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Large solutions for cooperative logistic systems: existence and uniqueness in star-shaped domains

Luis Maire [†]

Departamento de Matemática Aplicada, Universidad Complutense de Madrid, Madrid
Spain

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Abstract

We extend Theorem 1.1 of [J. Math. Anal. Appl. **435** (2016), 1738–1752] to show the uniqueness of large solutions for the system of (1) in star-shaped domains. This result is due to the maximum principle for cooperative systems of [J. López-Gómez and M. Molina-Meyer, Diff. Int. Eq. **7** (1994), 383–398], which allows us to establish the uniqueness without invoking to the blow-up rates of the solutions.

Keywords: Large positive solution. Cooperative system. Logistic equation. Uniqueness. Keller–Osserman.
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1 Introduction

This paper studies the uniqueness of the solution of the singular elliptic problem

$$\begin{cases} -\Delta u_i = \sum_{j=1}^{n} a_{ij}(x)u_j - b_i(x)f_i(u_i) & \text{in} \quad \Omega, \\ u_i = +\infty & \text{on} \quad \partial\Omega, \end{cases} \qquad 1 \le i \le n, \tag{1}$$

where Ω is a bounded subdomain of $\mathbb{R}^N$, $N \ge 1$ whose boundary is sufficiently regular (e.g. of class $\mathscr{C}^1$) and the heterogeneous terms satisfy $a_{ij}, b_i \in \mathscr{C}(\bar{\Omega})$ for all $1 \le i, j \le n$,

$$\begin{array}{ll} a_{ij}(x) > 0 \; 1 \le i \ne j \le n, \\ b_i(x) > 0 \quad 1 \le i \le n, \end{array} \quad \text{for all } \; x \in \Omega. \tag{2}$$

[†]Corresponding author.
Email address: luismaire@ucm.es

As far as concerns the nonlinear terms, it is assumed that $f_i \in \mathscr{C}^1[0,\infty)$ is a nondecreasing function such that $f_i(0) = 0$, for all $1 \le i \le n$. A function $u := (u_1,\dots,u_n) \in \left[\mathscr{C}^{2+\nu}(\Omega)\right]^n$, $\nu \in (0,1)$, is a solution of (1) if it satisfies the system of (1) and

$$\lim_{\substack{x\to z\\ x\in\Omega, z\in\partial\Omega}} u_i(x) = +\infty, \quad 1 \le i \le n.$$

These functions will be referred as *large solutions*. Its study goes back to the pioneering works of J. B. Keller [8] and R. Osserman [20], who considered the case of the single equation $\Delta u = f(u)$ when f is a monotone function. Since them, many works have dealt with large solutions of elliptic equations (see, e.g. the lists of references of [11]), but almost all of them are focused in the study of the single equation. Some of the few existing references for systems are [6, 7, 12–14], although, except [12], they only study the special case in which $n = 2$. Moreover, the majority of the works are restricted to the case in which the nonlinearities are of power-type.

In order to ensure the existence of large solutions of (1) one should ask for the following Keller-Osserman type condition:

(KO) There exists $f \in \mathscr{C}^1[0,+\infty)$ such that $f(u) < f_i(u)$ for every $1 \le i \le n$ and, for every $a,b > 0$,

$$I(c) := \int_c^{+\infty} \frac{d\theta}{\sqrt{\int_c^\theta bf(s) - asds}} < +\infty \quad \text{for some } c > 0.$$

This condition is a generalization of the one made by P. Álvarez-Caudevilla and J. López-Gómez in [1]. According to S. Dumont et al. [4], by the monotonicity of f we have that

$$\liminf_{c\to+\infty} I(c) = 0,$$

so the second part of the *Keller-Osserman* condition described in [1] is satisfied. The reader is sent to [11, Chapter 3] and [4] for a detailed discussion on Keller–Ossermann conditions. Essentially, (KO) is a condition on the growth of f_i at infinity. It is not hard to check that the existence of $p > 1$ and $C > 0$ such that

$$f_i(u) \ge Cu^p, \quad 1 \le i \le n, \tag{3}$$

for every $u > 0$ sufficiently large, entails (KO).

