified with its topological dual H' using the Riesz theorem.

Consider $T \in (\tau,\infty)$ fixed and let $B_i : (\tau,T) \times V_i \to V_i'$ be, for $i = 1,\dots,m$, operators, in general nonlinear, such that

A1) The mapping $(\tau,T) \ni t \mapsto B_i(t,v) \in V_i'$ is measurable for each $v \in V$.

A2) Each operator B_i is hemicontinuous, i.e. for all $t \in (\tau,T)$ and for all $u,v,w \in V_i$, the application $\theta \in (\tau,T) \mapsto \langle B_i(t,u+\theta v),w\rangle \in \mathbb{R}$ is continuous.

Suppose also that there exist $1 < p_i < \infty$, $i = 1,\dots,m$, at least one of them greater than or equal to 2, constants $c > 0$, $\alpha > 0$ and $\lambda \geq 0$, and a nonnegative function $C \in L^1(\tau,T)$, such that for all $t \in (\tau,T)$ it satisfies

A3) $B_i(t,\cdot)$ is bounded in V_i', i.e.
$$\|B_i(t,v)\|_{*_i} \leq c(1+\|v\|_i^{p_i-1}) \quad \forall v \in V_i.$$

A4) $B_i(t,\cdot)$ is monotone, i.e.
$$\langle B_i(t,v) - B_i(t,w), v-w\rangle + \lambda|v-w|^2 \geq 0 \quad \forall v,w \in V_i.$$

A5) $B_i(t,\cdot)$ is coercive, i.e.
$$\langle B_i(t,v),v\rangle + \lambda|v|^2 + C(t) \geq \alpha\|v\|_i^{p_i} \quad \forall v \in V_i.$$

Suppose given functions $h_i \in L^{p_i'}(\tau,T;V_i')$ for $i = 1,\dots,m$, and an initial datum $u_\tau \in H$.

In what follows we denote

$$B(t,v) = \sum_{i=1}^m B_i(t,v) \quad \forall v \in V,$$
$$h(t) = \sum_{i=1}^m h_i(t).$$

Now, we consider the problem

$$\begin{cases} u \in \bigcap_{i=1}^m L^{p_i}(\tau,T;V_i) \quad \forall T > \tau, \\ u'(t) + B(t,u(t)) = h(t) \quad \text{in } \mathscr{D}'(\tau,T;V'), \\ u(\tau) = u_\tau. \end{cases} \tag{8}$$

Then, we have the following result (see [19, Théorème 1.4, p. 168]). Observe that in the proof, it can be used any numerable family formed by linearly independent elements such that the vector space generated by this family is dense in V.

Theorem 1. *Under the above assumptions, there exists a unique solution u to the problem (8). In addition, this solution satisfies*

$$u \in C([\tau,T];H), \quad u' \in \sum_{i=1}^m L^{p_i'}(\tau,T;V_i').$$

3 Existence and uniqueness

In this section we will prove the existence of a weak solution to (1) using iterations, compactness arguments and the monotonicity method for solving nonlinear PDEs. Due to the presence of the nonlocal operator in the diffusion term, the monotonicity method recalled in Section 2 does not seem possible to be applied directly to problem (1). However we first apply this method to a non-autonomous reaction-diffusion equation in which the diffusion term has a viscosity which depends on time, but it does not depend on the solution. Then through iterations and appropriate estimates, we can prove the existence and uniqueness of a weak solution to (1) using compactness arguments.

For each $n \geq 1$, we denote by u^n the weak solution to

$$(P_n) \begin{cases} u \in L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega)) \cap C([\tau,T];L^2(\Omega)), \\ \dfrac{d}{dt}(u(t),v) + a(l(u^{n-1}(t)))((u(t),v)) = (f(u(t)),v) + \langle h(t),v\rangle, \\ u(\tau) = u_\tau, \end{cases}$$

with $u^0 \equiv 0$ and u^n is the solution to (P_n) if $n \geq 1$, where the equation in (P_n) must be understood in the sense of the distributions $\mathscr{D}'(\tau,\infty)$ for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$.

Corollary 2. *Suppose that (2)–(5) hold and $h \in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$. Then, for any $u_\tau \in L^2(\Omega)$ there exists a unique solution u^n to (P_n) for all $n \geq 1$.*

Proof. The existence and uniqueness of the solution to problem (P_n) is due to Theorem 1. Namely, take $H = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$, $p_1 = 2$, $V_2 = L^p(\Omega)$ and $p_2 = p$, and define

$$\begin{aligned} B_1(t,v) &= -a(l(u^{n-1}(t)))\Delta v \quad \forall v \in H_0^1(\Omega) \quad \forall t \in (\tau,T), \\ B_2(t,v) &= -f(v) \quad \forall v \in L^p(\Omega) \quad \forall t \in (\tau,T), \\ h_1(t) &= h(t) \quad \forall t \in (\tau,T), \\ h_2(t) &= 0 \quad \forall t \in (\tau,T). \end{aligned}$$

Then it is not difficult to check that B_1 and B_2 satisfy A1)-A5). As a result, there exists a unique solution to (P_n) for all $n \geq 1$.

