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Computing the two first probability density functions of the random Cauchy-Euler differential equation: Study about regular-singular points

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Abstract

In this paper the randomized Cauchy-Euler differential equation is studied. With this aim, from a statistical point of view, both the first and second probability density functions of the solution stochastic process are computed. Then, the main statistical functions, namely, the mean, the variance and the covariance functions are determined as well. The study includes the computation of the first and second probability density functions of the regular-singular infinite point via an adequate mapping transforming the problem about the origin. The study is strongly based upon the Random Variable Transformation technique along with some results that have been recently published by some of authors to the random homogeneous linear second-order differential equation. Finally, an illustrative example is shown.

Keywords: random Cauchy-Euler differential equation, Random Variable Transformation technique, first and second probability density functions
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1 Introduction

Deterministic differential equations play a key role in many disciplines to model numerous phenomena. In practice, the application of models based on differential equations requires setting their inputs such as coefficients and initial conditions. These parameters are usually obtained by experiments where measurement errors are involved. In addition, there are external sources which can affect the physical system to be modelled. These facts motivate the treatment of inputs parameters as random variables (RV's) or stochastic processes (SP's) rather

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than deterministic constants or functions, respectively. In this contribution, we will focus on random differential equations (RDE's), that is, differential equations whose coefficients are RV's or SP's.

As a main difference with respect to the deterministic framework, solving a RDE means not only to obtain its solution but also probabilistic information associated to the solution SP. Indeed, if $W(u)$ denotes the solution SP of a RDE, then it is also important to compute its mean, $\mu_W(u) = \mathbb{E}[W(u)]$, and its variance, $\sigma_W^2(u) = \mathbb{V}[W(u)]$. Additionally, the computation of the first probability density function (1-PDF), $\hat{f}_1(w,u)$, is also important because from it one gets a full probabilistic description of the solution SP in each time instant u. Furthermore, from the 1-PDF one can compute all the one-dimensional statistical moments of $W(u)$,

$$\mathbb{E}\left[(W(u))^k\right] = \int_{-\infty}^{\infty} w^k \hat{f}_1(w,u)\,\mathrm{d}w, \quad k = 0,1,2,\ldots$$

Therefore, from it the mean and the variance can be straightforwardly obtained

$$\mathbb{E}[W(u)] = \int_{-\infty}^{\infty} w\hat{f}_1(w,u)\,\mathrm{d}w, \quad \mathbb{V}[W(u)] = \int_{-\infty}^{\infty} w^2 \hat{f}_1(w,u)\,\mathrm{d}w - (\mathbb{E}[W(u)])^2. \tag{1}$$

In general, to get more probabilistic information, the n-dimensional PDF's of the solution SP might be computed, but it usually involves complex computations. For example, the 2-PDF, $\hat{f}_2(w_1,u_1;w_2,u_2)$, provides a full probabilistic description of $W(u)$ at every arbitrary pair of times, u_1 and u_2. In particular, from it the correlation function, $\Gamma_W(u_1,u_2)$, can be computed. This function gives a measure of linear statistical interdependence between $W(u_1)$ and $W(u_2)$ and it is given by

$$\Gamma_W(u_1,u_2) = \mathbb{E}[W(u_1)W(u_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_1 w_2 \hat{f}_2(w_1,u_1;w_2,u_2)\,\mathrm{d}w_1\mathrm{d}w_2.$$

Furthermore, $\Gamma_W(u_1,u_2)$ allows us the computation of the covariance function

$$C_W(u_1,u_2) = \Gamma_W(u_1,u_2) - \mathbb{E}[W(u_1)]\,\mathbb{E}[W(u_2)]. \tag{2}$$

The Cauchy-Euler differential equation is adequate to model a number of phenomena in Engineering, particularly in Mechanics and in Theory of Potential Fields. For example, in this latter context it is applied to model the electric potential field between two concentric spheres [8, Sec. 2.5]. The formulation of this problem depends on the potential fields of both spheres, which usually are not known in a deterministic way but randomly due the heterogeneity of the surrounding medium. This motivates the study of the random Cauchy-Euler differential equation. Besides, the deterministic Cauchy-Euler differential equation has two regular-singular points, zero and infinity. Therefore, the goal of this paper is to obtain the 1-PDF and the 2-PDF of the solution SP of each one of two random IVP's, on the one hand we shall consider a random IVP based on a Cauchy-Euler differential equation, and on the other hand, using an adequate transformation, infinity is moved to the origin. With this aim the results presented in [1] will be applied. In [1], the 1-PDF and 2-PDF of the solution SP of a random homogeneous linear second-order differential equation has been computed. To conduct our analysis, we will take advantage of the Random Variable Transformation (RVT) method that is stated below in Theorem 1. This technique allows us to obtain the PDF of a random vector obtained from the mapping of another random vector whose PDF is known.

Theorem 1 (Multidimensional RVT method [11]). *Let us consider* $\mathbf{X} = [X_1,\ldots,X_m]^\top$ *and* $\mathbf{Y} = [Y_1,\ldots,Y_m]^\top$ *two m-dimensional absolutely continuous random vectors defined on a complete probability space* $(\Omega,\mathfrak{F},\mathbb{P})$*. Let* $\mathbf{r}: \mathbb{R}^m \to \mathbb{R}^m$ *be a one-to-one deterministic transformation of* $\mathbf{X}$ *into* $\mathbf{Y}$*, i.e.,* $\mathbf{Y} = \mathbf{r}(\mathbf{X})$*. Assume that* $\mathbf{r}$ *is continuous in* $\mathbf{X}$ *and has continuous partial derivatives with respect to each* X_i*,* $1 \le i \le m$*. Then, if* $f_{\mathbf{X}}(\mathbf{x})$ *denotes the joint probability density function of vector* $\mathbf{X}$*, and* $\mathbf{s} = \mathbf{r}^{-1} = (s_1(y_1,\ldots,y_m),\ldots,s_m(y_1,\ldots,y_m))$ *represents the inverse mapping of* $\mathbf{r} = (r_1(x_1,\ldots,x_m),\ldots,r_m(x_1,\ldots,x_m))$*, the joint probability density function of vector* $\mathbf{Y}$ *is given by*