The assumption made on the first inequalities of (2) guarantees that the system of (1) is *cooperative*, so the maximum principle for cooperative systems of J. López-Gómez and M. Molina-Meyer, [16], which is a fundamental tool for the analysis carried out in this paper, is available. Thus, the usual comparison principle works in our context (see Theorem 2 of Section 2). In particular, we can adapt the construction of a maximal and a minimal large solution given in [1, Sections 3 and 4]. Then, under the general hypotheses of this paper and (KO), there exists a minimal and a maximal solution to (1) for every subdomain $\Omega \subset \mathbb{R}^N$ with $\partial\Omega$ sufficiently regular.

The uniqueness of solutions of (1) is still a widely open question, even when (1) reduces to a single equation, and the usual uniqueness argument for the equation via the *blow-up rates* does not have a trivial extension to cover (1) (see [14]). Our main result establishes the uniqueness of large solution of (1) when Ω is a star-shaped domain, i.e. when there exists a point $x_0 \in \Omega$, called the *center* of Ω, such that the line segment between x_0 and x belongs to Ω for every $x \in \Omega$. It can be stated as follows.

Theorem 1. *Suppose that Ω is star-shaped. Without loss of generality, we can suppose that the center of Ω is the origin; otherwise, the change of variables $y = x - x_0$ transforms x_0 into 0. Let D_0 be an open neighborhood of $\partial\Omega$ with the next property:*

- *There exists $\rho_0 > 1$ such that for every $1 \le \rho \le \rho_0$ and $x \in \Omega \cap D_0$ with $\rho x \in \Omega \cap D_0$,*

$$\begin{array}{c} \rho^2 a_{ij}(\rho x) \ge a_{ij}(x), \\ b_i(\rho x) \le b_i(x), \end{array} \quad 1 \le i,j \le n. \tag{4}$$

Assume that each f_i is super-homogeneous of degree $p_i > 1$, in the sense that for every $1 \le i \le n$ there exists $p_i > 1$ such that

$$f_i(tu) \ge t^p f_i(u) \quad \text{for all } t > 1,\ u > 0. \tag{5}$$

Then, (1) *has a unique positive solution.*

Theorem 1 is a substancial extension of [12, Theorem 1.1] and [15, Theorem 1.1]. Indeed, the main result of [12] establishes the uniqueness of solution for the radially symmetric counterpart of (1) with constant coupling coefficients, $a_{ij} \in \mathbb{R}_+$, $1 \le i \ne j \le n$, while [15, Theorem 1.1] only deals with (1) in the restricted case $n = 1$ and $a_{11} \in \mathbb{R}$. On the other hand, the case where Ω is an annular region is covered in [12] and [15].

The hypothesis (5) goes back to [10, Eq. (11)] and [3, Eq. (6)]. It is easily seen that (5) implies (KO), which ensures the existence of positive solutions of (1): Assuming (5), we have that (3) is satisfied for the choice $p := \min\{p_i \,:\, 1 \le i \le n\}$ and $C := \min\{f_i(1) \,:\, 1 \le i \le n\}$. Moreover, (5) implies that

$$f_i(u)/u \ \text{ is increasing}, \quad 1 \le i \le n, \tag{6}$$

because

$$\frac{f_i(\theta u) - f_i(u)}{\theta u - u} \ge \frac{\theta^{p_i} f_i(u) - f_i(u)}{\theta u - u} = \frac{\theta^{p_i} - 1}{\theta - 1}\frac{f_i(u)}{u} > \frac{f_i(u)}{u}, \quad 1 \le i \le n,$$

for every $\theta > 1$ and $u > 0$. The last inequalities entail that $f_i'(u) \ge f_i(u)/u$, and hence, $(f_i(u)/u)' \ge 0$ for all $1 \le i \le n$. In the proof of Theorem 1, we will not use (5) directly, but the following equivalent condition.

$$\exists \alpha > 0 \ \text{ such that } \ \rho^{2+\alpha} f_i(\rho^{-\alpha} v) \le f_i(v) \quad \text{for all } \ \rho > 1,\ v > 0,\ 1 \le i \le n. \tag{7}$$

Indeed, the change of variables

$$v = tu, \quad t = \rho^\alpha, \quad \alpha = \frac{2}{p-1},$$

transforms (5) into (7).