Remark 2. Observe that in the above result the function a does not need to fulfil any Lipschitz condition in order to guarantee the uniqueness since in the problems (P_n) the viscosity $a(l(u^{n-1}))$ works *as a constant* on each iteration, arising the time-depending monotone operator B_1.

We are ready to prove the existence of weak solutions to (1).

Theorem 3. *Suppose that (2)–(5) hold and $h \in L^2_{loc}(\mathbb{R};H^{-1}(\Omega))$. Then for each $u_\tau \in L^2(\Omega)$, problem (1) possesses at least a weak solution.*

Proof. Given $u^0 \equiv 0$, by Corollary 2, problem (P_1) can be solved, arising u^1. Inductively, given u^{n-1}, denote by u^n the solution to (P_n). Then, we have

$$\frac{1}{2}\frac{d}{dt}|u^n(t)|^2 + a(l(u^{n-1}(t)))\|u^n(t)\|^2 = (f(u^n(t)),u^n(t)) + \langle h(t),u^n(t)\rangle \quad \text{a.e. } t \in (\tau,T).$$

Now, making use of (2) and (5), we obtain

$$\frac{d}{dt}|u^n(t)|^2 + m\|u^n(t)\|^2 + 2\alpha_2\|u^n(t)\|^p_{L^p(\Omega)} \leq 2\kappa|\Omega| + \frac{1}{m}\|h(t)\|_*^2 \quad \text{a.e. } t \in (\tau,T).$$

Integrating between τ and $t \in [\tau, T]$, we deduce that $\{u^n\}$ is bounded in $L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$. From this, taking into account that each $u^n \in C([\tau,T];L^2(\Omega))$, we deduce that there exists a positive constant $\widehat{C}$ such that

$$|u^n(t)| \leq \widehat{C} \quad \forall t \in [\tau,T] \quad \forall n \geq 1.$$

Then, using that the function $a \in C(\mathbb{R};\mathbb{R}_+)$ and $l \in L^2(\Omega)$, there exists a positive constant $M_{\widehat{C}}$ such that

$$a(l(u^{n-1}(t))) \leq M_{\widehat{C}} \quad \forall t \in [\tau,T] \quad \forall n \geq 1.$$

Therefore, it fulfils that $\{-a(l(u^{n-1}))\Delta u^n\}$ is bounded in $L^2(\tau,T;H^{-1}(\Omega))$. Moveover, making use of the boundedness of $\{u_n\}$ in $L^p(\tau,T;L^p(\Omega))$, we obtain that the sequence $\{f(u_n)\}$ is bounded in $L^q(\tau,T;L^q(\Omega))$.

Now, since it satisfies for all $n \geq 1$

$$\frac{du^n}{dt} - a(l(u^{n-1}))\Delta u^n = f(u^n) + h \quad \text{in } \mathscr{D}'(\tau,T;H^{-1}(\Omega)+L^q(\Omega)),$$

the sequence $\{(u^n)'\}$ is bounded in $L^2(\tau,T;H^{-1}(\Omega)) + L^q(\tau,T;L^q(\Omega))$. Therefore, there exist a subsequence of $\{u^n\}$ (relabeled the same), a function $u \in L^\infty(\tau,T;L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega))$ with $u' \in L^2(\tau,T;H^{-1}(\Omega)) + L^q(\tau,T;L^q(\Omega))$, $\xi_1 \in L^q(\tau,T;L^q(\Omega))$, and $\xi_2 \in L^2(\tau,T;H_0^1(\Omega))$, such that

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weakly-star in } L^\infty(\tau,T;L^2(\Omega)), \\ u^n \rightharpoonup u & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)), \\ u^n \rightharpoonup u & \text{weakly in } L^p(\tau,T;L^p(\Omega)), \\ u^n \to u & \text{strongly in } L^2(\tau,T;L^2(\Omega)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(\tau,T;H^{-1}(\Omega)) + L^q(\tau,T;L^q(\Omega)), \\ f(u^n) \rightharpoonup \xi_1 & \text{weakly in } L^q(\tau,T;L^q(\Omega)), \\ a(l(u^{n-1}))u^n \rightharpoonup \xi_2 & \text{weakly in } L^2(\tau,T;H_0^1(\Omega)). \end{cases} \tag{9}$$

Now, we will prove that $\xi_1 = f(u)$ and $\xi_2 = a(l(u))u$. From (9), we deduce

$$u^n(x,t) \to u(x,t) \quad \text{strongly} \quad \forall (x,t) \in (\Omega \times (\tau,T)) \setminus N_1, \tag{10}$$

$$u^n(t) \to u(t) \quad \text{strongly in } L^2(\Omega) \quad \forall t \in (\tau,T) \setminus N_2, \tag{11}$$

where N_1 is a null set in $\mathbb{R}^{N+1}$ and N_2 is a null set in $\mathbb{R}$.

