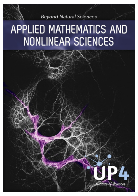
CrossMark

Applied Mathematics and Nonlinear Sciences 2(1) (2017) 13–20



Applied Mathematics and Nonlinear Sciences

http://journals.up4sciences.org

A sufficient condition for the existence of a k-factor excluding a given r-factor

Sizhong Zhou[1]†, Lan Xu[2] and Yang Xu[3].

[1]School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, P. R. CHINA

[2]Department of Mathematics, Changji University, Changji, Xinjiang 831100, P. R. CHINA

[3]Department of Mathematics, Qingdao Agricultural University, Qingdao, Shandong 266109, P. R. CHINA

Submission Info

Communicated by Juan L.G. Guirao
Received 4th November 2016
Accepted 10th January 2017
Available online 10th January 2017

Abstract

Let G be a graph, and let k,r be nonnegative integers with $k \geq 2$. A k-factor of G is a spanning subgraph F of G such that $d_F(x) = k$ for each $x \in V(G)$, where $d_F(x)$ denotes the degree of x in F. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{x \in S} N_G(x)$. The binding number of G is defined by $bind(G) = \min\left\{\frac{|N_G(S)|}{|S|} : \emptyset \neq S \subset V(G), N_G(S) \neq V(G)\right\}$. In this paper, we obtain a binding number and neighborhood condition for a graph to have a k-factor excluding a given r-factor. This result is an extension of the previous results.

Keywords: graph; binding number; neighborhood; k-factor
AMS 2010 codes: 05C70

1 Introduction

For motivation and background to this work see [1]. In this paper, we consider only finite and simple graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. A graph is Hamiltonian if it admits a Hamiltonian cycle. For each $x \in V(G)$, the neighborhood $N_G(x)$ of x is the set of vertices of G adjacent to x, and the degree $d_G(x)$ of x is $|N_G(x)|$. For $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{x \in S} N_G(x)$. $G[S]$ denotes the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. A vertex subset S of G is called independent if $G[S]$ has no edges. The symbol $\delta(G)$ denotes the minimum degree of G. The binding number

†Corresponding author.
Email address: zsz_cumt@163.com

of G is defined by $bind(G) = \min\left\{\frac{|N_G(S)|}{|S|} : \emptyset \neq S \subset V(G), N_G(S) \neq V(G)\right\}$. A spanning subgraph F of G with $d_F(x) = k$ for each $x \in V(G)$ is called a k-factor of G.

Many authors studied graph factors [1–10]. Anderson [11] gave a binding number condition for graphs to have 1-factors. Woodall [12] showed a binding number condition for a graph to have a Hamiltonian cycle (or a 2-factor). Katerinis and Woodall [13] obtained a binding number condition for graphs to have k-factors. The following theorems of k-factors in terms of binding number are known.

Theorem 1 (Anderson [11])**.** *Let G be a graph of order n. If n is even and $bind(G) \geq \frac{4}{3}$, then G has a 1-factor.*

Theorem 2 (Woodall [12])**.** *Let G be a graph. If $bind(G) \geq \frac{3}{2}$, then G has a Hamiltonian cycle (or a 2-factor).*

Theorem 3 (Katerinis and Woodall [13])**.** *Let $k \geq 2$ be an integer and let G be a graph of order $n \geq 4k-6$ and binding number $bind(G)$ such that kn is even and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$. Then G has a k-factor.*

In this paper, we obtain a binding number condition for a graph to have a k-factor excluding a given r-factor, which is an extension of Theorems 1, 2, and 3. The main result will be given in the following section.

2 Main Theorems

In this section, we give our main results, which are the following theorems.

Theorem 4. *Let k and r be two nonnegative integers with $k \geq 2$, and let G be a graph of order n with $n \geq \frac{(2k-1)(2k-3)}{k}$, and let G have an r-factor Q. Suppose that kn is even, $bind(G) \geq \frac{(2k-1)(n-1)}{kn-(r+1)(2k-1)+1}$ and $|N_G(X)| > \frac{(k-1)n+(2rk-r+1)|X|-2}{2k-1}$ for any nonempty independent subset X of $V(G)$. Then G has a k-factor excluding a given r-factor Q if $G-E(Q)$ is connected.*

If $r = 0$ in Theorem 4, then we obtain the following corollary.