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{s}(\mathbf{y}))\,|J_m|,$$

where $|J_m|$, which is assumed to be different from zero, denotes the absolute value of the Jacobian defined by the determinant

$$J_m = \det \begin{bmatrix} \frac{\partial s_1(y_1,\ldots,y_m)}{\partial y_1} & \cdots & \frac{\partial s_m(y_1,\ldots,y_m)}{\partial y_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_1(y_1,\ldots,y_m)}{\partial y_m} & \cdots & \frac{\partial s_m(y_1,\ldots,y_m)}{\partial y_m} \end{bmatrix}.$$

The following result, that will be required later, is a direct consequence of Theorem 1.

Corollary 2. *Let $\mathbf{X} = (X_0, C, B, A_2)$ be an absolutely continuous random vector with joint PDF $f_{\mathbf{X}}(x_0, c, b, a_2)$ and let $\mathbf{Y} = \mathbf{r}(\mathbf{X}) = (X_0, X_1, A_1, A_2)$ with $X_1 = kC$, $A_1 = B - 1$ and $k \neq 0$ an arbitrary constant. Then, the joint PDF of random vector $\mathbf{Y}$ is given by*

$$f_{\mathbf{Y}}(x_0, x_1, a_1, a_2) = f_{\mathbf{X}}(x_0, x_1/k, a_1 + 1, a_2)|1/k|. \tag{3}$$

Finally, we want to notice that the techniques that will be applied throughout this manuscript have been successfully used to study other random differential equations and discrete and continuous dynamic models [2–7, 9, 10, 12].

This contribution is organized as follows. Section 2 is devoted to compute the 1-PDF and the 2-PDF of the solution SP of the problem under study. As the study is based upon a number of results already established, these findings are previously introduced for the sake of completeness in the presentation. In Section 3, the 1-PDF and 2-PDF of the solution SP of the corresponding random IVP associated to the analysis of the infinity regular-singular point will be addressed. In Section 4, we show an illustrative numerical example where the 1-PDF and the 2-PDF of the solution SP and, the mean, the variance and the covariance functions are computed as well. In this example, the 1-PDF, the mean and the variance functions of the random IVP problem to study the infinity point will be also shown. Conclusions are drawn in Section 5.

2 Computing the 1-PDF and the 2-PDF of the solution stochastic process of the randomized Cauchy-Euler differential equation about the regular-singular point u_0

Let us consider the following random IVP (4) based on a Cauchy-Euler differential equation

$$\left.\begin{aligned} u^2 W''(u) + BuW'(u) + A_2 W(u) &= 0, \quad u > u_0 > 0, \\ W(u_0) &= X_0, \\ W'(u_0) &= C, \end{aligned}\right\} \tag{4}$$

where the input parameters X_0, C, B and A_2 are assumed to be absolutely continuous RV's, defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with a joint PDF $f_{X_0,C,B,A_2}(x_0, c, b, a_2)$. Thus, for the sake of generality we are implicitly assuming that involved RV's are probabilistically dependent. As usual, hereinafter the notation $X \equiv X(\omega)$ will be used indistinctly. Our subsequent analysis is strongly related to the problem studied in [1], this motivates the notation used in IVP (4) for coefficients, initial conditions and unknown. In order to make clearer our notation, we will assume that the domain of every random inputs is an interval, although it is not necessary since we are assuming that X_0, C, B and A_2 are probabilistically dependent RV's

$$\begin{aligned} \mathscr{D}_{X_0} &= \{x_0 = X_0(\omega),\, \omega \in \Omega : x_{0,1} < x_0 < x_{0,2}\}, \\ \mathscr{D}_C &= \{c = C(\omega),\, \omega \in \Omega : c_1 < c < c_2\}, \\ \mathscr{D}_B &= \{b = B(\omega),\, \omega \in \Omega : b_1 < b < b_2\}, \\ \mathscr{D}_{A_2} &= \{a_2 = A_2(\omega),\, \omega \in \Omega : a_{2,1} < a_2 < a_{2,2}\}. \end{aligned}$$

With the transformation $u = \mathrm{e}^t u_0$, $\forall u > u_0$, IVP (4) is equivalent to IVP (5)

$$\left.\begin{aligned} Z''(t) + A_1 Z'(t) + A_2 Z(t) &= 0, \quad t > 0, \\ Z(0) &= X_0, \\ Z'(0) &= X_1, \end{aligned}\right\} \tag{5}$$

where $A_1 = B-1$, $X_1 = Cu_0$ and $Z(t) = W(\mathrm{e}^t u_0)$. Then, the domains of RV's A_1 and X_1 are defined as follows

$$\mathscr{D}_{X_1}=\{x_1 = X_1(\omega),\, \omega\in\Omega : c_1u_0 < x_1 < c_2u_0\},$$
$$\mathscr{D}_{A_1}=\{a_1 = A_1(\omega),\, \omega\in\Omega : b_1-1 < a_1 < b_2-1\}.$$

This section is addressed to compute the 1-PDF and the 2-PDF of the solution SP to IVP (4), $W(u)$. With this aim, first we will apply the theoretical results obtained in [1] to compute the 1-PDF and 2-PDF of the solution SP to IVP (5), $Z(t)$. In [1], the 1-PDF and the 2-PDF of the random homogeneous linear second-order differential equation was determined and the mean, the variance and the covariance functions as well. Secondly, we will take advantage of Corollary 2, to get probabilistic information in terms of the joint PDF $f_{X_0,C,B,A_2}(x_0,c,b,a_2)$, which is known. Finally, we will undo the change of variable $u = \mathrm{e}^t u_0$ to obtain the 1-PDF and 2-PDF of the solution SP to the random IVP (4).