The assumption (4) is a condition on the growth of the heterogeneous terms, a_{ij}, b_i, along the rays of Ω as x approximates $\partial\Omega$. More precisely, if for every $z \in \partial\Omega$ we define the functions

$$\begin{array}{l} a_{ij}^z(t) := a_{ij}(tz), \\ b_i^z(t) := b_i(tz), \end{array} \quad t \in [0,1],\ 1 \le i, j \le n,$$

then, the second line of (4) means that $b_i^z(t)$ is non-increasing when $t \sim 1$, for every $1 \le i \le n$. Note that, if the following conditions are satisfied,

$$\begin{array}{l} a_{ii}^z(t) \ge 0, \\ a_{ij}^z(t) \le a_{ij}^z(s) \end{array} \quad \text{for all } \ t_0 < t < s < 1,\ z \in \partial\Omega,\ 1 \le i, j \le n,$$

for some $t_0 \in (0,1)$, then the inequalities of the first line of (4) hold. It is remarkable that (4) is weaker than hypothesis (ii) established in [12, Remark 4.2] if we assume Ω is a ball. Indeed, suppose $\Omega = B_R(0) := \{x \in \mathbb{R}^N \,:\, \|x\| < R\}$ and

$$a_{ij}(x) := a_{ij}^*(\mathrm{dist}(x, \partial\Omega)), \quad 1 \le i \ne j \le n,$$

for some positive continuous non-increasing functions a_{ij}^*. Then,

$$\rho^2 a_{ij}(\rho x) > a_{ij}(\rho x) = a_{ij}^*(\mathrm{dist}(\rho x, \partial\Omega)) \ge a_{ij}^*(\mathrm{dist}(x, \partial\Omega)) = a_{ij}(x)$$

for every $1 \le i \ne j \le n$, $\rho > 1$ and $x \in \Omega$ with $\rho x \in \Omega$.

The distribution of this paper is the following. Section 2 sketches the existence of a minimal and a maximal solution to (1) and provides us with some necessary results for proving the main result. Finally, in Section 3 we show the proof Theorem 1.

2 Existence and previous results

The existence of a minimal and a maximal solution to (1) can be obtained simply by adapting the abstract results of J. López-Gómez and P. Álvarez-Caudevilla [1, Section 3] to the case of n equations. For our purpose, it is enough if we show a scheme of this construction, with special attention in the construction of a supersolution for the non singular counterpart of (1).

Given a regular subdomain $D \subset \mathbb{R}^N$, we define the operator $\mathfrak{L} : [\mathscr{C}^{2+\nu}(D)]^n \to [\mathscr{C}^{\nu}(D)]^n$ by

$$(\mathfrak{L}u)_i = -\Delta u_i - \sum_{j=1}^{n} a_{ij}(\cdot)u_j, \quad 1 \le i \le n.$$

Thanks to the cooperative structure of $\mathfrak{L}$, given by (2), it is well known that there exists a unique $\sigma \in \mathbb{R}$ such that the linear eigenvalue problem

$$\begin{cases} \mathfrak{L}\varphi = \sigma\varphi \text{ in } & D, \\ \varphi = 0 \qquad \text{on} & \partial D, \end{cases}$$

admits a positive eigenfunction $\varphi \in [\mathscr{C}^{2+\nu}(D)]^n$, $\varphi_i(x) > 0$ for all $x \in D$, $1 \le i \le n$. This value $\sigma \in \mathbb{R}$ is called the *principal eigenvalue* of $\mathfrak{L}$ under Dirichlet homogeneous boundary conditions. From here on, it will be denoted by $\sigma[\mathfrak{L}, D]$.

The following theorem goes back to M. Molina-Meyer, [17–19]. It can be obtained by adapting the classical theory of sub and supersolution provided in H. Amann [2] and the maximum principle for cooperative systems of J. López-Gómez and M. Molina-Meyer, [16].

Theorem 2. *Suppose that* (2) *and* (5) *are satisfied. Then, for every* $w \in [\mathscr{C}(\partial\Omega)]^n$, $w \ge 0$, *the boundary value problem*

$$\begin{cases} -\Delta u_i = \displaystyle\sum_{j=1}^{n} a_{ij}(x)u_j - b_i(x)f_i(u_i) \text{ in } & \Omega, \\ u_i = w_i & \text{on } \partial\Omega, \end{cases} \quad 1 \le i \le n, \tag{1}$$

has a unique positive solution, throughout denoted by $\theta_{[\Omega,w]}$. *Moreover, for every positive supersolution* $\bar{u}$ *(resp. subsolution* $\underline{u}$*) of* (1), *one gets*

$$\bar{u} \ge \theta_{[\Omega,w]} \qquad (\textit{resp. } \underline{u} \le \theta_{[\Omega,w]}).$$

Sketch of the proof. By the general assumptions concerning to the nonlinear terms, $\underline{u} := \vec{0}$ is a (strict) subsolution of (1), for every $w > 0$. Then, for the existence of a positive solution, it only remains to construct a supersolution of (1).