Corollary 5. *Let k be a nonnegative integer with $k \geq 2$, and let G be a graph of order n with $n \geq \frac{(2k-1)(2k-3)}{k}$. Suppose that kn is even, $bind(G) \geq \frac{(2k-1)(n-1)}{kn-(2k-2)}$ and $|N_G(X)| > \frac{(k-1)n+|X|-2}{2k-1}$ for any nonempty independent subset X of $V(G)$. Then G has a k-factor.*

If Q is a Hamiltonian cycle in Theorem 4, then we obtain the following corollary.

Corollary 6. *Let k be a nonnegative integer with $k \geq 2$, and let G be a Hamiltonian graph of order n with $n \geq \frac{(2k-1)(2k-3)}{k}$. Suppose that kn is even, $bind(G) \geq \frac{(2k-1)(n-1)}{kn-2(3k-2)}$ and $|N_G(X)| > \frac{(k-1)n+(4k-1)|X|-2}{2k-1}$ for any nonempty independent subset X of $V(G)$. Then G has a k-factor excluding a given Hamiltonian cycle C if $G-E(C)$ is connected.*

Unfortunately, the authors do not know whether the conditions in Theorem 4 are the best possible or not. Hence, we pose the following conjecture.

Conjecture 7. *Let k and r be two nonnegative integers with $k \geq 2$, and let G be a graph of order n with $n \geq \frac{(2k-1)(2k-3)}{k}$, and let G have an r-factor Q. Suppose that kn is even, $bind(G) \geq \frac{(2k-1)(n-1)}{kn-(r+1)(2k-1)+2}$ and $|N_G(X)| \geq \frac{(k-1)n+(2rk-r+1)|X|-2}{2k-1}$ for any nonempty independent subset X of $V(G)$. Then G has a k-factor excluding a given r-factor Q if $G-E(Q)$ is connected.*

Using Theorem 4, we obtain a binding number condition for a graph to have a k-factor including a given r-factor.

Theorem 8. *Let k and r be two nonnegative integers with $k \geq r+2$, and let G be a graph of order n with $n \geq \frac{(2k-2r-1)(2k-2r-3)}{k-r}$, and let G have an r-factor Q. Suppose that kn and rn are both even, $bind(G) \geq \frac{(2k-2r-1)(n-1)}{(k-r)n-(r+1)(2k-2r-1)+1}$ and $|N_G(X)| > \frac{(k-r-1)n+(2rk-2r^2-r+1)|X|-2}{2k-2r-1}$ for any nonempty independent subset X of $V(G)$. Then G has a k-factor including a given r-factor Q if $G-E(Q)$ is connected.*

Proof. By the assumption of Theorem 8, G has an r-factor Q. Let $m = k-r$. Then we have $m \geq 2$, mn even, $n \geq \frac{(2m-1)(2m-3)}{m}$, $bind(G) \geq \frac{(2m-1)(n-1)}{mn-(r+1)(2m-1)+1}$, $|N_G(X)| > \frac{(m-1)n+(2rm-r+1)|X|-2}{2m-1}$ for any nonempty independent subset X of $V(G)$, and $G-E(Q)$ connected. According to Theorem 4, G has an m-factor F' excluding a given r-factor Q, and G has a k-factor F $(F = E(F') \cup E(Q))$ including a given r-factor Q. This completes the proof of Theorem 8.

If Q is a Hamiltonian cycle in Theorem 8, then we obtain the following corollary.