2.1 1-PDF: First probability density function of the solution stochastic process about the regular-singular point u_0

As it is detailed in [1], the solution SP of the linear second-order random differential equation (5) depends on the real and complex nature of the roots of the associated characteristic equation. These roots are given by

$$\left.\begin{aligned} \alpha_1(A_1,A_2) &= \frac{-A_1+\sqrt{\Delta}}{2},\\ \alpha_2(A_1,A_2) &= \frac{-A_1-\sqrt{\Delta}}{2}, \end{aligned}\right\} \quad \Delta = (A_1)^2 - 4A_2. \tag{6}$$

In the deterministic theory, depending on the value of the discriminant $\Delta = \Delta(\omega)$, $\omega\in\Omega$, the roots can be real or complex. As A_1 and A_2 are assumed to be absolutely continuous RV's, this happens with certain probabilities

$$\begin{cases} p_1 = \mathbb{P}\left[\omega\in\Omega : \Delta(\omega) > 0\right],\\ p_2 = \mathbb{P}\left[\omega\in\Omega : \Delta(\omega) < 0\right],\\ p_3 = \mathbb{P}\left[\omega\in\Omega : \Delta(\omega) = 0\right]. \end{cases} \tag{7}$$

Notice that, as A_1 and A_2 are absolutely continuous RV's, the probability p_3 is zero. Then, only the case $0 < p_1, p_2 < 1$ with $p_1 + p_2 = 1$ must be considered. As we have two events, depending on the real or complex nature of $\alpha_i(A_1,A_2)$, $i = 1,2$, the 1-PDF of the solution SP, $Z(t)$, will be split into two pieces, $f_{1\mathrm{R}}(z,t)$ and $f_{1\mathrm{C}}(z,t)$, corresponding to the contribution of real or imaginary roots whose associated probabilities are p_1 and p_2, respectively. Then, the complete 1-PDF to the random IVP (5) will be expressed as

$$f_1(z,t) = f_{1\mathrm{R}}(z,t) + f_{1\mathrm{C}}(z,t).$$

Notice that for $t > 0$, $f_{1\mathrm{R}}(z,t)$ and $f_{1\mathrm{C}}(z,t)$ are not PDF since their integrals are p_1 and p_2, respectively. As a consequence, the 1-PDF to the target random IVP (4) is

$$\hat{f}_1(w,u) = \hat{f}_{1\mathrm{R}}(w,u) + \hat{f}_{1\mathrm{C}}(w,u). \tag{8}$$

Now we will compute $\hat{f}_{1\mathrm{R}}(w,u)$. To do that, first we will determine the 1-PDF of random IVP (5), $f_{1\mathrm{R}}(z,t)$. We need to assume that $p_1 > 0$ and then applying Eq. (2.10) of [1], i.e.,

$$f_{1\mathrm{R}}(z,t) = \int_{c_1u_0}^{c_2u_0}\int_{b_1-1}^{b_2-1}\int_{\min\left[a_{2,1},\frac{a_1^2}{4}\right]}^{\min\left[a_{2,2},\frac{a_1^2}{4}\right]} \frac{f_{X_0,X_1,A_1,A_2}\left(\frac{z-g_{\mathrm{R}}(t)x_1}{h_{\mathrm{R}}(t)},x_1,a_1,a_2\right)}{|h_{\mathrm{R}}(t)|}\mathrm{d}a_2\,\mathrm{d}a_1\,\mathrm{d}x_1, \tag{9}$$

being

$$g_{\mathrm{R}}(t) = \frac{\mathrm{e}^{\alpha_1 t}-\mathrm{e}^{\alpha_2 t}}{\alpha_1-\alpha_2}, \quad h_{\mathrm{R}}(t) = \frac{\alpha_1\,\mathrm{e}^{\alpha_2 t}-\alpha_2\,\mathrm{e}^{\alpha_1 t}}{\alpha_1-\alpha_2}, \tag{10}$$

where $\alpha_i \equiv \alpha_i(a_1,a_2)$, $i=1,2$, are given in (6).

Applying Corollary 2 with $k=u_0$, the 1-PDF (9) can be written as

$$f_{1\mathrm{R}}(z,t)=\int_{c_1u_0}^{c_2u_0}\int_{b_1-1}^{b_2-1}\int_{\min\left[a_{2,1},\frac{b^2}{4}\right]}^{\min\left[a_{2,2},\frac{b^2}{4}\right]}\frac{f_{X_0,C,B,A_2}\left(\frac{z-g_{\mathrm{R}}(t)c}{h_{\mathrm{R}}(t)},\frac{c}{u_0},b+1,a_2\right)}{|h_{\mathrm{R}}(t)u_0|}\mathrm{d}a_2\,\mathrm{d}b\,\mathrm{d}c.$$