Then, since the sequence $\{f(u^n)\}$ is bounded in $L^q(\tau,T;L^q(\Omega))$, $f \in C(\mathbb{R})$ and (10) holds, it fulfils that $\xi_1 = f(u)$, thanks to [19, Lemme 1.3, p. 12].

Finally, we will prove $\xi_2 = a(l(u))u$. Since $a \in C(\mathbb{R};\mathbb{R}_+)$, $l \in L^2(\Omega)$ and (11) holds, we have

$$a(l(u^{n-1}(t))) \to a(l(u(t))) \quad \forall t \in (\tau,T) \setminus N_2.$$

Therefore,

$$a(l(u^{n-1}(t)))u^n(x,t) \to a(l(u(t)))u(x,t) \quad \forall (x,t) \in (\Omega \times (\tau,T)) \setminus (N_1 \cup (\Omega \times N_2)),$$

where $N_1 \cup (\Omega \times N_2)$ is a null set in $\mathbb{R}^{N+1}$. From this and the boundedness of $\{a(l(u^{n-1}))u^n\}$ in $L^2(\tau,T;H_0^1(\Omega))$, it fulfils that $\xi_2 = a(l(u))u$, applying again [19, Lemme 1.3, p. 12].

Thereupon, to prove (7) for all $v \in H_0^1(\Omega) \cap L^p(\Omega)$, we fix $T > \tau$ and $\varphi \in \mathscr{D}(\tau,T)$. Since u^n is a solution to (P_n), it satisfies for all $n \geq 1$

$$\int_\tau^T (u^n(t),v)\varphi'(t)dt + \int_\tau^T a(l(u^{n-1}(t)))\langle -\Delta u^n(t),v\rangle \varphi(t)dt = \int_\tau^T \langle f(u^n(t)) + h(t),v\rangle \varphi(t)dt.$$

Taking limit when $n \to \infty$ in the previous expression and making use of (9), (7) holds.

Finally, to prove the existence of a weak solution to (1), we only need to check that $u(\tau) = u_\tau$. Observe that this equality makes complete sense since $u \in C([\tau,T];L^2(\Omega))$ (cf. [18, Théorème 2, p. 575]). To do this, consider fixed $v \in H_0^1(\Omega) \cap L^p(\Omega)$ and $\varphi \in H^1(\tau,T)$, with $\varphi(T) = 0$ and $\varphi(\tau) \neq 0$. Since u^n is a weak solution to (P_n), it fulfils

$$\frac{d}{dt}(u^n(t),v) + a(l(u^{n-1}(t)))((u^n(t),v)) = (f(u^n(t)),v) + \langle h(t),v\rangle \quad \text{a.e. } t \in (\tau,T).$$

Now, multiplying by φ in the previous expression and integrating between τ and T, we have

$$\begin{aligned} &-(u_\tau,v)\varphi(\tau) + \int_\tau^T (u^n(t),v)\varphi'(t)dt + \int_\tau^T a(l(u^{n-1}(t)))((u^n(t),v))\varphi(t)dt \\ &= \int_\tau^T (f(u^n(t)),v)\varphi(t)dt + \int_\tau^T \langle h(t),v\rangle\varphi(t)dt. \end{aligned}$$

Thereupon, taking limit in n, we obtain

$$\begin{aligned} &-(u_\tau,v)\varphi(\tau) + \int_\tau^T (u(t),v)\varphi'(t)dt + \int_\tau^T a(l(u(t)))((u(t),v))\varphi(t)dt \\ &= \int_\tau^T (f(u(t)),v)\varphi(t)dt + \int_\tau^T \langle h(t),v\rangle\varphi(t)dt. \end{aligned}$$

Otherwise, we deduce from (7)

$$\begin{aligned} &-(u(\tau),v)\varphi(\tau) + \int_\tau^T (u(t),v)\varphi'(t)dt + \int_\tau^T a(l(u(t)))((u(t),v))\varphi(t)dt \\ &= \int_\tau^T (f(u(t)),v)\varphi(t)dt + \int_\tau^T \langle h(t),v\rangle\varphi(t)dt. \end{aligned}$$

Comparing these last two equalities we deduce that $(u(\tau),v)\varphi(\tau) = (u_\tau,v)\varphi(\tau)$. Finally, since $\varphi(\tau) \neq 0$ and $H_0^1(\Omega) \cap L^p(\Omega)$ is dense in $L^2(\Omega)$, the equality $u(\tau) = u_\tau$ holds.