Corollary 9. *Let k be a nonnegative integer with $k \geq 4$, and let G be a Hamiltonian graph of order n with $n \geq \frac{(2k-5)(2k-7)}{k-2}$. Suppose that kn is even, $bind(G) \geq \frac{(2k-5)(n-1)}{(k-2)n-2(3k-8)}$ and $|N_G(X)| > \frac{(k-3)n+(4k-9)|X|-2}{2k-5}$ for any nonempty independent subset X of $V(G)$. Then G has a k-factor including a given Hamiltonian cycle C if $G-E(C)$ is connected.*

The previous results on a graph to have a k-factor including a given Hamiltonian cycle are shown in the following

Theorem 10 (Matsuda [14])**.** *Let $k \geq 2$ be an integer and let G be a graph of order $n > 8k^2 - 2(\alpha+12)k + 3\alpha + 16$, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Suppose that kn is even and the minimum degree $\delta(G)$ of G is at least k. If for any nonadjacent vertices x and y of G, $d_G(x)+d_G(y) \geq n+\alpha$, then G has a k-factor including a given Hamiltonian cycle.*

Theorem 11 (Gao, Li, and Li [15])**.** *Let $k \geq 2$ be an integer and let G be a graph of order $n > 12(k-2)^2 + 2(5-\alpha)(k-2) - \alpha$. Suppose that kn is even, $\delta(G) \geq k$ and $\max\{d_G(x), d_G(y)\} \geq \frac{n+\alpha}{2}$ for each pair of nonadjacent vertices x and y in G, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Then G has a k-factor including a given Hamiltonian cycle C if $G-E(C)$ is connected.*

3 The Proof of Theorem 4

Let G be a graph, and $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. We use $e_G(S,T)$ to denote the number of edges that join S and T. For an integer $k \geq 1$, a component C of $G-(S \cup T)$ is called an odd component if $k|V(C)| + e_G(V(C),T)$ is odd. We write

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| - h_G(S,T),$$

where $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $h_G(S,T)$ is the number of odd components of $G-(S \cup T)$.

The proof of Theorem 4 relies heavily on the following lemmas.

Lemma 12 (Tutte [16])**.** *Let G be a graph of order n and k a positive integer. Then for any disjoint subsets S and T of $V(G)$, the following statements hold:*

1. *G has a k-factor if and only if $\delta_G(S,T) \geq 0$.*
2. *$\delta_G(S,T) \equiv kn (mod\ 2)$.*

Lemma 13 (Katerinis and Woodall [13])**.** *Let G be a graph of order n and k a positive integer with kn even. Suppose that there exists a pair of disjoint subsets S and T of $V(G)$ such that*

$$\delta_G(S,T) \leq -2. \quad (1)$$

Let $W = G-S-T$ and let ω be the number of components of W. If $|S \cup T|$ is maximal subject to (1), then $|V(C)| \geq 3$ for every component C of W, so that $|V(W)| \geq 3\omega$.

Lemma 14 (Woodall [12]). *Let G be a graph of order n with $bind(G) \geq c$. Then $\delta(G) \geq n - \frac{n-1}{c}$.*

Proof of Theorem 4. According to the assumption of Theorem 4, G has an r-factor Q. Set $H = G - E(Q)$. Then $V(H) = V(G)$. Hence G has a desired factor if and only if H has a k-factor. By way of contradiction, we assume that H has no k-factor. Then, by Lemma 12, there exist two disjoint subsets S and T of $V(H) = V(G)$ such that

$$\delta_H(S,T) = k|S| + d_{H-S}(T) - k|T| - h_H(S,T) \leq -2, \tag{2}$$

where $h_H(S,T)$ denotes the number of odd components of $H - (S \cup T)$. And subject to (2), we choose S and T such that $|S \cup T|$ is as large as possible. From (2), we have

$$k|S| + d_{H-S}(T) - k|T| - \omega \leq -2, \tag{3}$$

where ω denotes the number of components of $H - (S \cup T)$. Obviously,

$$\omega \leq n - |S| - |T|. \tag{4}$$

If $\omega > 0$, then let m denote the minimum order of components of $H - (S \cup T)$. We shall make use of the obvious facts that

$$\delta(H) \leq m - 1 + |S| + |T| \tag{5}$$

and

$$m\omega \leq |V(H)| - |S| - |T| = n - |S| - |T| \leq n. \tag{6}$$

Moreover, it follows from Lemma 13 and the choice of S and T that $m \geq 3$.