Finally, as $u=\mathrm{e}^t u_0$, $\forall u>u_0$, then $t=\log\left(\frac{u}{u_0}\right)$. Therefore, for values of $u>u_0>0$, we obtain the expression of the real part of the 1-PDF to the solution SP, $W(u)$,

$$\hat{f}_{1\mathrm{R}}(w,u)=\int_{c_1u_0}^{c_2u_0}\int_{b_1-1}^{b_2-1}\int_{\min\left[a_{2,1},\frac{b^2}{4}\right]}^{\min\left[a_{2,2},\frac{b^2}{4}\right]}\frac{f_{X_0,C,B,A_2}\left(\frac{w-\hat{g}_{\mathrm{R}}(u)c}{\hat{h}_{\mathrm{R}}(u)},\frac{c}{u_0},b+1,a_2\right)}{|\hat{h}_{\mathrm{R}}(u)u_0|}\mathrm{d}a_2\,\mathrm{d}b\,\mathrm{d}c. \tag{11}$$

being

$$\hat{g}_{\mathrm{R}}(u)=g_{\mathrm{R}}\left(\log\left(\frac{u}{u_0}\right)\right),\quad \hat{h}_{\mathrm{R}}(u)=h_{\mathrm{R}}\left(\log\left(\frac{u}{u_0}\right)\right), \tag{12}$$

with $\alpha_i \equiv \alpha_i(b-1,a_2)$, $i=1,2$, given in (6).

Assuming $p_2>0$, the contribution corresponding to the complex part can be obtained similarly. In this case, one obtains

$$\hat{f}_{1\mathrm{C}}(w,u)=\int_{c_1u_0}^{c_2u_0}\int_{b_1-1}^{b_2-1}\int_{\max\left[a_{2,1},\frac{b^2}{4}\right]}^{\max\left[a_{2,2},\frac{b^2}{4}\right]}\frac{f_{X_0,C,B,A_2}\left(\frac{w-\hat{g}_{\mathrm{C}}(u)c}{\hat{h}_{\mathrm{C}}(u)},\frac{c}{u_0},b+1,a_2\right)}{|\hat{h}_{\mathrm{C}}(u)u_0|}\mathrm{d}a_2\,\mathrm{d}b\,\mathrm{d}c, \tag{13}$$

being

$$\hat{g}_{\mathrm{C}}(u)=g_{\mathrm{C}}\left(\log\left(\frac{u}{u_0}\right)\right),\quad \hat{h}_{\mathrm{C}}(u)=h_{\mathrm{C}}\left(\log\left(\frac{u}{u_0}\right)\right), \tag{14}$$

with

$$\begin{aligned} g_{\mathrm{C}}(t)&=\frac{\mathrm{e}^{\mathrm{Re}(\alpha_1)t}}{\mathrm{Im}(\alpha_1)}\sin(\mathrm{Im}(\alpha_1)t),\\ h_{\mathrm{C}}(t)&=\mathrm{e}^{\mathrm{Re}(\alpha_1)t}\left[\cos(\mathrm{Im}(\alpha_1)t)-\frac{\mathrm{Re}(\alpha_1)}{\mathrm{Im}(\alpha_1)}\sin(\mathrm{Im}(\alpha_1)t)\right], \end{aligned} \tag{15}$$

where $\alpha_1\equiv\alpha_1(b-1,a_2)$ being

$$\alpha_1(A_1,A_2)=\mathrm{Re}(\alpha_1)+\mathrm{i}\,\mathrm{Im}(\alpha_1),\quad \mathrm{i}=\sqrt{-1},\quad \begin{cases}\mathrm{Re}(\alpha_1)=\dfrac{-A_1}{2},\\[2ex] \mathrm{Im}(\alpha_1)=\dfrac{\sqrt{-A_1^2+4A_2}}{2}.\end{cases} \tag{16}$$

2.2 2-PDF: Second probability density function of the solution stochastic process about the regular-singular point u_0

To compute the 2-PDF of the solution SP, $W(u)$, the same reasoning of Subsection 2.1 will be applied. Therefore, let $u_1,u_2\geq u_0$, the 2-PDF is given by the sum of the real and complex part which are given by the following expressions

$$\begin{aligned}\hat{f}_{2R}(w_1,u_1;w_2,u_2)=\int_{b_1-1}^{b_2-1}\int_{\min\left[a_{2,1},\frac{b^2}{4}\right]}^{\min\left[a_{2,2},\frac{b^2}{4}\right]} f_{X_0,C,B,A_2}\left(\frac{w_1\hat{g}_R(u_2)-w_2\hat{g}_R(u_1)}{\hat{g}_R(u_2)\hat{h}_R(u_1)-\hat{g}_R(u_1)\hat{h}_R(u_2)},\right.\\ \left.\left(\frac{w_2\hat{h}_R(u_1)-w_1\hat{h}_R(u_2)}{\hat{g}_R(u_2)\hat{h}_R(u_1)-\hat{g}_R(u_1)\hat{h}_R(u_2)}\right)\frac{1}{u_0},b+1,a_2\right)\times\frac{1}{|\left(\hat{g}_R(u_2)\hat{h}_R(u_1)-\hat{g}_R(u_1)\hat{h}_R(u_2)\right)u_0|}\mathrm{d}a_2\,\mathrm{d}b.\end{aligned}$$

$$\hat{f}_{2C}(w_1,u_1;w_2,u_2) = \int_{b_1-1}^{b_2-1}\int_{\max\left[a_{2,1},\frac{b^2}{4}\right]}^{\max\left[a_{2,2},\frac{b^2}{4}\right]} f_{X_0,C,B,A_2}\left(\frac{w_1\hat{g}_C(u_2)-w_2\hat{g}_C(u_1)}{\hat{g}_C(u_2)\hat{h}_C(u_1)-\hat{g}_C(u_1)\hat{h}_C(u_2)},\right.$$

$$\left.\left(\frac{w_2\hat{h}_C(u_1)-w_1\hat{h}_C(u_2)}{\hat{g}_C(u_2)\hat{h}_C(u_1)-\hat{g}_C(u_1)\hat{h}_C(u_2)}\right)\frac{1}{u_0},b+1,a_2\right)\times\frac{1}{|\left(\hat{g}_C(u_2)\hat{h}_C(u_1)-\hat{g}_C(u_1)\hat{h}_C(u_2)\right)u_0|}\mathrm{d}a_2\,\mathrm{d}b,$$

where the functions $\hat{g}_R(u)$, $\hat{h}_R(u)$, $\hat{g}_C(u)$ and $\hat{h}_C(u)$ are given by expression (12) and (14).