In the special case when

$$\min_{z \in \partial\Omega} b_i(z) > 0, \quad 1 \le i \le n, \tag{2}$$

the function $\bar{u} := (M, \ldots, M)$ provides us with a supersolution of (1) for M sufficiently large. Indeed, by (5),

$$f_i(m) \ge f_i(1)m^{p_i}, \quad \text{for all } m > 1.$$

Thus, owing to (2), there exists $m_0 > 0$ such that for every $m > m_0$,

$$\sum_{j=1}^{n} ||a_{ij}||_\infty m - \min_{x \in \bar{\Omega}} b_i(x)f_i(m) \le m \sum_{j=1}^{n} ||a_{ij}||_\infty - \min_{x \in \bar{\Omega}} b_i(x)f_i(1)m^{p_i} < 0,$$

because $p_i > 1$ for all $1 \le i \le n$. Hence,

$$-\Delta m = 0 > \sum_{j=1}^{n} ||a_{ij}||_\infty m - \min_{x \in \bar{\Omega}} b_i(x)f_i(m) \ge \sum_{j=1}^{n} a_{ij}(x)m - b_i(x)f_i(m),$$

which shows what we claimed above. But in general (2) might fail, so we proceed as follows. Define

$$a := \max_{1\le i,j\le n} ||a_{ij}||_\infty,$$

and consider the operator

$$(\bar{\mathfrak{L}}u)_i := -\Delta u_i - a\sum_{j=1}^n u_j, \quad 1\le i\le n.$$

By the monotonicity with respect to the coupling terms of the operator $\mathfrak{L}$, it is clear that

$$\sigma[\mathfrak{L},D] > \sigma[\bar{\mathfrak{L}},D],$$

(see [16, Theorem 3.2] if necessary). Moreover, by the uniqueness of the principal eigenvalue, we have

$$\sigma[\bar{\mathfrak{L}},D] = \lambda_1[-\Delta - na, D] = \lambda_1[-\Delta, D] - na, \tag{3}$$

where $\lambda_1[-\Delta,D]$ stands for the classical first eigenvalue of $-\Delta$ in D under Dirichlet homogeneous boundary conditions. On the other hand, thanks to the Faber–Krahn inequality, [5,9],

$$\lambda_1[-\Delta,D]\to+\infty, \quad \text{as } |D|\downarrow 0,$$

where $|D|$ denotes the Lebesgue measure of D. Therefore, by (3), there exists $\delta_0>0$, depending on a_{ij}, such that

$$\sigma[\mathfrak{L},D]>0, \tag{4}$$

for every regular subdomain $D\subset\mathbb{R}^N$ such that $|D|<\delta_0$.

Set D' a neighborhood of $\partial\Omega$ satisfying (4) and $\varphi=(\varphi_1,\ldots,\varphi_n)$ an eigenfunction associated to $\sigma[\mathfrak{L},D']$, i.e.,

$$-\Delta\varphi_i - \sum_{j=1}^n a_{ij}(x)\varphi_j = \sigma[-\mathfrak{L},D']\varphi_i > 0, \quad x\in D', \; 1\le i\le n, \tag{5}$$

and $\varphi_i(x)>0$ for every $x\in D'$. Clearly, we can consider another open neighborhood of $\partial\Omega$, namely D^*, such that $\bar{D}^*\subset D'$ and

$$\varphi_i(x)>0 \quad \text{for all } x\in\bar{D}^*. \tag{6}$$

The last property allows us to define the next function,

$$\Phi := \begin{cases} \varphi \text{ in } & \Omega\cap D^*, \\ g \text{ in } & \Omega\setminus D^*, \end{cases}$$

where g is any positive regular extension of φ to $\Omega\setminus D^*$, i.e., such that