According to Lemma 14 and $bind(G) \geq \frac{(2k-1)(n-1)}{kn-(r+1)(2k-1)+1}$, we have

$$\delta(G) \geq n - \frac{n-1}{\frac{(2k-1)(n-1)}{kn-(r+1)(2k-1)+1}} = \frac{(k-1)n + (r+1)(2k-1) - 1}{2k-1}. \tag{7}$$

Claim 1. $T \neq \emptyset$.

Proof. Suppose that $T = \emptyset$.

If $S = \emptyset$, then by (3) we obtain

$$\omega \geq k|S| + d_{H-S}(T) - k|T| + 2 = 2,$$

which contradicts the assumption that $H = G - E(Q)$ is connected.

If $S \neq \emptyset$, then from (3) and (4) we deduce

$$0 < k|S| + 2 \leq \omega \leq n - |S|. \tag{8}$$

Using (8), we have

$$n - 1 - k|S| - |S| \geq 1. \tag{9}$$

In view of (5), (6) and (8), we obtain

$$\begin{aligned}
\delta(H) &\leq m - 1 + |S| \leq \frac{n - |S|}{\omega} - 1 + |S| \\
&\leq \frac{n - |S|}{k|S| + 2} - 1 + |S| < \frac{n - |S|}{k|S| + 1} - 1 + |S| \\
&= \frac{n-1}{k+1} - \frac{(n - 1 - k|S| - |S|)(k|S| - k)}{(k+1)(k|S| + 1)}.
\end{aligned}$$

Combining this with (9) and $|S| \geq 1$, we have

$$\delta(H) < \frac{n-1}{k+1} - \frac{(n-1-k|S|-|S|)(k|S|-k)}{(k+1)(k|S|+1)} \leq \frac{n-1}{k+1}. \tag{10}$$

Note that $\delta(H) = \delta(G) - r$. Using (7) and (10), we have

$$\frac{n-1}{k+1} > \delta(H) = \delta(G) - r \geq \frac{(k-1)n+(r+1)(2k-1)-1}{2k-1} - r = \frac{(k-1)n+2k-2}{2k-1},$$

which is a contradiction since $k \geq 2$. Hence, $T \neq \emptyset$. The proof of Claim 3 is complete.
Since $T \neq \emptyset$, we define

$$h = \min\{d_{H-S}(x) : x \in T\}.$$

The following proof splits into four cases by the value of h.
Case 1. $h = 0$.
Set $X = \{x : x \in T, d_{H-S}(x) = 0\}$. Clearly, $X \neq \emptyset$ since $h = 0$ and X is an independent subset of $V(H)$. It is easy to see that

$$|S| \geq |N_H(X)|. \tag{11}$$

Note that $|N_H(X)| \geq |N_G(X)| - r|X|$. According to (11) and the assumption of Theorem 4, we obtain

$$|S| \geq |N_H(X)| \geq |N_G(X)| - r|X| > \frac{(k-1)n+|X|-2}{2k-1}. \tag{12}$$

In view of (3), (4), (12), $|S|+|T| \leq n$ and $k \geq 2$, we deduce

$$\begin{aligned}
-2 &\geq k|S| + d_{H-S}(T) - k|T| - \omega \\
&\geq k|S| + d_{H-S}(T) - k|T| - (n-|S|-|T|) \\
&\geq k|S| + |T| - |X| - k|T| - n + |S| + |T| \\
&= (k+1)|S| - (k-2)|T| - |X| - n \\
&\geq (k+1)|S| - (k-2)(n-|S|) - |X| - n \\
&= (2k-1)|S| - (k-1)n - |X| \\
&> (2k-1) \cdot \frac{(k-1)n+|X|-2}{2k-1} - (k-1)n - |X| \\
&= -2.
\end{aligned}$$