3 Computing the 1-PDF and the 2-PDF of the solution stochastic process of the randomized Cauchy-Euler differential equation about the infinity regular-singular point

This section is addressed in the study of the infinity point of the randomized Cauchy-Euler differential equation. In the deterministic theory, in order to analyse $u=+\infty$, the first step is to introduce the change of variable $u=1/s$, and then to study a neighbourhood about the point $s=0$ in the resulting expression. In our case, the transformed random IVP is the following

$$\left.\begin{array}{l} s^2V''(s)+DsV'(s)+A_2V(s)=0, \quad 0<s<s_0,\\ V(s_0)\ =X_0,\\ V'(s_0)=E, \end{array}\right\} \tag{17}$$

where $D=2-B$, $E=-C/s_0^2$ and $V(s)=W(1/s)$, $\forall s: 0<s<s_0$.

The 1-PDF and 2-PDF of the solution SP, $V(s)$, of the IVP (17) can be obtained following the same strategy exhibited in the previous section, because we are dealing with a random Cauchy-Euler differential equation too. We use the change of variable $s=\mathrm{e}^{-t}s_0$, $\forall s: 0<s<s_0$, and we obtain the following random IVP

$$\left.\begin{array}{l} Z''(t)+A_1Z'(t)+A_2Z(t)=0, \quad t>0,\\ Z(0)\ =X_0,\\ Z'(0)=X_1, \end{array}\right\} \tag{18}$$

where $A_1=(1-D)=B-1$, $X_1=C/s_0$, $t=\log(s_0/s)$, $\forall t>0$ and $Z(t)=V(s_0\,\mathrm{e}^{-t})$. Then, the domains of RV's A_1 and X_1 are defined as follows

$$\begin{array}{l}\mathscr{D}_{A_1}=\{a_1=A_1(\omega),\,\omega\in\Omega: b_1-1<a_1<b_2-1\},\\ \mathscr{D}_{X_1}=\{x_1=X_1(\omega),\,\omega\in\Omega: \frac{c_1}{s_0}<x_1<\frac{c_2}{s_0}\}.\end{array}$$

3.1 1-PDF: First probability density function of the solution stochastic process about the infinite regular-singular point

Following the same reasoning shown in Subsection 2.1, the 1-PDF of the solution SP to IVP (17), $V(s)$, is given by

$$\bar{f}_1(v,s)=\bar{f}_{1\mathrm{R}}(v,s)+\bar{f}_{1\mathrm{C}}(v,s), \tag{19}$$

where

$$\bar{f}_{1\mathrm{R}}(v,s)=\int_{\frac{c_1}{s_0}}^{\frac{c_2}{s_0}}\int_{b_1-1}^{b_2-1}\int_{\min\left[a_{2,1},\frac{b^2}{4}\right]}^{\min\left[a_{2,2},\frac{b^2}{4}\right]}\frac{f_{X_0,C,B,A_2}\left(\frac{v-\bar{g}_\mathrm{R}(s)c}{\bar{h}_\mathrm{R}(s)},cs_0,b+1,a_2\right)s_0}{|\bar{h}_\mathrm{R}(s)|}\mathrm{d}a_2\,\mathrm{d}b\,\mathrm{d}c, \tag{20}$$

being

$$\bar{g}_\mathrm{R}(s)=g_\mathrm{R}\left(\log\left(\frac{s_0}{s}\right)\right),\quad \bar{h}_\mathrm{R}(s)=h_\mathrm{R}\left(\log\left(\frac{s_0}{s}\right)\right), \tag{21}$$

with $\alpha_i \equiv \alpha_i(b-1, a_2)$, $i = 1, 2$, given in (6), and

$$\bar{f}_{1\mathrm{C}}(v,s) = \int_{\frac{c_1}{s_0}}^{\frac{c_2}{s_0}} \int_{b_1-1}^{b_2-1} \int_{\max\left[a_{2,1},\frac{b^2}{4}\right]}^{\max\left[a_{2,2},\frac{b^2}{4}\right]} \frac{f_{X_0,C,B,A_2}\left(\frac{v-\bar{g}_{\mathrm{C}}(s)c}{\bar{h}_{\mathrm{C}}(s)}, cs_0, b+1, a_2\right) s_0}{|\bar{h}_{\mathrm{C}}(s)|} \mathrm{d}a_2\, \mathrm{d}b\, \mathrm{d}c, \tag{22}$$

being

$$\bar{g}_{\mathrm{C}}(s) = g_{\mathrm{C}}\left(\log\left(\frac{s_0}{s}\right)\right), \quad \bar{h}_{\mathrm{C}}(s) = h_{\mathrm{C}}\left(\log\left(\frac{s_0}{s}\right)\right), \tag{23}$$

where the functions g_R, h_R, g_C and h_C are given by expressions (10), (15) and (16).