$$\min_{\Omega\setminus D^*} g>0,$$

which exists because of (6). Then, $\tau\Phi$ provides us with a supersolution of (1) if $\tau>1$ is sufficiently large. Indeed, by (6),

$$\tau\Phi_i(z)=\varphi_i(z)>w_i(z) \quad \text{for every } z\in\partial\Omega, \;\; \tau>\tau_0, \;\; 1\le i\le n,$$

for every $\tau>1$ sufficiently large. On the other side, using (4) and (5) we get that, in $\Omega\cap D^*$,

$$\begin{aligned} -\Delta(\tau\Phi_i) = \tau(-\Delta\varphi_i) &= \tau\sum_{j=1}^n a_{ij}(\cdot)\varphi_j + \tau\sigma[-\Delta,D']\varphi_i \\ &\ge \sum_{j=1}^n a_{ij}(\cdot)\tau\varphi_j - b_i(\cdot)f_i(\tau\varphi_i), \qquad 1\le i\le n, \end{aligned}$$

while, in $\Omega \setminus D^*$, we can take $\tau > 1$ sufficiently large so that

$$-\Delta(\tau\Phi_i) = \tau(-\Delta g_i) \geq \tau \sum_{j=1}^{n} a_{ij}(\cdot)g_j - b_i(\cdot)\tau^{p_i} f_i(g_i), \quad 1 \leq i \leq n,$$

because $b_i(x) > 0$ for every $x \in \Omega \setminus D$. Lastly, applying (5) in the last inequality yields

$$-\Delta(\tau\Phi) \geq \sum_{j=1}^{n} a_{ij}(\cdot)\tau g_j - b_i(\cdot) f_i(\tau g_i), \quad x \in \Omega \setminus D^*.$$

This finishes the construction of a supersolution. The last assertion of the theorem is due to the uniqueness of the solution of (1), which is a consequence of the maximum principle and (6). □

From Theorem 2 we deduce that the mapping

$$\begin{aligned} (0,+\infty) &\longrightarrow \left[\mathscr{C}^{2+\nu}(\bar{\Omega})\right]^n \\ m &\longmapsto \theta_{[\Omega,\vec{m}]}, \end{aligned}$$

where $\vec{m} := (m,\ldots,m)$, is increasing. Moreover, by adapting the construction provided in [11, Chapter 3] for the single equation, we obtain the following result, which in case $n = 2$ is given by [1, Theorem 4.10].

Theorem 3. *Under condition* (KO)*, the point-wise limit*

$$\theta_{[\Omega,\infty]}(x) := \lim_{m\uparrow+\infty} \theta_{[\Omega,\vec{m}]}(x), \qquad x \in \Omega,$$

is well defined, and it provides us with the minimal solution of (1)*, throughout denoted by L_Ω^{min}. Furthermore, the maximal solution of* (1) *is given by*

$$L_\Omega^{max} = \lim_{\delta\downarrow 0} \theta_{[\Omega_\delta,\infty]},$$

where we have denoted

$$\Omega_\delta := \{x \in \Omega \,:\, d(x,\partial\Omega) > \delta\}, \qquad \delta > 0.$$

3 Proofs of the main results

3.1 Proof of Theorem 1

It suffices to show that $L_\Omega^{\min} = L_\Omega^{\max}$. Consider

$$D := D_0 \cap D^*,$$

where D_0 is the set mentioned in the statement of Theorem 1 and D^* is the one arisen in the previous section. Let us define the sets

$$\Omega_\rho := \{x \in \Omega \,:\, \rho x \in \Omega\}, \quad \rho > 0,$$

and

$$\Gamma_\rho := \Omega_\rho \setminus \Omega_{\rho_0},$$

where it is assumed we have fixed a $\rho_0 > 1$ sufficiently small so that

$$\Gamma_\rho \subset D, \quad \text{for all } 1 \leq \rho \leq \rho_0. \tag{1}$$

Note that the component of $\partial\Gamma_\rho$ are $\partial\Omega_{\rho_0}$ and $\partial\Omega_\rho$, and $\partial\Omega_\rho$ approximates $\partial\Omega$ as $\rho \downarrow 1$.