Corollary 4. *Under assumptions of Theorem 3, if additionally the function a is locally Lipschitz, then there exists a unique weak solution to (1), denoted by $u(\cdot) = u(\cdot;\tau,u_\tau)$. Moreover, this solution behaves continuously in $L^2(\Omega)$ with respect to initial data.*

Proof. Both assertions will be proved simultaneously since the same estimates are valid for both purposes. Suppose that u_1 and u_2 are two weak solutions to (1) corresponding to initial values $u_{1\tau}$, $u_{2\tau} \in L^2(\Omega)$ respectively. Then, from the energy equality we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u_1(t)-u_2(t)|^2 + a(l(u_1(t)))\|u_1(t)-u_2(t)\|^2 \\ &\leq |a(l(u_1(t))) - a(l(u_2(t)))|\|u_2(t)\|\|u_1(t)-u_2(t)\| + (f(u_1(t)) - f(u_2(t)), u_1(t)-u_2(t)) \ a.e.\ t > \tau. \end{aligned}$$

Since $u_1, u_2 \in C([\tau,T];L^2(\Omega))$, therefore $\{u_i(t)\}_{i=1,2} \subset S$ for all $t \in [\tau,T]$, where S is a bounded set of $L^2(\Omega)$. Moreover, it satisfies $\{l(u_i(t))\}_{i=1,2} \subset [-R,R]$ for all $t \in [\tau,T]$, for some $R > 0$, since $l \in L^2(\Omega)$. Therefore, using (2), (4) and the locally Lipschitz continuity of the function a, we have

$$\frac{d}{dt}|u_1(t)-u_2(t)|^2 \leq \frac{(L_a(R))^2|l|^2\|u_2(t)\|^2 + 4\eta m}{2m}|u_1(t)-u_2(t)|^2 \quad \text{a.e. } t \in (\tau,T),$$

where $L_a(R)$ is the Lipschitz constant of the function a in $[-R,R]$. Then, both statements, uniqueness and continuity w.r.t. initial data, hold.

Remark 3. Under the locally Lipschitz assumption for a, thanks to the uniqueness of weak solution to (1), if fulfils that the whole sequence $\{u^n\}$ converges to u weakly in $L^2(\tau,T;H_0^1(\Omega))\cap L^p(\tau,T;L^p(\Omega))$ and weakly-star in $L^\infty(\tau,T;L^2(\Omega))$. Analogously, the whole sequence $\{(u^n)'\}$ converges to u' weakly in $L^2(\tau,T;H^{-1}(\Omega))+L^q(\tau,T;L^q(\Omega))$.

Conclusions and final comments

We have presented a procedure to obtain existence of solution for a nonlocal reaction-difussion problem through a combination of different methods, namely monotonicity, iterations and compactness arguments. To go further on the study of the long-time behaviour and regularity issues related to (1) and variations we refer to [3–6] and the cited references in Section 1. As referred in that paragraph, these results ensure the global-in-time existence of solutions to some nonlocal models settled in Biology and Physics among other areas.

It is delicate how to generalise the method established in this paper to other nonlocal problems with monotone operators but nonlinear (see e.g. [5]). Now, we provide some extensions for linear operators.

The previous procedure can be applied to equations with more general linear diffusion terms with divergence form like $a(l(u))Au$, where $A=\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial}{\partial x_j}\right)$, with $b_{ij}\in L^\infty(\Omega)$ for all $i,j=1,\dots,N$ and $\sum_{i,j=1}^N b_{ij}(x)\xi_i\xi_j \geq \zeta|\xi|^2_{\mathbb{R}^N}$, where $\zeta>0$. In fact, it is possible to apply this procedure to nonlocal equations given by a sum of a finite family of terms under the above assumptions, namely

$$\frac{du}{dt}-\sum_{k=1}^m a_k(l_k(u))A_k u=\sum_{k=1}^m (f_k(u)+h_k(t)),$$

where the nonlocal operator $a_k(l_k(u))$ can be different for each k, f_k fulfils

$$-\kappa-\alpha_1|s|^{p_k}\leq f_k(s)s\leq \kappa-\alpha_2|s|^{p_k}\quad \forall s\in\mathbb{R},$$

and $\sum_k h_k$ belongs to $L^2_{loc}(\mathbb{R};H^{-1}(\Omega))+\sum_{k=1}^m L^{p_k'}_{loc}(\mathbb{R};L^{p_k'}(\Omega))$.

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