3.2 2-PDF: Second probability density function of the solution stochastic process about the infinite regular-singular point

To compute the 2-PDF of the solution SP $V(s)$, a direct adaptation of the arguments exhibited in Subsection 2.1 will be applied. Therefore, let $s_1, s_2 \geq s_0$, the 2-PDF is given by the sum of the real and complex part which are given by the following expressions

$$\bar{f}_{2R}(v_1,s_1;v_2,s_2) = \int_{b_1-1}^{b_2-1} \int_{\min\left[a_{2,1},\frac{b^2}{4}\right]}^{\min\left[a_{2,2},\frac{b^2}{4}\right]} f_{X_0,C,B,A_2}\left(\frac{v_1\bar{g}_R(s_2)-v_2\bar{g}_R(s_1)}{\bar{g}_R(s_2)\bar{h}_R(s_1)-\bar{g}_R(s_1)\bar{h}_R(s_2)},\right.$$

$$\left.\left(\frac{v_2\bar{h}_R(s_1)-v_1\bar{h}_R(s_2)}{\bar{g}_R(s_2)\bar{h}_R(s_1)-\bar{g}_R(v_1)\bar{h}_R(v_2)}\right) s_0, b+1, a_2\right) \times \frac{s_0}{|\bar{g}_R(s_2)\bar{h}_R(s_1)-\bar{g}_R(s_1)\bar{h}_R(s_2)|} \mathrm{d}a_2\, \mathrm{d}b,$$

$$\bar{f}_{2C}(v_1,s_1;v_2,s_2) = \int_{b_1-1}^{b_2-1} \int_{\max\left[a_{2,1},\frac{b^2}{4}\right]}^{\max\left[a_{2,2},\frac{b^2}{4}\right]} f_{X_0,C,B,A_2}\left(\frac{v_1\bar{g}_C(s_2)-v_2\bar{g}_C(s_1)}{\bar{g}_C(s_2)\bar{h}_C(s_1)-\bar{g}_C(s_1)\bar{h}_C(s_2)},\right.$$

$$\left.\left(\frac{v_2\bar{h}_C(s_1)-v_1\bar{h}_C(s_2)}{\bar{g}_C(s_2)\bar{h}_C(s_1)-\bar{g}_C(s_1)\bar{h}_C(s_2)}\right) s_0, b+1, a_2\right) \times \frac{s_0}{|\bar{g}_C(s_2)\bar{h}_C(s_1)-\bar{g}_C(s_1)\bar{h}_C(s_2)|} \mathrm{d}a_2\, \mathrm{d}b,$$

where the functions $\bar{g}_R(u)$, $\bar{h}_R(u)$, $\bar{g}_C(u)$ and $\bar{h}_C(u)$ are given by expression (21) and (23).

4 An illustrative example

In this section we will show an example where the results obtained in Sections 2 and 3 are illustrated. As we have pointed out in (8), the 1-PDF and 2-PDF in both regular-singular problems depend on real and complex nature of the roots of the involved characteristic equation, having as associated probabilities p_1 and p_2, respectively. These probabilities are defined in (7). To account for the most interesting cases regarding our previous analysis, we shall consider the following three possibles scenarios

- **Case I**. Where $p_1 \gg p_2$, i.e., real and distinct roots are more probable than imaginary roots. Then, the probabilistic contribution of $\hat{f}_{1\mathrm{R}}(w,u)$ ($\bar{f}_{1\mathrm{R}}(v,s)$) to $\hat{f}_1(w,u)$ ($\bar{f}_1(v,s)$) is greater than $\hat{f}_{1\mathrm{C}}(w,u)$ ($\bar{f}_{1\mathrm{C}}(v,s)$).

- **Case II**. Where $p_1 \approx p_2 \approx \frac{1}{2}$, i.e., real and distinct roots and imaginary roots are equiprobable. Then, the probabilistic contribution of $\hat{f}_{1\mathrm{R}}(w,u)$ ($\bar{f}_{1\mathrm{R}}(v,s)$) to $\hat{f}_1(w,u)$ ($\bar{f}_1(v,s)$) is similar than $\hat{f}_{1\mathrm{C}}(w,u)$ ($\bar{f}_{1\mathrm{C}}(v,s)$).

- **Case III**. Where $p_1 \ll p_2$, i.e., real and distinct roots are less probable than imaginary roots. Then, the probabilistic contribution of $\hat{f}_{1\mathrm{R}}(w,u)$ ($\bar{f}_{1\mathrm{R}}(v,s)$) to $\hat{f}_1(w,u)$ ($\bar{f}_1(v,s)$)is smaller than $\hat{f}_{1\mathrm{C}}(w,u)$ ($\bar{f}_{1\mathrm{C}}(v,s)$).

In the three cases we will consider a joint Gaussian distribution for the input parameters $J_i = (X_0, C, B, A_2)^\top \sim \mathrm{N}(\mu_i; \Sigma)$, $i = 1, 2, 3$ (i corresponding to Cases I, II and III, respectively), where

$$\mu_i = \begin{cases} (2,1,4,1)^\top \text{ if } i = 1, \\ (1,1,3,1)^\top \text{ if } i = 2, \\ (1,1,3,2)^\top \text{ if } i = 3, \end{cases} \qquad \Sigma = \frac{1}{10} \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}. \tag{24}$$

Probabilities p_1 and $p_2 = 1 - p_1$ are collected in Table 1. Notice that

$$\begin{aligned} p_1 &= \mathbb{P}[\omega \in \Omega : \Delta(\omega) > 0] = \mathbb{P}\left[\omega \in \Omega : (B(\omega) - 1)^2 - 4A_2(\omega) > 0\right] \\ &= \mathbb{P}\left[\omega \in \Omega : \frac{(B(\omega)-1)^2}{4} > A_2(\omega)\right] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\frac{(b-1)^2}{4}} f_{B,A_2}(b,c)\mathrm{d}a_2\right] \mathrm{d}b, \end{aligned} \tag{25}$$

where $f_{B,A_2}(b,a_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_0,C,B,A_2}(x_0,c,b,a_2)\,\mathrm{d}x_0\,\mathrm{d}c$.