That is a contradiction.
Case 2. $1 \leq h \leq k-1$.
Note that $\delta(H) \leq |S| + h$ and $\delta(H) = \delta(G) - r$. And using (7), we obtain

$$|S| \geq \delta(G) - r - h \geq \frac{(k-1)n+2k-2}{2k-1} - h. \tag{13}$$

According to (3), (4), $|S|+|T| \leq n$ and $1 \leq h \leq k-1$, we have

$$\begin{aligned}
-2 &\geq k|S| + d_{H-S}(T) - k|T| - \omega \\
&\geq k|S| + h|T| - k|T| - (n-|S|-|T|) \\
&= (k+1)|S| - (k-1-h)|T| - n \\
&\geq (k+1)|S| - (k-1-h)(n-|S|) - n \\
&= (2k-h)|S| - (k-h)n,
\end{aligned}$$

that is,

$$-2 \geq (2k-h)|S| - (k-h)n. \tag{14}$$

Multiplying (14) by $(2k-1)$ and rearranging, and then using (13),

$$\begin{aligned}0 &\geq (2k-h)(2k-1)|S| - (2k-1)(k-h)n + 2(2k-1)\\ &\geq (2k-h)((k-1)n + 2k - 2 - (2k-1)h) - (2k-1)(k-h)n + 2(2k-1)\\ &= (h-1)(kn - (2k-1)(2k-h) + 1) + 2k - 1,\end{aligned}$$

that is,

$$0 \geq (h-1)(kn - (2k-1)(2k-h) + 1) + 2k - 1. \tag{15}$$

If $h = 1$, then from (15) we obtain

$$0 \geq 2k - 1 > 0,$$

which is a contradiction.
If $h = 2$, then by (15) and $n \geq \frac{(2k-1)(2k-3)}{k}$, we obtain

$$\begin{aligned}0 &\geq (h-1)(kn - (2k-1)(2k-h) + 1) + 2k - 1\\ &= kn - (2k-1)(2k-2) + 1 + 2k - 1\\ &\geq (2k-1)(2k-3) - (2k-1)(2k-2) + 1 + 2k - 1\\ &= 1,\end{aligned}$$

a contradiction.
If $3 \leq h \leq k-1$, then using (15) and $n \geq \frac{(2k-1)(2k-3)}{k}$, we have

$$\begin{aligned}0 &\geq (h-1)(kn - (2k-1)(2k-h) + 1) + 2k - 1\\ &\geq (h-1)(kn - (2k-1)(2k-3) + 1) + 2k - 1\\ &\geq h - 1 + 2k - 1\\ &> 2k - 1 > 0,\end{aligned}$$

that is a contradiction.
Case 3. $h = k$.
According to (3), we obtain

$$\begin{aligned}-2 &\geq k|S| + d_{H-S}(T) - k|T| - \omega\\ &\geq k|S| + h|T| - k|T| - \omega\\ &= k|S| - \omega,\end{aligned}$$

which implies

$$\omega \geq k|S| + 2. \tag{16}$$

In view of (6) and Claim 3, we have

$$\omega \leq \frac{n - |S| - |T|}{m} \leq \frac{n - |S| - 1}{m}.$$

Combining this with (16) and $m \geq 3$, we infer

$$k|S| + 2 \leq \omega \leq \frac{n - |S| - 1}{m} \leq \frac{n - |S| - 1}{3}, \tag{17}$$

which implies

$$|S| \leq \frac{n-7}{3k+1}. \tag{18}$$

Using (7), $h = k$, $\delta(H) = \delta(G) - r$ and $\delta(H) \leq |S| + h$, we deduce

$$|S| \geq \frac{(k-1)n + 2k - 2}{2k-1} - k,$$

which contradicts (18) since $k \geq 2$ and $n \geq \frac{(2k-1)(2k-3)}{k}$.

Case 4. $h \geq k+1$.

According to (3), we have

$$\begin{aligned}
-2 &\geq k|S| + d_{H-S}(T) - k|T| - \omega \\
&\geq k|S| + h|T| - k|T| - \omega \\
&\geq k|S| + |T| - \omega.
\end{aligned}$$

Combining this with Claim 3, we obtain

$$\omega \geq k|S| + |T| + 2 \geq |S| + |T| + 2 \geq 3. \tag{19}$$

In view of (5), (6), and (19), $m \geq 3$ and $\delta(G) = \delta(H) + r$, we obtain

$$\begin{aligned}
\delta(G) &= \delta(H) + r \leq m - 1 + |S| + |T| + r \\
&\leq m - 1 + \omega - 2 + r = m + \omega - 3 + r \\
&\leq m + \omega - 3 + \frac{(m-3)(\omega-3)}{3} + r \\
&= \frac{m\omega}{3} + r \leq \frac{n}{3} + r,
\end{aligned}$$

which contradicts (7).