Set $u = (u_1,\ldots,u_n)$ a positive solution of (1) and consider the functions defined by

$$\bar{u}_{\rho,i}(x) := \rho^\alpha u_i(\rho x) + \tau\varphi_i(\rho x), \quad x \in \Gamma_\rho,\ \tau > 1,\ 1 \leq i \leq n, \tag{2}$$

where φ is the eigenfunction that satisfies (5). Then, the following result of technical nature holds.

Lemma 4. *There exists $\tau > 1$ such that $\bar{u}_\rho$ is a supersolution of the singular problem*

$$\begin{cases} -\Delta v_i = \sum_{j=1}^{n} a_{ij}(x)v_j - b_i(x)f_i(v_i) \text{ in } \Gamma_\rho, \\ v_i = +\infty & \text{on } \partial\Omega_\rho, \\ v_i = L^{max}_{\Omega,i} & \text{on } \partial\Omega_{\rho_0}, \end{cases} \tag{3}$$

for every $1 < \rho < \frac{\rho_0}{2}$.

Proof. As u is a solution of (1), by (2)

$$\bar{u}_i = +\infty \text{ on } I_\rho, \quad \text{for every } 1 \le \rho \le \frac{\rho_0}{2}, \; 1 \le i \le n.$$

On the other hand, using (1), we have that

$$F := \{\rho z \,:\, z \in \partial\Omega_{\rho_0}, \, 1 \le \rho \le \frac{\rho_0}{2}\} \subset \Omega \cap D.$$

Thus, thanks to (6), there exists $\tau > 1$ such that

$$\tau\varphi_i(x) > L^{\max}_{\Omega}(x) \quad \text{for every } x \in F,$$

which ensures that $\bar{u}$ satisfies the required inequalities on the boundary. Finally, owing to (5), for every $1 < \rho < \rho_0$ and $x \in \Gamma_\rho$,

$$\begin{aligned} -\Delta\bar{u}_{\rho,i}(x) &= \rho^{2+\alpha}(-\Delta)u_i(\rho x) + \rho^2\tau(-\Delta)\varphi_i(\rho x) \\ &= \rho^2 \sum_{j=1}^{n} a_{ij}(\rho x)[\rho^\alpha u_j(\rho x) + \tau\varphi_j(\rho x)] + \rho^2\tau\sigma[-\Delta,\Gamma]\varphi_i(\rho x) \\ &\quad - \rho^{2+\alpha}b_i(\rho x)f_i(u_i(\rho x)) \\ &\ge \rho^2 \sum_{j=1}^{n} a_{ij}(\rho x)\bar{u}_j(x) - \rho^{2+\alpha}b_i(\rho x)f_i(u_i(\rho x)). \end{aligned}$$

Hence, invoking to (4) and (7) yields

$$\begin{aligned} -\Delta\bar{u}_{\rho,i}(x) &\ge \rho^2 \sum_{j=1}^{n} a_{ij}(\rho x)\bar{u}_j(x) - \rho^{2+\alpha}b_i(\rho x)f_i(u_i(\rho x)) \\ &\ge \sum_{j=1}^{n} a_{ij}(x)\bar{u}_j(x) - \rho^{2+\alpha}b_i(x)f_i(u_i(\rho x)) \\ &= \sum_{j=1}^{n} a_{ij}(x)\bar{u}_j(x) - \rho^{2+\alpha}b_i(x)f_i(\rho^{-\alpha}\rho^\alpha u_i(\rho x)) \\ &= \sum_{j=1}^{n} a_{ij}(x)\bar{u}_j(x) - b_i(x)f_i(\rho^\alpha u_i(x)) \ge \sum_{j=1}^{n} a_{ij}(x)\bar{u}_j(x) - b_i(x)f_i(\bar{u}_i(x)) \end{aligned}$$

for every $0 < \rho < \rho_0$ and $x \in \Gamma_\rho$. Therefore, $\bar{u}_\rho$ is a supersolution of (3) for all $1 < \rho < \rho_0$. □

By the construction of the sets Γ_ρ, it is clear that, for every $1 < \rho < \rho_0/2$

$$L^{\max}(x) < +\infty, \quad \text{for all } x \in \bar{\Gamma}_\rho \subset \Omega.$$