Cases	p_1	p_2	p_s
I	0.978524	0.021476	0.966055
II	0.530394	0.469606	0.966054
III	0.045171	0.954829	0.999866

Table 1 Columns p_1 and $p_2 = 1 - p_1$ collect the values of the probabilities given by (25) corresponding to Cases I–III, when $J_i \sim \mathrm{N}(\mu_i, \Sigma)$, being μ_i and Σ specified in (24). Values of p_s represent the probabilities associated with asymptotic stability according to (26).

Based on the well-known condition that characterizes the asymptotic stability of the zero-steady state solution, $Z(t) \equiv 0$, to the deterministic counterpart of random IVP's (5) and (18), that is $A_1 > 0$ and $A_2 > 0$, we will study the asymptotic stability of the solution $W(u) \equiv 0$ to the random IVP (4), hence also of the solution $V(s) \equiv 0$. This analysis relies on the computation of the following probability, p_s,

$$\begin{aligned} p_s &= \mathbb{P}[\omega \in \Omega : A_1(\omega) > 0, A_2(\omega) > 0] = \mathbb{P}[\omega \in \Omega : (B-1)(\omega) > 0, A_2(\omega) > 0] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{1}^{\infty}\int_{0}^{\infty} f_{X_0,C,B,A_2}(x_0,c,b,a_2)\,\mathrm{d}a_2\,\mathrm{d}b\,\mathrm{d}x_0\,\mathrm{d}c. \end{aligned} \tag{26}$$

Values of p_s in the Cases I–III are shown in Table 1.

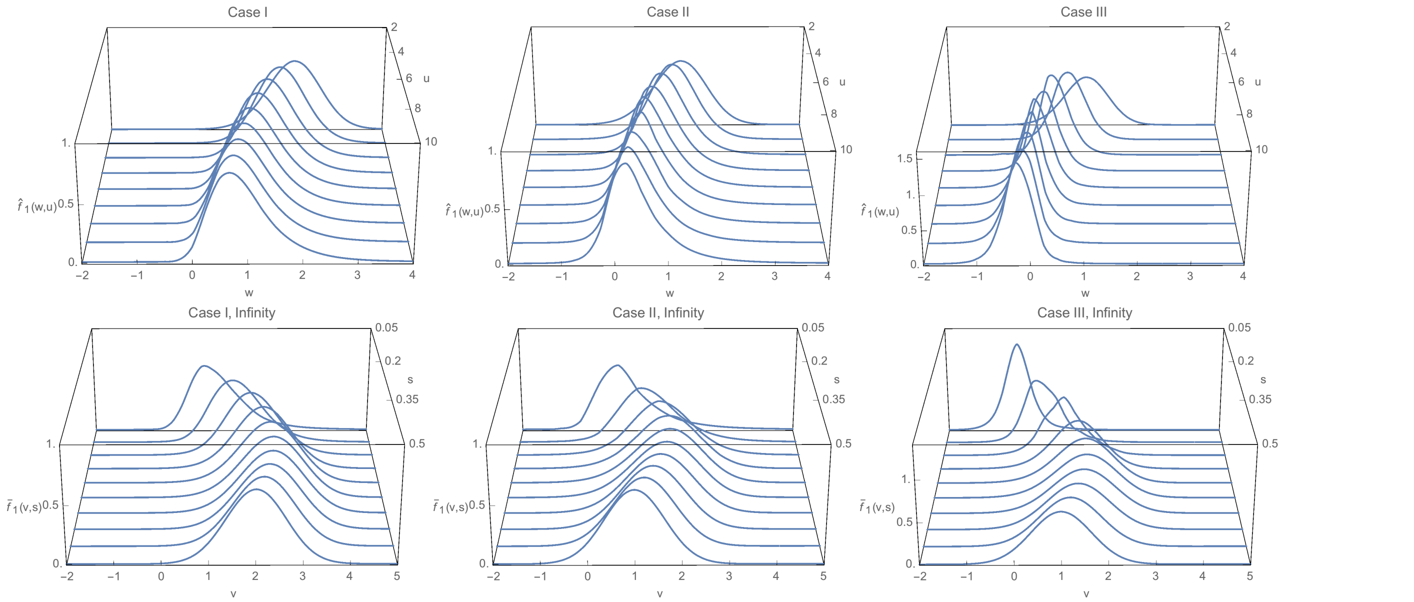

Fig. 1 Top: Plots of the 1-PDF of the solution SP, $W(u)$, to the random IVP (4) with $u_0 = 1$ given in (8), (11), (13) in Cases I–III at different values of $u \in \{2,3,...,10\}$. Bottom: Plots of the 1-PDF of the solution SP, $V(s)$, to the random IVP (17) with $s_0 = 0.5$ given in (19), (20), (22) in Cases I–III at different values of $s \in \{0.05, 0.1, ..., 0.5\}$.