Hence, G has a desired factor, that is, G has a k-factor excluding a given r-factor. This completes the proof of Theorem 4.

Acknowledgements

The authors would like to thank the anonymous referees for their comments on this paper. This work is supported by the National Natural Science Foundation of China (Grant No. 11371009, 11501256, 61503160) and the National Social Science Foundation of China (Grant No. 14AGL001), and sponsored by 333 Project of Jiangsu Province.

References

[1] W. Gao, Y. Gao, T. Xu and L. Liang, (2014), *Vulnerability of Networks and Existence of Fractional Factor Avoiding Certain Channels and Sites*, Journal of Computers 9, No 7, 1712-1722. doi 10.4304/jcp.9.7.1712-1722

[2] W. Gao and W. Wang, (2015), *Toughness and fractional critical deleted graph*, Utilitas Mathematica 98, 295-310.

[3] W. Gao, L. Liang, T. Xu and J. Zhou, (2014), *Tight Toughness Condition for Fractional* (g, f, n)*-Critical Graphs*, Journal of the Korean Mathematical Society 51, No 1, 55-65. doi 10.4134/JKMS.2014.51.1.055

[4] J. Cai, G. Liu and J. Hou, (2008), *Binding number and Hamiltonian* (g, f)*-factors in graphs II*, International Journal of Computer Mathematics 85, No 9, 1325-1331. doi 10.1080/00207160701524368

[5] S. Zhou, (2009), *Independence number, connectivity and* (a, b, k)*-critical graphs*, Discrete Mathematics 309, No 12, 4144-4148. doi 10.1016/j.disc.2008.12.013

[6] S. Zhou, (2011), *A sufficient condition for graphs to be fractional* (k,m)*-deleted graphs*, Applied Mathematics Letters 24, No 9, 1533-1538. doi 10.1016/j.aml.2011.03.041
[7] S. Zhou, (2010), *A sufficient condition for a graph to be an* (a,b,k)*-critical graph*, International Journal of Computer Mathematics 87, No 10, 2202-2211. doi 10.1080/00207160902777914
[8] S. Zhou, (2009), *A minimum degree condition of fractional* (k,m)*-deleted graphs*, Comptes Rendus Mathematique 347, No 21-22, 1223-1226. doi 10.1016/j.crma.2009.09.022
[9] S. Zhou, (2011), *Some new sufficient conditions for graphs to have fractional k-factors*, International Journal of Computer Mathematics 88, No 3, 484-490. doi 10.1080/00207161003681286
[10] S. Zhou, (2014), *Remarks on orthogonal factorizations of digraphs*, International Journal of Computer Mathematics 91, No 10, 2109-2117. doi 10.1080/00207160.2014.881993
[11] I. Anderson, (1971), *Perfect matchings of a graph*, Journal of Combinatorial Theory, Series B 10, No 3, 183-186. doi 10.1016/0095-8956(71)90041-4
[12] D.R Woodall, (1973), *The binding number of a graph and its Anderson number*, Journal of Combinatorial Theory Series B 15, No 3, 225-255. doi 10.1016/0095-8956(73)90038-5
[13] P. Katerinis and D.R Woodall, (1987), *Binding numbers of graphs and the existence of k-factors*, The Quarterly Journal of Mathematics 38, No 2, 221-228. doi 10.1093/qmath/38.2.221
[14] H. Matsuda, (2005), *Regular Factors Containing a Given Hamiltonian Cycle*, in: Combinatorial Geometry and Graph Theory, Vol 3330 of the series Lecture Notes in Computer Science, 123-132. doi 10.1007/978-3-540-30540-8_14
[15] Y. Gao, G. Li and X. Li, (2009), *Degree condition for the existence of a k-factor containing a given Hamiltonian cycle*, Discrete Mathematics 309, No 8, 2373-2381. doi 10.1016/j.disc.2008.05.014
[16] W. T. Tutte, (1952), *The factors of graphs*, Canadian Journal of Mathematics 4, No 3, 314-328. doi 10.4153/CJM-1952-028-2