Hence, applying Theorem 2 to the solution $L_\Omega^{\max}$ and the supersolution $\bar{u}_\rho$, we obtain that, for every $1 \le i \le n$,

$$L_{\Omega,i}^{\max}(x) \le \bar{u}_{\rho,i}(x) = \rho^\alpha u_i(\rho x) + \tau\varphi_i(\rho x), \quad x \in \Gamma_\rho, \;\; 1 < \rho < \rho_0.$$

Making the choice $u(x) = L_\Omega^{\min}(x)$ and letting $\rho \downarrow 1$, we can infer that

$$L_{\Omega,i}^{\max}(x) \le L_{\Omega,i}^{\min}(x) + \tau\varphi_i(x), \quad x \in \Gamma_1 = \Omega \setminus \Omega_{\rho_0}, \;\; 1 \le i \le n.$$

In particular, dividing by $L_{\Omega,i}^{\min}$ yields

$$1 \le \frac{L_{\Omega,i}^{\max}(x)}{L_{\Omega,i}^{\min}(x)} \le 1 + \frac{\tau\varphi_i(x)}{L_{\Omega,i}^{\min}(x)}, \quad x \in \Gamma_1, \;\; 1 \le i \le n,$$

and using that $\tau\varphi$ is bounded in $\bar{\Gamma}_1$, we get

$$\lim_{\substack{x \to \partial\Omega \\ x \in \Omega}} \frac{L_{\Omega,i}^{\max}(x)}{L_{\Omega,i}^{\min}(x)} = 1, \quad 1 \le i \le n.$$

Thanks to the last inequalities, the following maps

$$q_i(x) := \begin{cases} L_{\Omega,i}^{\max}(x)/L_{\Omega,i}^{\min}(x) & x \in \Omega, \\ 1 & x \in \partial\Omega, \end{cases} \quad 1 \le i \le n,$$

are continuous. In particular, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|q_i(x) - 1| = \frac{L_{\Omega,i}^{\max}(x)}{L_{\Omega,i}^{\min}(x)} - 1 < \varepsilon, \quad \text{for all} x \in \Omega, \; z \in \partial\Omega \text{ such that } |z - x| < \delta.$$

Setting

$$Q_\eta := \{x \in \bar{\Omega} : \mathrm{dist}(x, \partial\Omega) \le \eta\}, \quad \eta > 0,$$

we find that

$$L_{\Omega,i}^{\max}(x) < (1+\varepsilon) L_{\Omega,i}^{\min}(x), \quad x \in Q_\delta. \tag{4}$$

To conclude the proof, note that $(1+\varepsilon)L_{\Omega,i}^{\min}$ is a supersolution of the problem

$$\begin{cases} -\Delta u_i = \sum_{j=1}^n a_{ij}(x) u_j - b_i(x) f_i(u_i) \text{ in } \Omega \setminus Q_\delta, \\ u_i = L_{\Omega,i}^{\max} \qquad\qquad\qquad\qquad \text{on } \partial(\Omega \setminus Q_\delta), \end{cases} \quad 1 \le i \le n,$$

Indeed, by (4), the inequalities on the boundary are satisfied, and

$$\begin{aligned} -\Delta((1+\varepsilon)L_{\Omega,i}^{\min}) &= \sum_{j=1}^n a_{ij}(x)(1+\varepsilon)L_{\Omega,j}^{\min} - (1+\varepsilon) b_i(x) f_i(L_{\Omega,i}^{\min}) \\ &= \sum_{j=1}^n a_{ij}(x)(1+\varepsilon)L_{\Omega,j}^{\min} - b_i(x) \frac{f_i(L_{\Omega,i}^{\min})}{L_{\Omega,i}^{\min}} (1+\varepsilon) L_{\Omega,i}^{\min} \\ &\ge \sum_{j=1}^n a_{ij}(x)(1+\varepsilon)L_{\Omega,j}^{\min} - b_i(x) f_i((1+\varepsilon)L_{\Omega,i}^{\min}). \end{aligned}$$

Therefore, by Theorem 2,

$$(1+\varepsilon)L_{\Omega,i}^{\min}(x) \ge L_{\Omega,i}^{\max}(x), \quad x \in \Omega \setminus Q_\delta, \; 1 \le i \le n,$$

which, together with (4), and letting $\varepsilon \downarrow 0$, provides us with with the desired equality,

$$L_\Omega^{\min} = L_\Omega^{\max} \quad \text{in } \Omega.$$

This ends the proof. □

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