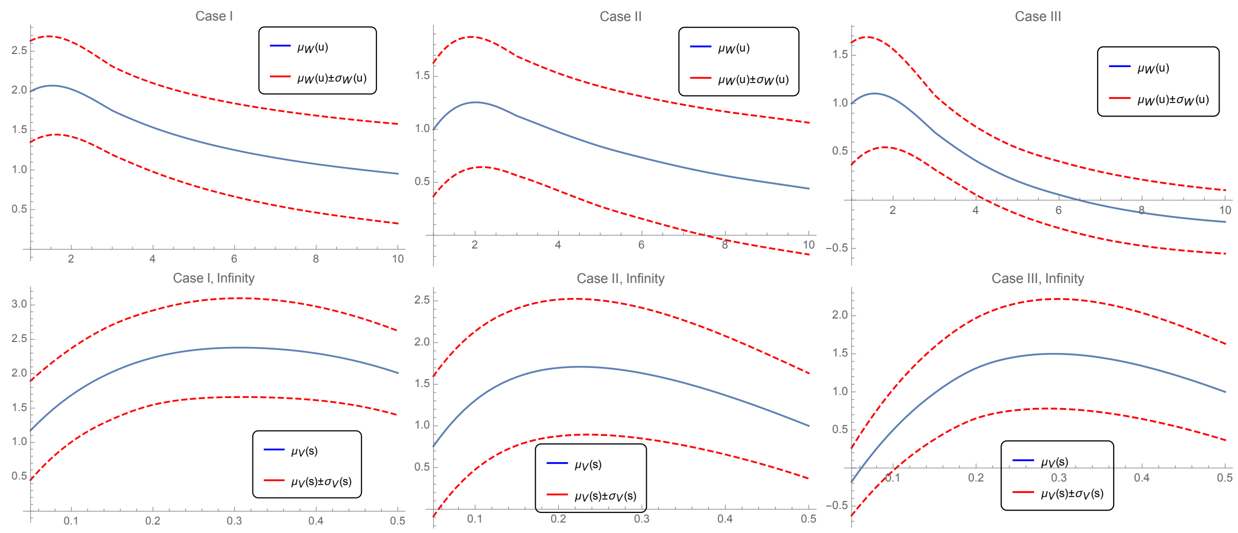

Fig. 2 Top: Plots of the mean, $\mu_W(u)$, and plus/minus the standard deviation, $\sigma_W(u)$, of the solution SP, $W(u)$, to IVP (4) in Cases I–III at different values of $u \in [1, 10]$. Bottom: Plots of the mean, $\mu_V(s)$, and plus/minus the standard deviation, $\sigma_V(s)$, of the solution SP, $V(s)$, to IVP (17) in Cases I–III at different values of $s \in [0.05, 0.5]$.

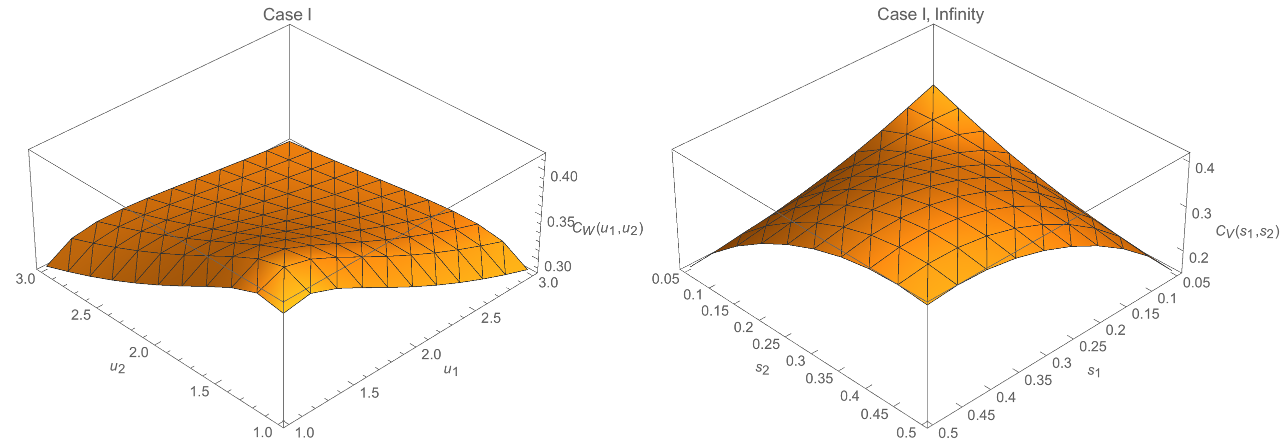

Fig. 3 Covariance function given by (3) in the Case I to both problems, IVP (4) (left) and IVP (17) (right) for the values of $u_1, u_2 \in [1,3]$ and $s_1, s_2 \in [0.05, 0.5]$.

In Figure 1, the graphical representations for the 1-PDF, $\hat{f}_1(w,u)$, at $u \in \{2,3,\ldots,10\}$ and $\bar{f}(v,s)$, at $v \in \{0.05, 0.1, \ldots, 0.5\}$ in Cases I–III are shown. In these graphical representations, we can see how the PDFs evolves over the times u and s in each Case I–III. The behaviour of the PDFs is in full agreement with the plots shown in Figure 2 where the means $\mu_W(u)$ and $\mu_V(s)$ plus/minus the standard deviations $\sigma_W(u)$ and $\sigma_V(s)$ have been plotted for both problems. Regarding the IVP (4), in the three cases we observe as the mean function decreases as u increases and the standard deviation increases slowly. To the IVP (17) we can observe that the mean increase at first and then decrease. In addition, in Figure 3 the covariance function given by the expression (2) has been plotted to Case I for both problems.

5 Conclusions

In this work we have given a full probabilistic solution to the randomized Cauchy-Euler differential equation under very general conditions. Specifically, we have obtained closed explicit formulas for the first and the second probability density functions of the solution stochastic process of the random Cauchy-Euler differential equation assuming that input parameters (coefficients and initial conditions) are absolutely continuous random variables with a joint probability density function. In this manner, all one-dimensional statistical moments of the solution can be computed including the mean, the variance and the covariance functions. The method allows us to construct punctual and probabilistic predictions in practical applications where the random Cauchy-Euler model appears. Furthermore, the technique applied in our study could be extended to perform an analogous analysis of other important randomized differential equations.

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